2.1 Introduction to Linear Programming

A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + …. + a_nx_n$, where $a_1, a_2, …, a_n$ are constants and $x_1, x_2 … x_n$ are variables. The term Programming refers to the process of determining a particular program or plan of action. So Linear Programming (LP) is one of the most important optimization (maximization / minimization) techniques developed in the field of Operations Research (OR).

The methods applied for solving a linear programming problem are basically simple problems; a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n-variables ($x_1, x_2 … x_n$), at least one of them is non-zero, can be obtained if there are exactly *n* relations. When the number of relations is greater than or less than *n*, a unique solution does not exist but a number of trial solutions can be found.

In various practical situations, the problems are seen in which the number of relations is not equal to the number of the number of variables and many of the relations are in the form of inequalities (≤ or ≥) to maximize or minimize a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problem (LPP).

Definition – The general LPP calls for optimizing (maximizing / minimizing) a linear function of variables called the '**Objective function**' subject to a set of linear equations and / or inequalities called the '**Constraints**' or '**Restrictions'**.

2.2 General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for $x_1, x_2 … x_n$ so as to maximize or minimize

$$Z = c_1x_1 + c_2x_2 +…………+c_nx_n$$

subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_1nx_n\ (\leq \text{or} \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_2nx_n\ (\leq \text{or} \geq)\ b_2$

.
.
.

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n\ (\leq \text{or} \geq)\ b_m$

and

$x_1 \geq 0,\ x_2 \geq 0, \ldots,\ x_n \geq 0$

Where

Z = value of overall measure of performance

x_j = level of activity (for j = 1, 2, ..., n)

c_j = increase in Z that would result from each unit increase in level of activity j

b_i = amount of resource i that is available for allocation to activities (for i = 1,2, …, m)

a_{ij} = amount of resource i consumed by each unit of activity j

Resource	Resource usage per unit of activity	Amount of resource available
	Activity	
	1 2 …………………….. n	
1	a_{11} a_{12} …………………….a_{1n}	b_1
2	a_{21} a_{22} …………………….a_{2n}	b_2
.	.	.
.	.	.
.	.	.
m	a_{m1} a_{m2} …………………….a_{mn}	b_m
Contribution to Z per unit of activity	c_1 c_2 ………………………..c_n	

Data needed for LP model

- The level of activities x_1, x_2………x_n are called **decision variables**.

- The values of the c_j, b_i, a_{ij} (for i=1, 2 … m and j=1, 2 … n) are the **input constants** for the model. They are called as **parameters** of the model.
- The function being maximized or minimized $Z = c_1x_1 + c_2x_2 +\dots +c_nx_n$ is called **objective function**.
- The restrictions are normally called as **constraints**. The constraint $a_{i1}x_1 + a_{i2}x_2 \dots a_{in}x_n$ are sometimes called as **functional constraint** (L.H.S constraint). $x_j \geq 0$ restrictions are called **non-negativity constraint**.

2.3 Assumptions in LPP

1. **Proportionality**

   The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable i.e. if resource availability increases by some percentage, then the output shall also increase by same percentage.

2. **Additivity**

   Sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all resources individually or collectively.

3. **Divisibility**

   The variables are not restricted to integer values

4. **Deterministic**

   Coefficients in the objective function and constraints are completely known and do not change during the period under study in all the problems considered.

5. **Finiteness**

   Variables and constraints are finite in number.

6. **Optimality**

   In LPP, we determine the decision variables so as to optimize the objective function of the LPP.

7. The problem involves only one objective, profit maximization or cost minimization.

2.4 Applications of Linear Programming

Personnel Assignment Problem

Suppose we are given 'm' persons, 'n' jobs and the expected productivity c_{ij} of i^{th} person on the j^{th} job. We want to find an assignment of person's $x_{ij} \geq 0$ for all i and j, to 'n' jobs so that the average productivity of person assigned is maximum, subject to the conditions

$$\sum_{j=1}^{n} x_{ij} \leq a_i \quad \text{and} \quad \sum_{i=1}^{m} x_{ij} \leq b_j$$

Where a_i is the number of persons in personnel category i

b_j is the number of jobs in personnel category j

Transportation Problem

Suppose that 'm' factories (sources) supply 'n' warehouses (destinations) with certain product. Factory F_i (i=1, 2 … m) produces a_i units and warehouse W_j (j=1, 2, 3 … n) requires b_j units. Suppose that the cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped and that the unit cost is c_{ij}. Let the decision variables x_{ij} be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \, c_{ij} \quad \text{is minimized}$$

The supply and demand must be satisfied exactly.

Mathematically, this problem is to find x_{ij} (i=1, 2 … m; j=1, 2 … n) in order to minimize the total transportation cost

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij}$$

Subject to constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, .., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{for } j = 1, 2, .., n$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \text{ and } x_{ij} \geq 0$$

Efficiency on Operation of system of Dams

In this problem, we determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

Optimum Estimation of Executive Compensation

The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

Agriculture Applications

Linear programming can be applied in agricultural planning for allocating the limited resources such as labour, water supply and working capital etc, so as to maximize the net revenue.

Military Applications

These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets and the problem of community defense against disaster to find the number of defense units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

Production Management

Linear programming can be applied in production management for determining product mix, product smoothing and assembly time-balancing.

Marketing Management

Linear programming helps in analyzing the effectiveness of advertising campaign and time based on the available advertising media. It also helps in travelling salesman in finding the shortest route for his tour.

Manpower Management

Linear programming allows the personnel manager to analyze personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

Physical distribution

Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centers for physical distribution.

2.5 Advantages of Linear Programming Techniques

1. It helps us in making the optimum utilization of productive resources.
2. The quality of decisions may also be improved by linear programming techniques.
3. Provides practically solutions.

4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique.

2.6 Limitations of Linear Programming

Some limitations are associated with linear programming techniques

1. In some problems, objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.
2. There is no guarantee of getting integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.
3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
4. Sometimes large scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.
5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.
6. Linear programming deals with only single objective, whereas in real life situation problems come across with multi objectives. Goal programming and multi-objective programming deals with such problems.

2.7 Formulation of LP Problems

Example 1

A firm manufactures two types of products A and B and sells them at a profit of Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires 1 minute of processing time on G and 2 minutes on H; type B requires 1 minute on G and 1 minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Solution

Let

x_1 be the number of products of type A

x_2 be the number of products of type B

After understanding the problem, the given information can be systematically arranged in the form of the following table.

	Type of products (minutes)		
Machine	Type A (x_1 units)	Type B (x_2 units)	Available time (mins)
G	1	1	400
H	2	1	600
Profit per unit	Rs. 2	Rs. 3	

Since the profit on type A is Rs. 2 per product, 2 x_1 will be the profit on selling x_1 units of type A. similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of type B is given by

Maximize $Z = 2x_1 + 3x_2$ (**objective function**)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + 3x_2$.

But, machine G is not available for more than 6 hours 40 minutes (400 minutes). Therefore,

$$x_1 + x_2 \leq 400 \text{ (\textbf{first constraint})}$$

Also, the machine H is available for 10 hours (600 minutes) only, therefore,

$$2x_1 + 3x_2 \leq 600 \text{ (\textbf{second constraint})}$$

Since it is not possible to produce negative quantities

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ (\textbf{non-negative restrictions})}$$

Hence

Maximize $Z = 2x_1 + 3x_2$

Subject to restrictions

$$x_1 + x_2 \leq 400$$

$$2x_1 + 3x_2 \leq 600$$

and non-negativity constraints

$$x_1 \geq 0 , x_2 \geq 0$$

Example 2

A company produces two products A and B which possess raw materials 400 quintals and 450 labour hours. It is known that 1 unit of product A requires 5 quintals of raw materials and 10 man hours and yields a profit of Rs 45. Product B requires 20 quintals of raw materials, 15 man hours and yields a profit of Rs 80. Formulate the LPP.

Solution

Let

x_1 be the number of units of product A

x_2 be the number of units of product B

	Product A	Product B	Availability
Raw materials	5	20	400
Man hours	10	15	450
Profit	Rs 45	Rs 80	

Hence

Maximize $Z = 45x_1 + 80x_2$

Subject to

$5x_1 + 20x_2 \leq 400$

$10x_1 + 15x_2 \leq 450$

$x_1 \geq 0$, $x_2 \geq 0$

Example 3

A firm manufactures 3 products A, B and C. The profits are Rs. 3, Rs. 2 and Rs. 4 respectively. The firm has 2 machines and below is given the required processing time in minutes for each machine on each product.

	Products		
Machine	A	B	C
X	4	3	5
Y	2	2	4

Machine X and Y have 2000 and 2500 machine minutes. The firm must manufacture 100 A's, 200 B's and 50 C's type, but not more than 150 A's.

Solution

Let

x_1 be the number of units of product A

x_2 be the number of units of product B

x_3 be the number of units of product C

	Products			
Machine	A	B	C	Availability
X	4	3	5	2000
Y	2	2	4	2500
Profit	3	2	4	

Max Z = $3x_1 + 2x_2 + 4x_3$

Subject to

$4x_1 + 3x_2 + 5x_3 \leq 2000$

$2x_1 + 2x_2 + 4x_3 \leq 2500$

$100 \leq x_1 \leq 150$

$x_2 \geq 200$

$x_3 \geq 50$

Example 4

A company owns 2 oil mills A and B which have different production capacities for low, high and medium grade oil. The company enters into a contract to supply oil to a firm every week with 12, 8, 24 barrels of each grade respectively. It costs the company Rs 1000 and Rs 800 per day to run the mills A and B. On a day A produces 6, 2, 4 barrels of each grade and B produces 2, 2, 12 barrels of each grade. Formulate an LPP to determine number of days per week each mill will be operated in order to meet the contract economically.

Solution

Let

x_1 be the no. of days a week the mill A has to work

x_2 be the no. of days per week the mill B has to work

Grade	A	B	Minimum requirement
Low	6	2	12

High	2	2	8
Medium	4	12	24
Cost per day	Rs 1000	Rs 800	

Minimize $Z = 1000x_1 + 800\,x_2$

Subject to

$6x_1 + 2x_2 \geq 12$

$2x_1 + 2x_2 \geq 8$

$4x_1 + 12x_2 \geq 24$

$x_1 \geq 0$, $x_2 \geq 0$

Example 5

A company has 3 operational departments weaving, processing and packing with the capacity to produce 3 different types of clothes that are suiting, shirting and woolen yielding with the profit of Rs. 2, Rs. 4 and Rs. 3 per meters respectively. 1m suiting requires 3mins in weaving 2 mins in processing and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving 1 min in processing and 3 mins in packing while 1m of woolen requires 3 mins in each department. In a week total run time of each department is 60, 40 and 80 hours for weaving, processing and packing department respectively. Formulate a LPP to find the product to maximize the profit.

Solution

Let

x_1 be the number of units of suiting

x_2 be the number of units of shirting

x_3 be the number of units of woolen

	Suiting	Shirting	Woolen	Available time
Weaving	3	4	3	60
Processing	2	1	3	40

Packing	1	3	3	80
Profit	2	4	3	

Maximize $Z = 2x_1 + 4x_2 + 3x_3$

Subject to

$3x_1 + 4x_2 + 3x_3 \leq 60$

$2x_1 + 1x_2 + 3x_3 \leq 40$

$x_1 + 3x_2 + 3x_3 \leq 80$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Example 6

ABC Company produces both interior and exterior paints from 2 raw materials m1 and m2. The following table produces basic data of problem.

	Exterior paint	Interior paint	Availability
M1	6	4	24
M2	1	2	6
Profit per ton	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also maximum daily demand for interior paint is 2 tons. Formulate LPP to determine the best product mix of interior and exterior paints that maximizes the daily total profit.

Solution

Let

x_1 be the number of units of exterior paint

x_2 be the number of units of interior paint

Maximize $Z = 5x_1 + 4x_2$

Subject to

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$x_2 - x_1 \leq 1$

$x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0$

b) The maximum daily demand for exterior paint is atmost 2.5 tons

$x_1 \leq 2.5$

c) Daily demand for interior paint is atleast 2 tons

$x_2 \geq 2$

d) Daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

$x_2 > x_1 + 1$

Example 7

A company produces 2 types of hats. Each hat of the I type requires twice as much as labour time as the II type. The company can produce a total of 500 hats a day. The market limits daily sales of I and II types to 150 and 250 hats. Assuming that the profit per hat are Rs.8 for type A and Rs. 5 for type B. Formulate a LPP models in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution

Let x_1 be the number of hats produced by type A

Let x_2 be the number of hats produced by type B

Maximize $Z = 8x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 500$ (labour time)

$x_1 \leq 150$

$x_2 \leq 250$

$x_1 \geq 0, x_2 \geq 0$

Example 8

A manufacturer produces 3 models (I, II and III) of a certain product. He uses 2 raw materials A and B of which 4000 and 6000 units respectively are available. The raw materials per unit of 3 models are given below.

Raw materials	I	II	III
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and thrice that of model III. The entire labour force of factory can produce an equivalent of 2500 units of model I. A model survey indicates that the minimum demand of 3 models is 500, 500 and 375 units respectively. However the ratio of number of units produced must be equal to 3:2:5. Assume that profits per unit of model are 60, 40 and 100 respectively. Formulate a LPP.

Solution

Let

x_1 be the number of units of model I

x_2 be the number of units of model II

x_3 be the number of units of model III

Raw materials	I	II	III	Availability
A	2	3	5	4000
B	4	2	7	6000
Profit	60	40	100	

$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$ [Labour time]

$x_1 \geq 500$, $x_2 \geq 500$, $x_3 \geq 375$ [Minimum demand]

The given ratio is x_1: x_2: x_3 = 3: 2: 5

$x_1 / 3 = x_2 / 2 = x_3 / 5 = k$

$x_1 = 3k; x_2 = 2k; x_3 = 5k$

$x_2 = 2k \rightarrow k = x_2 / 2$

Therefore $x_1 = 3\ x_2 / 2 \rightarrow 2x_1 = 3x_2$

Similarly $2x_3 = 5x_2$

Maximize $Z= 60x_1 + 40x_2 + 100x_3$

Subject to $2x_1 + 3x_2 + 5x_3 \leq 4000$

$4x_1 + 2x_2 + 7x_3 \leq 6000$

$x_1 + 1/2x_2 + 1/3x_3 \leq 2500$

$2\ x_1 = 3x_2$

$2\ x_3 = 5x_2$

and $x_1 \geq 500, x_2 \geq 500, x_3 \geq 375$

Example 9

A person wants to decide the constituents of a diet which will fulfill his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the table.

Food Type	Yield/unit			Cost/Unit Rs
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum Requirement	800	200	700	

Formulate the LP for the problem.

Solution

Let

x_1 be the number of units of food type l

x_2 be the number of units of food type 2

x_3 be the number of units of food type 3

x_4 be the number of units of food type 4

Minimize $Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$

Subject to

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$

$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$

$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0$$

Exercise

1. Define the terms used in LPP.
2. Mention the advantages of LPP.
3. What are the assumptions and limitations of LPP?
4. A firm produces three products. These products are processed on three different machines. The time required manufacturing one unit of each of the three products and the daily capacity of the three machines are given in the table.

Machine	Time per unit (mins)			Machine capacity Min /day
	Product 1	Product 2	Product 3	
M1	2	3	2	440
M2	4	-	3	470
M3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs. 3 and Rs. 6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical model for the model.

5. A chemical firm produces automobiles cleaner X and polisher Y and realizes Rs. 10 profit on each batch of X and Rs. 30 on Y. Both products require processing through the same machines, A and B but X requires 4 hours in A and 8 hours in B, whereas Y requires 6 hours in A and 4 hours in B. during the fourth coming week machines A and B have 12 and 16 hours of available capacity, respectively. Assuming that demand exists for both products, how many batches of each should be produce to realize the optimal profit Z?

6. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is formed by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine fro providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

Unit 3

3.4.2 No optimal solution

3.4.3 Unbounded solution

3.1 Graphical Solution Procedure

The graphical solution procedure

1. Consider each inequality constraint as equation.
2. Plot each equation on the graph as each one will geometrically represent a straight line.
3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is '≤' then the region below the line lying in the first quadrant is shaded. Similarly for '≥' the region above the line is shaded. The points lying in the common region will satisfy the constraints. This common region is called **feasible region.**
4. Choose the convenient value of Z and plot the objective function line.
5. Pull the objective function line until the extreme points of feasible region.
    a. In the maximization case this line will stop far from the origin and passing through at least one corner of the feasible region.
    b. In the minimization case, this line will stop near to the origin and passing through at least one corner of the feasible region.
6. Read the co-ordinates of the extreme points selected in step 5 and find the maximum or minimum value of Z.

3.2 Definitions

1. **Solution** – Any specification of the values for decision variable among (x_1, x_2… x_n) is called a solution.

2. **Feasible solution** is a solution for which all constraints are satisfied.
3. **Infeasible solution** is a solution for which at least one constraint is not satisfied.
4. **Feasible region** is a collection of all feasible solutions.
5. **Optimal solution** is a feasible solution that has the most favorable value of the objective function.
6. **Most favorable value** is the largest value if the objective function is to be maximized, whereas it is the smallest value if the objective function is to be minimized.
7. **Multiple optimal solution** – More than one solution with the same optimal value of the objective function.
8. **Unbounded solution** – If the value of the objective function can be increased or decreased indefinitely such solutions are called unbounded solution.
9. **Feasible region** – The region containing all the solutions of an inequality
10. **Corner point feasible solution** is a solution that lies at the corner of the feasible region.

3.3 Example problems

Example 1

Solve $3x + 5y < 15$ graphically

Solution

Write the given constraint in the form of equation i.e. $3x + 5y = 15$

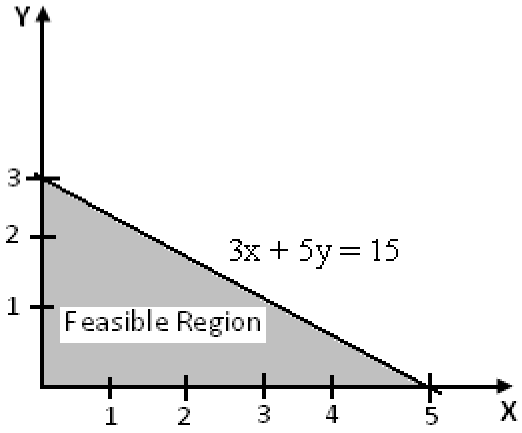
Put $x=0$ then the value $y=3$

Put $y=0$ then the value $x=5$

Therefore the coordinates are (0, 3) and (5, 0). Thus these points are joined to form a straight line as shown in the graph.

Put $x=0$, $y=0$ in the given constraint then

$0<15$, the condition is true. (0, 0) is solution nearer to origin. So shade the region below the line, which is the feasible region.



Example 2

Solve 3x + 5y >15

Solution

Write the given constraint in the form of equation i.e. 3x + 5y = 15

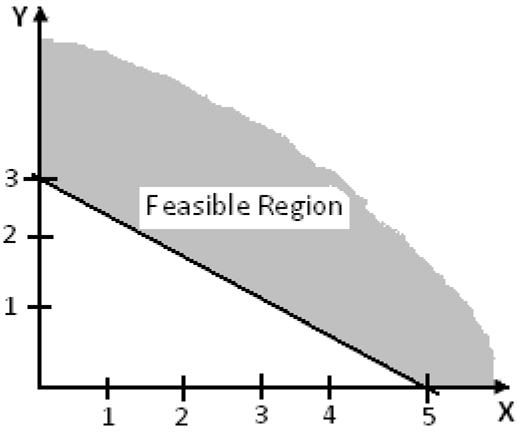
Put x=0, then y=3

Put y=0, then x=5

So the coordinates are (0, 3) and (5, 0)

Put x =0, y =0 in the given constraint, the condition turns out to be false i.e. 0 > 15 is false.

So the region does not contain (0, 0) as solution. The feasible region lies on the outer part of the line as shown in the graph.



Example 3

Max Z = $80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $4x_1 + 2x_2 \leq 40$, written in a form of equation

$4x_1 + 2x_2 = 40$

Put $x_1 = 0$, then $x_2 = 20$

Put $x_2 = 0$, then $x_1 = 10$

The coordinates are (0, 20) and (10, 0)

The second constraint $2x_1 + 4x_2 \leq 32$, written in a form of equation

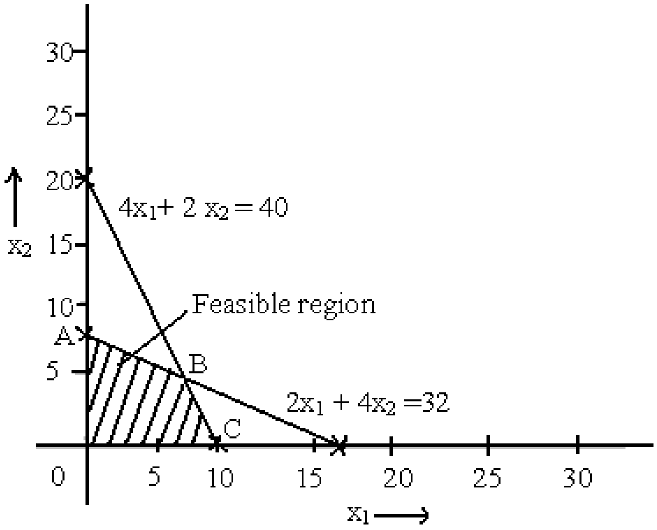
$2x_1 + 4x_2 = 32$

Put $x_1 = 0$, then $x_2 = 8$

Put $x_2 = 0$, then $x_1 = 16$

The coordinates are (0, 8) and (16, 0)

The graphical representation is



The corner points of feasible region are A, B and C. So the coordinates for the corner points are

A (0, 8)

B (8, 4) (Solve the two equations $4x_1 + 2x_2 = 40$ and $2x_1 + 4x_2 = 32$ to get the coordinates)

C (10, 0)

We know that Max $Z = 80x_1 + 55x_2$

At A (0, 8)

$Z = 80(0) + 55(8) = 440$

At B (8, 4)

$Z = 80(8) + 55(4) = 860$

At C (10, 0)

$Z = 80(10) + 55(0) = 800$

The maximum value is obtained at the point B. Therefore Max Z = 860 and $x_1 = 8$, $x_2 = 4$

Example 4

Minimize $Z = 10x_1 + 4x_2$

Subject to

$3x_1 + 2x_2 \geq 60$

$7x_1 + 2x_2 \geq 84$

$3x_1 + 6x_2 \geq 72$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $3x_1 + 2x_2 \geq 60$, written in a form of equation

$3x_1 + 2x_2 = 60$

Put $x_1 = 0$, then $x_2 = 30$

Put $x_2 = 0$, then $x_1 = 20$

The coordinates are (0, 30) and (20, 0)

The second constraint $7x_1 + 2x_2 \geq 84$, written in a form of equation

$7x_1 + 2x_2 = 84$

Put $x_1 = 0$, then $x_2 = 42$

Put $x_2 = 0$, then $x_1 = 12$

The coordinates are (0, 42) and (12, 0)

The third constraint $3x_1 + 6x_2 \geq 72$, written in a form of equation

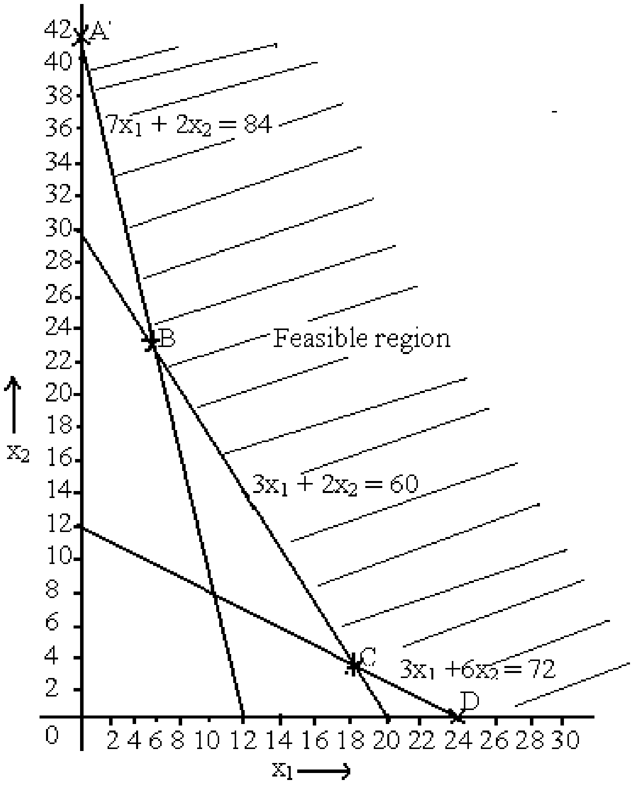
$3x_1 + 6x_2 = 72$

Put $x_1 = 0$, then $x_2 = 12$

Put $x_2 = 0$, then $x_1 = 24$

The coordinates are (0, 12) and (24, 0)

The graphical representation is



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 42)

B (6, 21) (Solve the two equations $7x_1 + 2x_2 = 84$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

C (18, 3) Solve the two equations $3x_1 + 6x_2 = 72$ and $3x_1 + 2x_2 = 60$ to get the coordinates)

D (24, 0)

We know that Min $Z = 10x_1 + 4x_2$

At A (0, 42)

$Z = 10(0) + 4(42) = 168$

At B (6, 21)

$Z = 10(6) + 4(21) = 144$

At C (18, 3)

Z = 10(18) + 4(3) = 192

At D (24, 0)

Z = 10(24) + 4(0) = 240

The minimum value is obtained at the point B. Therefore Min Z = 144 and $x_1 = 6$, $x_2 = 21$

Example 5

A manufacturer of furniture makes two products – chairs and tables. Processing of this product is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair and a table is Rs 2 and Rs 10 respectively. What should be the daily production of each of two products?

Solution

Let x_1 denotes the number of chairs

Let x_2 denotes the number of tables

	Chairs	Tables	Availability
Machine A	2	5	16
Machine B	6	0	30
Profit	Rs 2	Rs 10	

LPP

Max $Z = 2x_1 + 10x_2$

Subject to

$$2x_1 + 5x_2 \leq 16$$

$6x_1 + 0x_2 \leq 30$

$x_1 \geq 0$, $x_2 \geq 0$

Solving graphically

The first constraint $2x_1 + 5x_2 \leq 16$, written in a form of equation

$2x_1 + 5x_2 = 16$

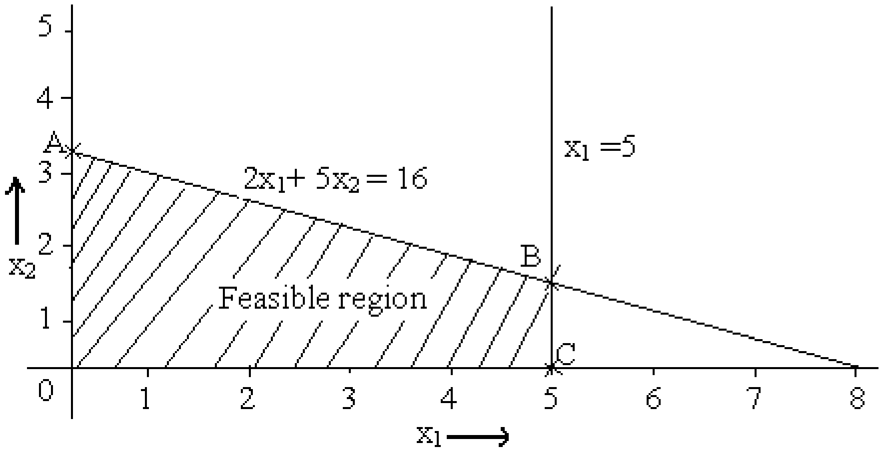
Put $x_1 = 0$, then $x_2 = 16/5 = 3.2$

Put $x_2 = 0$, then $x_1 = 8$

The coordinates are (0, 3.2) and (8, 0)

The second constraint $6x_1 + 0x_2 \leq 30$, written in a form of equation

$6x_1 = 30 \rightarrow x_1 = 5$



The corner points of feasible region are A, B and C. So the coordinates for the corner points are

A (0, 3.2)

B (5, 1.2) (Solve the two equations $2x_1 + 5x_2 = 16$ and $x_1 = 5$ to get the coordinates)

C (5, 0)

We know that Max $Z = 2x_1 + 10x_2$

At A (0, 3.2)

$Z = 2(0) + 10(3.2) = 32$

At B (5, 1.2)

Z = 2(5) + 10(1.2) = 22

At C (5, 0)

Z = 2(5) + 10(0) = 10

Max Z = 32 and $x_1 = 0$, $x_2 = 3.2$

The manufacturer should produce approximately 3 tables and no chairs to get the max profit.

3.4 Special Cases in Graphical Method

3.4.1 Multiple Optimal Solution

Example 1

Solve by using graphical method

Max $Z = 4x_1 + 3x_2$

Subject to

$4x_1 + 3x_2 \leq 24$

$x_1 \leq 4.5$

$x_2 \leq 6$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $4x_1 + 3x_2 \leq 24$, written in a form of equation

$4x_1 + 3x_2 = 24$

Put $x_1 = 0$, then $x_2 = 8$

Put $x_2 = 0$, then $x_1 = 6$

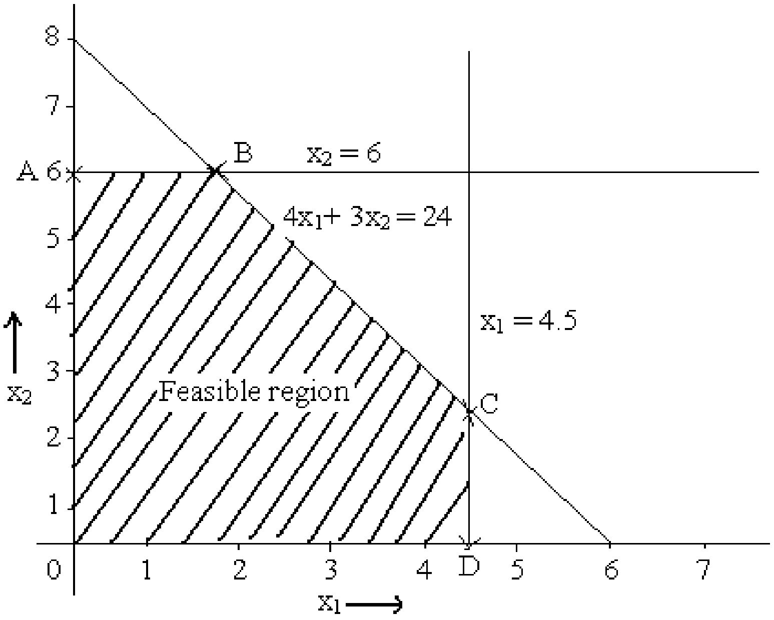
The coordinates are (0, 8) and (6, 0)

The second constraint $x_1 \leq 4.5$, written in a form of equation

$x_1 = 4.5$

The third constraint $x_2 \leq 6$, written in a form of equation

$x_2 = 6$



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 6)

B (1.5, 6) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_2 = 6$ to get the coordinates)

C (4.5, 2) (Solve the two equations $4x_1 + 3x_2 = 24$ and $x_1 = 4.5$ to get the coordinates)

D (4.5, 0)

We know that Max $Z = 4x_1 + 3x_2$

At A (0, 6)

$Z = 4(0) + 3(6) = 18$

At B (1.5, 6)

$Z = 4(1.5) + 3(6) = 24$

At C (4.5, 2)

Z = 4(4.5) + 3(2) = 24

At D (4.5, 0)

Z = 4(4.5) + 3(0) = 18

Max Z = 24, which is achieved at both B and C corner points. It can be achieved not only at B and C but every point between B and C. Hence the given problem has multiple optimal solutions.

3.4.2 No Optimal Solution

Example 1

Solve graphically

Max Z = $3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 1$

$x_1 + x_2 \geq 3$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $x_1 + x_2 \leq 1$, written in a form of equation

$x_1 + x_2 = 1$

Put $x_1 = 0$, then $x_2 = 1$

Put $x_2 = 0$, then $x_1 = 1$

The coordinates are (0, 1) and (1, 0)

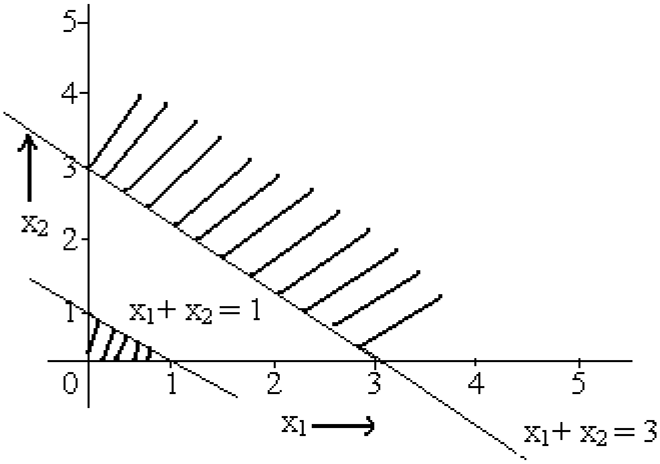
The first constraint $x_1 + x_2 \geq 3$, written in a form of equation

$x_1 + x_2 = 3$

Put $x_1 = 0$, then $x_2 = 3$

Put x_2 =0, then x_1 = 3

The coordinates are (0, 3) and (3, 0)



There is no common feasible region generated by two constraints together i.e. we cannot identify even a single point satisfying the constraints. Hence there is no optimal solution.

3.4.3 Unbounded Solution

Example

Solve by graphical method

Max Z = $3x_1 + 5x_2$

Subject to

$2x_1 + x_2 \geq 7$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

The first constraint $2x_1 + x_2 \geq 7$, written in a form of equation

$2x_1 + x_2 = 7$

Put x_1 =0, then x_2 = 7

Put x_2 =0, then x_1 = 3.5

The coordinates are (0, 7) and (3.5, 0)

The second constraint $x_1+ x_2 \geq 6$, written in a form of equation

$x_1+ x_2 = 6$

Put x_1 =0, then x_2 = 6

Put x_2 =0, then x_1 = 6

The coordinates are (0, 6) and (6, 0)

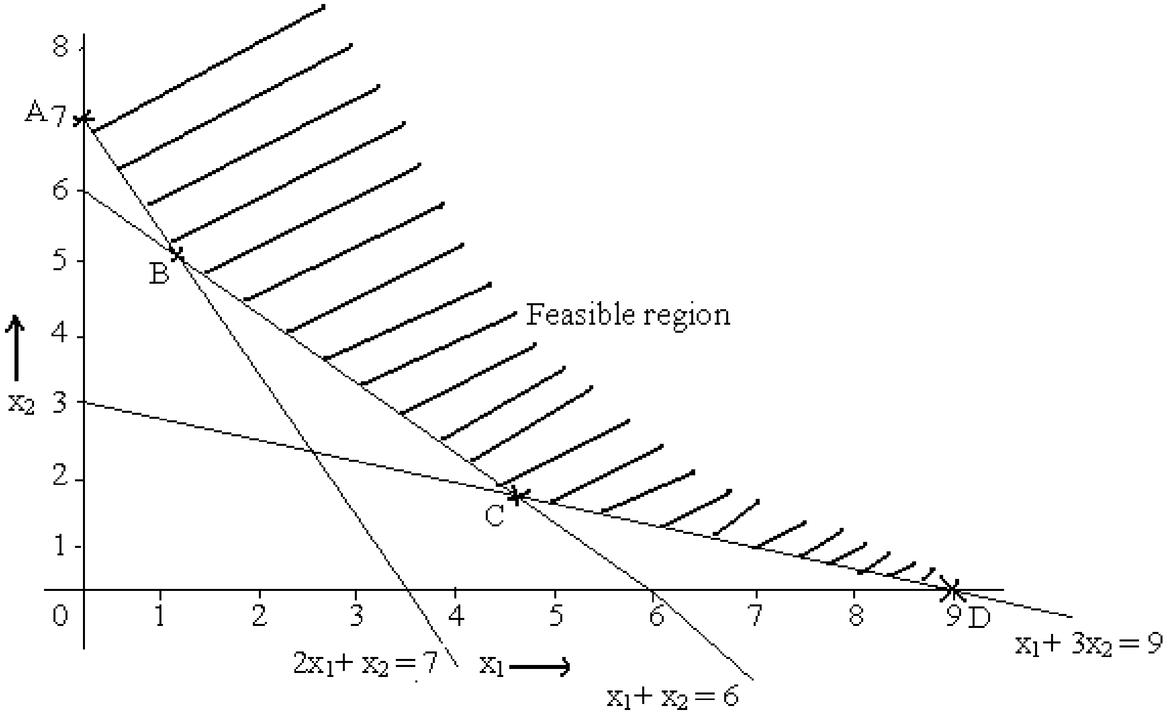
The third constraint $x_1+ 3x_2 \geq 9$, written in a form of equation

$x_1+ 3x_2 = 9$

Put x_1 =0, then x_2 = 3

Put x_2 =0, then x_1 = 9

The coordinates are (0, 3) and (9, 0)



The corner points of feasible region are A, B, C and D. So the coordinates for the corner points are

A (0, 7)

B (1, 5) (Solve the two equations $2x_1 + x_2 = 7$ and $x_1 + x_2 = 6$ to get the coordinates)

C (4.5, 1.5) (Solve the two equations $x_1 + x_2 = 6$ and $x_1 + 3x_2 = 9$ to get the coordinates)

D (9, 0)

We know that Max $Z = 3x_1 + 5x_2$

At A (0, 7)

$Z = 3(0) + 5(7) = 35$

At B (1, 5)

$Z = 3(1) + 5(5) = 28$

At C (4.5, 1.5)

$Z = 3(4.5) + 5(1.5) = 21$

At D (9, 0)

$Z = 3(9) + 5(0) = 27$

The values of objective function at corner points are 35, 28, 21 and 27. But there exists infinite number of points in the feasible region which is unbounded. The value of objective function will be more than the value of these four corner points i.e. the maximum value of the objective function occurs at a point at ∞. Hence the given problem has unbounded solution.

Exercise

1. A company manufactures two types of printed circuits. The requirements of transistors, resistors and capacitor for each type of printed circuits along with other data are given in table.

	Circuit		Stock available (units)
	A	B	
Transistor	15	10	180
Resistor	10	20	200
Capacitor	15	20	210
Profit	Rs.5	Rs.8	

How many circuits of each type should the company produce from the stock to earn maximum profit.

[Ans. Max Z = 82, 2 units of type A circuit and 9 units of type B circuit]

2. A company making cool drinks has 2 bottling plants located at towns T1 and T2. Each plant produces 3 drinks A, B and C and their production capacity per day is given in the table.

Cool drinks	Plant at	
	T1	T2
A	6000	2000

B	1000	2500
C	3000	3000

The marketing department of the company forecasts a demand of 80000 bottles of A, 22000 bottles of B and 40000 bottles of C during the month of June. The operating cost per day of plants at T1 and T2 are Rs. 6000 and Rs. 4000 respectively. Find graphically the number of days for which each plants must be run in June so as to minimize the operating cost while meeting the market demand.

[Ans. Min Z = Rs. 88000, 12 days for the plant T1 and 4 days for plant T2]

Solve the following LPP by graphical method

1. Max $Z = 3x_1 + 4x_2$

   Subject to

   $x_1 - x_2 \leq -1$

   $-x_1 + x_2 \leq 0$

   $x_1 \geq 0$, $x_2 \geq 0$

   [Ans. The problem has no solution]

2. Max $Z = 3x_1 + 2x_2$

   Subject to

   $-2x_1 + 3x_2 \leq 9$

   $x_1 - 5x_2 \geq -20$

   $x_1 \geq 0$, $x_2 \geq 0$

   [Ans. The problem has unbounded solution]

3. Max $Z = 45x_1 + 80x_2$

   Subject to

   $5x_1 + 20x_2 \leq 400$

   $10x_1 + 15x_2 \leq 450$

   $x_1 \geq 0$, $x_2 \geq 0$

   [Ans. Max Z = 2200, $x_1 = 24$, $x_2 = 14$]

Module 2

Unit 1

1.1 Introduction

General Linear Programming Problem (GLPP)

Maximize / Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots\ldots\ldots\ldots + c_nx_n$

Subject to constraints

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \ (\leq \text{ or } \geq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \ (\leq \text{ or } \geq)\ b_2$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \ (\leq \text{ or } \geq)\ b_m$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Where constraints may be in the form of any inequality (≤ or ≥) or even in the form of an equation (=) and finally satisfy the non-negativity restrictions.

1.2 Steps to convert GLPP to SLPP (Standard LPP)

Step 1 – Write the objective function in the maximization form. If the given objective function is of minimization form then multiply throughout by -1 and write Max $z' = \text{Min}(-z)$

Step 2 – Convert all inequalities as equations.

- If an equality of '≤' appears then by adding a variable called **Slack variable.** We can convert it to an equation. For example $x_1 + 2x_2 \leq 12$, we can write as

  $$x_1 + 2x_2 + s_1 = 12.$$

- If the constraint is of '≥' type, we subtract a variable called **Surplus variable** and convert it to an equation. For example

  $$2x_1 + x_2 \geq 15$$

  $$2x_1 + x_2 - s_2 = 15$$

Step 3 – The right side element of each constraint should be made non-negative

$$2x_1 + x_2 - s_2 = -15$$

$-2x_1 - x_2 + s_2 = 15$ (That is multiplying throughout by -1)

Step 4 – All variables must have non-negative values.

For example: $x_1 + x_2 \leq 3$

$x_1 > 0$, x_2 is unrestricted in sign

Then x_2 is written as $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$

Therefore the inequality takes the form of equation as $x_1 + (x_2' - x_2'') + s_1 = 3$

Using the above steps, we can write the GLPP in the form of SLPP.

Write the Standard LPP (SLPP) of the following

Example 1

Maximize $Z = 3x_1 + x_2$

Subject to

$2 x_1 + x_2 \leq 2$

$3 x_1 + 4 x_2 \geq 12$

and $x_1 \geq 0, x_2 \geq 0$

SLPP

Maximize $Z = 3x_1 + x_2$

Subject to

$2 x_1 + x_2 + s_1 = 2$

$3 x_1 + 4 x_2 - s_2 = 12$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Example 2

Minimize $Z = 4x_1 + 2 x_2$

Subject to

$3x_1 + x_2 \geq 2$

$x_1 + x_2 \geq 21$

$x_1 + 2x_2 \geq 30$

and $x_1 \geq 0, x_2 \geq 0$

SLPP

Maximize $Z' = - 4x_1 - 2 x_2$

Subject to

$3x_1 + x_2 – s_1 = 2$

$x_1 + x_2 – s_2 = 21$

$x_1 + 2x_2 – s_3 = 30$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

Example 3

Minimize $Z = x_1 + 2 x_2 + 3x_3$

Subject to

$2x_1 + 3x_2 + 3x_3 \geq – 4$

$3x_1 + 5x_2 + 2x_3 \leq 7$

and $x_1 \geq 0, x_2 \geq 0, x_3$ is unrestricted in sign

SLPP

Maximize $Z' = – x_1 – 2 x_2 – 3(x_3' – x_3'')$

Subject to

$–2x_1 – 3x_2 – 3(x_3' – x_3'') + s_1= 4$

$3x_1 + 5x_2 + 2(x_3' – x_3'') + s_2 = 7$

$x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, s_1 \geq 0, s_2 \geq 0$

1.3 Some Basic Definitions

Solution of LPP

Any set of variable (x_1, x_2… x_n) which satisfies the given constraint is called solution of LPP.

Basic solution

It is a solution obtained by setting any ‘n’ variable equal to zero and solving remaining ‘m’ variables. Such ‘m’ variables are called **Basic variables** and ‘n’ variables are called **Non-basic variables**.

Basic feasible solution

A basic solution that is feasible (all basic variables are non negative) is called basic feasible solution. There are two types of basic feasible solution.

1. **Degenerate basic feasible solution**

   If any of the basic variable of a basic feasible solution is zero than it is said to be degenerate basic feasible solution.
2. **Non-degenerate basic feasible solution**

   It is a basic feasible solution which has exactly ‘m’ positive x_i, where i=1, 2, … m. In other words all ‘m’ basic variables are positive and remaining ‘n’ variables are zero.

Optimum basic feasible solution

A basic feasible solution is said to be optimum if it optimizes (max / min) the objective function.

1.4 Introduction to Simplex Method

It was developed by G. Danztig in 1947. The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming.

The Simplex algorithm is an iterative procedure for solving LP problems in a finite number of steps. It consists of

- Having a trial basic feasible solution to constraint-equations

- Testing whether it is an optimal solution
- Improving the first trial solution by a set of rules and repeating the process till an optimal solution is obtained

Advantages

- Simple to solve the problems
- The solution of LPP of more than two variables can be obtained.

1.5 Computational Procedure of Simplex Method

Consider an example

Maximize $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Write the given GLPP in the form of SLPP

Maximize $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Step 2 – Present the constraints in the matrix form

$x_1 + x_2 + s_1 = 4$

$x_1 - x_2 + s_2 = 2$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Step 3 – Construct the starting simplex table using the notations

$C_j \rightarrow$ 3 2 0 0

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	
s_2	0	2	1	-1	0	1	
	$Z = C_B X_B$		Δ_j				

Step 4 – Calculation of Z and Δ_j and test the basic feasible solution for optimality by the rules given.

$Z = C_B X_B$

$= 0 * 4 + 0 * 2 = 0$

$\Delta_j = Z_j - C_j$

$= C_B X_{j} - C_j$

$\Delta_1 = C_B X_1 - C_j = 0 * 1 + 0 * 1 - 3 = -3$

$\Delta_2 = C_B X_2 - C_j = 0 * 1 + 0 * -1 - 2 = -2$

$\Delta_3 = C_B X_3 - C_j = 0 * 1 + 0 * 0 - 0 = 0$

$\Delta_4 = C_B X_4 - C_j = 0 * 0 + 0 * 1 - 0 = 0$

Procedure to test the basic feasible solution for optimality by the rules given

Rule 1 – If all $\Delta_j \geq 0$, the solution under the test will be **optimal**. Alternate optimal solution will exist if any non-basic Δ_j is also zero.

Rule 2 – If atleast one Δ_j is negative, the solution is not optimal and then proceeds to improve the solution in the next step.

Rule 3 – If corresponding to any negative Δ_j, all elements of the column X_j are negative or zero, then the solution under test will be **unbounded**.

In this problem it is observed that Δ_1 and Δ_2 are negative. Hence proceed to improve this solution

Step 5 – To improve the basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined.

- **Incoming vector**
  The incoming vector X_k is always selected corresponding to the most negative value of Δ_j. It is indicated by (↑).

- **Outgoing vector**
  The outgoing vector is selected corresponding to the least positive value of minimum ratio. It is indicated by (→).

Step 6 – Mark the key element or pivot element by ‘□‘.The element at the intersection of outgoing vector and incoming vector is the pivot element.

		Cj →	3	2	0	0	
Basic Variables	C_B	X_B	X_1 (X_k)	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	4	1	1	1	0	4 / 1 = 4

s_2	0 2	$\boxed{1}$ -1 0 1	2 / 1 = 2 ⟶ outgoing
	Z= C_B X_B = 0	↑incoming Δ_1= -3 Δ_2= -2 Δ_3=0 Δ_4=0	

- If the number in the marked position is other than unity, divide all the elements of that row by the key element.
- Then subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeroes in the remaining position of the column X_k.

Basic Variables	C_B X_B	X_1 X_2 (X_k) S_1 S_2	Min ratio X_B /X_k
s_1 x_1	0 2 3 2	(R_1=R_1 – R_2) 0 $\boxed{2}$ 1 -1 1 -1 0 1	2 / 2 = 1 ⟶ outgoing 2 / -1 = -2 (neglect in case of negative)
	Z=0*2+3*2= 6	↑incoming Δ_1=0 Δ_2= -5 Δ_3=0 Δ_4=3	

Step 7 – Now repeat step 4 through step 6 until an optimal solution is obtained.

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
x_2	2	1	$(R_1=R_1 / 2)$ 0	1	1/2	-1/2	
x_1	3	3	$(R_2=R_2 + R_1)$ 1	0	1/2	1/2	
	Z = 11		$\Delta_1=0$	$\Delta_2=0$	$\Delta_3=5/2$	$\Delta_4=1/2$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 11, $x_1 = 3$ and $x_2 = 1$

1.6 Worked Examples

Solve by simplex method

Example 1

Maximize $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 80x_1 + 55x_2 + 0s_1 + 0s_2$

Subject to

$4x_1 + 2x_2 + s_1 = 40$

$2x_1 + 4x_2 + s_2 = 32$

$x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$

$C_j \rightarrow$ 80 55 0 0

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	40	$\boxed{4}$	2	1	0	40 / 4 = 10⟶ outgoing
s_2	0	32	2	4	0	1	32 / 2 = 16
	Z= C_B X_B = 0		↑incoming Δ_1= -80	Δ_2= -55	Δ_3=0	Δ_4=0	
x_1	80	10	(R_1=R_1 / 4) 1	1/2	1/4	0	10/1/2 = 20
s_2	0	12	(R_2=R_2– 2R_1) 0	$\boxed{3}$	-1/2	1	12/3 = 4⟶ outgoing
	Z = 800		Δ_1=0	↑incoming Δ_2= -15	Δ_3=40	Δ_4=0	
x_1	80	8	(R_1=R_1– 1/2R_2) 1	0	1/3	-1/6	
x_2	55	4	(R_2=R_2 / 3) 0	1	-1/6	1/3	
	Z = 860		Δ_1=0	Δ_2=0	Δ_3=35/2	Δ_4=5	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max $Z = 860$, $x_1 = 8$ and $x_2 = 4$

Example 2

Maximize $Z = 5x_1 + 3x_2$
Subject to
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to

$3x_1 + 5x_2 + s_1 = 15$

$5x_1 + 2x_2 + s_2 = 10$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

$C_j \rightarrow$ 5 3 0 0

Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
s_1	0	15	3	5	1	0	15 / 3 = 5
s_2	0	10	$\boxed{5}$	2	0	1	10 / 5 = 2 → outgoing
	$Z = C_B X_B = 0$		↑incoming $\Delta_1 = -5$	$\Delta_2 = -3$	$\Delta_3 = 0$	$\Delta_4 = 0$	

			$(R_1=R_1-3R_2)$				
s_1	0	9	0	$\boxed{19/5}$	1	-3/5	9/19/5 = 45/19 →
			$(R_2=R_2/5)$				
x_1	5	2	1	2/5	0	1/5	2/2/5 = 5
				↑			
	Z = 10		$\Delta_1=0$	$\Delta_2=-1$	$\Delta_3=0$	$\Delta_4=1$	
			$(R_1=R_1/19/5)$				
x_2	3	45/19	0	1	5/19	-3/19	
			$(R_2=R_2-2/5R_1)$				
x_1	5	20/19	1	0	-2/19	5/19	
	Z = 235/19		$\Delta_1=0$	$\Delta_2=0$	$\Delta_3=5/19$	$\Delta_4=16/19$	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 235/19, x_1 = 20/19 and x_2 = 45/19

Example 3

Maximize Z = $5x_1 + 7x_2$

Subject to

$x_1 + x_2 \leq 4$

$3x_1 - 8x_2 \leq 24$

$10x_1 + 7x_2 \leq 35$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

SLPP

Maximize $Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 4$

$3x_1 - 8x_2 + s_2 = 24$

$10x_1 + 7x_2 + s_3 = 35$

$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0$

			$C_j \rightarrow$ 5	7	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	4	1	$\boxed{1}$	1	0	0	4 /1 = 4→outgoing
s_2	0	24	3	-8	0	1	0	–
s_3	0	35	10	7	0	0	1	35 / 7 = 5
	$Z = C_B X_B = 0$		-5	↑incoming -7	0	0	0	←Δ_j
x_2	7	4	1	1	1	0	0	
s_2	0	56	$(R_2 = R_2 + 8R_1)$ 11	0	8	1	0	
s_3	0	7	$(R_3 = R_3 - 7R_1)$ 3	0	-7	0	1	
	$Z = 28$		2	0	7	0	0	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z = 28$, $x_1 = 0$ and $x_2 = 4$

Example 4

Maximize Z = 2x – 3y + z

Subject to

$3x + 6y + z \leq 6$

$4x + 2y + z \leq 4$

$x - y + z \leq 3$

and $x \geq 0, y \geq 0, z \geq 0$

Solution

SLPP

Maximize $Z = 2x - 3y + z + 0s_1 + 0s_2 + 0s_3$

Subject to

$3x + 6y + z + s_1 = 6$

$4x + 2y + z + s_2 = 4$

$x - y + z + s_3 = 3$

$x \geq 0, y \geq 0, z \geq 0\ s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

$C_j \rightarrow$ 2 -3 1 0 0 0

Basic Variables	C_B	X_B	X	Y	Z	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	6	3	6	1	1	0	0	6 / 3 = 2
s_2	0	4	$\boxed{4}$	2	1	0	1	0	4 / 4 =1⟶ outgoing
s_3	0	3	1	-1	1	0	0	1	3 / 1 = 3
			↑incoming						

	Z = 0		-2	3	-1	0	0	0	←Δ_j
s_1	0	3	0	9/2	1/4	1	-3/4	0	3/1/4=12
x	2	1	1	1/2	1/4	0	1/4	0	1/1/4=4
s_3	0	2	0	-3/2	$\boxed{3/4}$	0	-1/4	1	8/3 = 2.6→
					↑incoming				
	Z = 2		0	4	1/2	0	1/2	0	←Δ_j
s_1	0	7/3	0	5	0	1	-2/3	-1/3	
x	2	1/3	1	1	0	0	1/3	-1/3	
z	1	8/3	0	-2	1	0	-1/3	4/3	
	Z = 10/3		0	3	0	0	1/3	2/3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 10/3, x = 1/3, y = 0 and z = 8/3

Example 5

Maximize $Z = 3x_1 + 5x_2$

Subject to

$3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Maximize $Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3x_1 + 2x_2 + s_1 = 18$

$x_1 + s_2 = 4$

$x_2 + s_3 = 6$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

		$C_j \rightarrow$	3	5	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	18	3	2	1	0	0	18 / 2 = 9
s_2	0	4	1	0	0	1	0	4 / 0 = ∞ (neglect)
s_3	0	6	0	$\boxed{1}$	0	0	1	6 / 1 = 6→
	Z = 0		-3	↑ -5	0	0	0	←Δ_j
				($R_1=R_1-2R_3$)				

s_1	0	6	$\boxed{3}$	0	1	0	-2	6 / 3 = 2 →
s_2	0	4	1	0	0	1	0	4 / 1 = 4
x_2	5	6	0	1	0	0	1	--
			↑					
	Z = 30		-3	0	0	0	5	←Δ_j
			(R_1=R_1 / 3)					
x_1	3	2	1	0	1/3	0	-2/3	
			(R_2=R_2 - R_1)					
s_2	0	2	0	0	-1/3	1	2/3	
x_2	5	6	0	1	0	0	1	
	Z = 36		0	0	1	0	3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 36, $x_1 = 2$, $x_2 = 6$

Example 6

Minimize $Z = x_1 – 3x_2 + 2x_3$

Subject to

$3x_1 – x_2 + 3x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

SLPP

Min (-Z) = Max $Z' = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3x_1 – x_2 + 3x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ $s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$

$C_j \rightarrow$ -1 3 -2 0 0 0

Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	7	3	-1	3	1	0	0	-
s_2	0	12	-2	$\boxed{4}$	0	0	1	0	3→
s_3	0	10	-4	3	8	0	0	1	10/3
	Z' = 0		1	↑ -3	2	0	0	0	←Δ_j
			$(R_1 = R_1 + R_2)$						
s_1	0	10	$\boxed{5/2}$	0	3	1	1/4	0	4→
			$(R_2 = R_2 / 4)$						
x_2	3	3	-1/2	1	0	0	1/4	0	-
			$(R_3 = R_3 – 3R_2)$						
s_3	0	1	-5/2	0	8	0	-3/4	1	-
	Z' = 9		↑ -5/2	0	0	0	3/4	0	←Δ_j
			$(R_1 = R_1 / 5/2)$						
x_1	-1	4	1	0	6/5	2/5	1/10	0	
			$(R_2 = R_2 + 1/2\ R_1)$						
x_2	3	5	0	1	3/5	1/5	3/10	0	
			$(R_3 = R_3 + 5/2R_1)$						
s_3	0	11	0	1	11	1	-1/2	1	
	Z' = 11		0	0	3/5	1/5	1/5	0	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Z' =11 which implies Z = -11, $x_1 = 4$, $x_2 = 5$, $x_3 = 0$

Example 7

Max Z = 2x + 5y

$x + y \leq 600$

$0 \leq x \leq 400$

$0 \leq y \leq 300$

Solution

SLPP

Max $Z = 2x + 5y + 0s_1 + 0s_2 + 0s_3$

$x + y + s_1 = 600$

$x + s_2 = 400$

$y + s_3 = 300$

$x_1 \geq 0,\ y \geq 0,\ s_1 \geq 0,\ s_2 \geq 0,\ s_3 \geq 0$

$C_j \rightarrow$ 2 5 0 0 0

Basic Variables	C_B	X_B	X	Y	S_1	S_2	S_3	Min ratio X_B / X_k
s_1	0	600	1	1	1	0	0	600 / 1 = 600
s_2	0	400	1	0	0	1	0	-
s_3	0	300	0	$\boxed{1}$	0	0	1	300 /1 = 300→
				↑				
	Z = 0		-2	-5	0	0	0	←Δ_j
				(R1 = R1 – R3)				
s_1	0	300	$\boxed{1}$	0	1	0	-1	300 /1 = 300→
s_2	0	400	1	0	0	1	0	400 / 1 = 400
y	5	300	0	1	0	0	1	-
			↑					
	Z = 1500		-2	0	0	0	5	←Δ_j
x	2	300	1	0	1	0	-1	
			(R2 = R2 – R1)					
s_2	0	100	0	0	-1	1	1	
y	5	300	0	1	0	0	1	

	Z = 2100	0	0	2	0	3	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Z = 2100, x = 300, y = 300

Exercise

Solve by simplex method

1. Maximize $Z = 5x_1 + 3x_2$

   Subject to

   $3x_1 + 5x_2 \leq 15$

   $5x_1 + 2x_2 \leq 10$

   and $x_1 \geq 0, x_2 \geq 0$

   [Ans. Max Z = 235/19, x_1= 20/19, x_2= 45/19]

2. Maximize $Z = 5x_1 + 7x_2$

   Subject to

   $x_1 + x_2 \leq 4$

   $3x_1 - 8x_2 \leq 24$

   $10x_1 + 7x_2 \leq 35$

   and $x_1 \geq 0, x_2 \geq 0$

   [Ans. Max Z = 28, x_1= 0, x_2= 4]

3. Maximize $Z = 2x_1 + 4x_2 + x_3 + x_4$

   Subject to

   $x_1 + 3x_2 + x_4 \leq 4$

   $2x_1 + x_2 \leq 3$

$x_2 + 4x_3 + x_4 \leq 3$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

[Ans. Max Z = 13/2, $x_1 = 1$, $x_2 = 1$, $x_3 = 1/2$, $x_4 = 0$]

4. Maximize $Z = 7x_1 + 5x_2$

   Subject to

   $-x_1 - 2x_2 \geq -6$

   $4x_1 + 3x_2 \leq 12$

   and $x_1 \geq 0, x_2 \geq 0$

[Ans. Max Z = 21, $x_1 = 3$, $x_2 = 0$]

5. Maximize $Z = 3x_1 + 2x_2$

   Subject to

   $2x_1 + x_2 \leq 10$

   $x_1 + 3x_2 \leq 6$

   $x_1 + x_2 \leq 21$

   and $x_1 \geq 0, x_2 \geq 0$

Unit 2

2.1 Computational Procedure of Big – M Method (Charne's Penalty Method)

2.1 Computational Procedure of Big – M Method (Charne's Penalty Method)

Step 1 – Express the problem in the standard form.

Step 2 – Add non-negative artificial variable to the left side of each of the equations corresponding to the constraints of the type '≥' or '='.

When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists).

On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by –M for maximization problems (+M for minimizing problem), where M > 0.

Step 3 – In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

2.2 Worked Examples

Example 1

Max $Z = -2x_1 - x_2$
Subject to
$3x_1 + x_2 = 3$
$4x_1 + 3x_2 \geq 6$
$x_1 + 2x_2 \leq 4$
and $x_1 \geq 0, x_2 \geq 0$

Solution

SLPP

Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 - s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

	C_j →		-2	-1	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	3	$\boxed{3}$	1	0	0	1	0	3 /3 = 1→
a_2	-M	6	4	3	-1	0	0	1	6 / 4 =1.5
s_2	0	4	1	2	0	1	0	0	4 / 1 = 4
	Z = -9M		↑ 2 – 7M	1 – 4M	M	0	0	0	←Δ_j
x_1	-2	1	1	1/3	0	0	X	0	1/1/3 =3
a_2	-M	2	0	$\boxed{5/3}$	-1	0	X	1	6/5/3 =1.2→
s_2	0	3	0	5/3	0	1	X	0	4/5/3=1.8
	Z = -2 – 2M		0	↑ $\frac{(-5M+1)}{3}$	0	0	X	0	←Δ_j
x_1	-2	3/5	1	0	1/5	0	X	X	
x_2	-1	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	Z = -12/5		0	0	1/5	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = -12/5, $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Max $Z = 3x_1 - x_2$
Subject to
$2x_1 + x_2 \geq 2$
$x_1 + 3x_2 \leq 3$
$x_2 \leq 4$
and $x_1 \geq 0$, $x_2 \geq 0$

Solution

SLPP

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$

Subject to

$2x_1 + x_2 – s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 3$

$x_2 + s_3 = 4$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$

	C_j →		3	-1	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k
a_1	-M	2	$\boxed{2}$	1	-1	0	0	1	2 / 2 = 1→
s_2	0	3	1	3	0	1	0	0	3 / 1 = 3
s_3	0	4	0	1	0	0	1	0	-
	Z = -2M		↑ -2M-3	-M+1	M	0	0	0	←Δ_j
x_1	3	1	1	1/2	-1/2	0	0	X	-
s_2	0	2	0	5/2	$\boxed{1/2}$	1	0	X	2/1/2 = 4→
s_3	0	4	0	1	0	0	1	X	-

					↑				
	Z = 3		0	5/2	-3/2	0	0	x	←Δ_j
x_1	3	3	1	3	0	1/2	0	x	
s_1	0	4	0	5	1	2	0	x	
s_3	0	4	0	1	0	0	1	x	
	Z = 9		0	10	0	3/2	0	x	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 9, $x_1 = 3$, $x_2 = 0$

Example 3

Min Z = $2x_1 + 3x_2$

Subject to

$x_1 + x_2 \geq 5$

$x_1 + 2x_2 \geq 6$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

SLPP

Min Z = Max Z$'$ = $-2x_1 - 3x_2 + 0s_1 + 0s_2 - M\,a_1 - M\,a_2$

Subject to

$x_1 + x_2 – s_1 + a_1 = 5$

$x_1 + 2x_2 – s_2 + a_2 = 6$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

	C_j →		-2	-3	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	5	1	1	-1	0	1	0	5 /1 = 5

a_2	-M	6	1	$\boxed{2}$	0	-1	0	1	6 / 2 = 3→
	$Z' = -11M$		-2M + 2	↑ - 3M+3	M	M	0	0	←Δ_j
a_1	-M	2	$\boxed{1/2}$	0	-1	1/2	1	X	2/1/2 = 4→
x_2	-3	3	1/2	1	0	-1/2	0	X	3/1/2 =6
	$Z' = -2M-9$		↑ (-M+1) / 2	0	M	(-M+3)/2	0	X	←Δ_j
x_1	-2	4	1	0	-2	1	X	X	
x_2	-3	1	0	1	1	-1	X	X	
	$Z' = -11$		0	0	1	1	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Z' = -11 which implies Max Z = 11, $x_1 = 4$, $x_2 = 1$

Example 4

Max Z =$3x_1 + 2x_2 + x_3$

Subject to

$2x_1 + x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

and x_1 is unrestricted

$x_2 \geq 0$, $x_3 \geq 0$

Solution

SLPP

Max Z = $3(x_1' - x_1'') + 2x_2 + x_3 - M\, a_1 - M\, a_2$

Subject to

$2(x_1' - x_1'') + x_2 + x_3 + a_1 = 12$

$3(x_1' - x_1'') + 4x_2 + a_2 = 11$

$x_1', x_1'', x_2, x_3, a_1, a_2 \geq 0$

Max $Z = 3x_1' - 3x_1'' + 2x_2 + x_3 - M\,a_1 - M\,a_2$

Subject to

$2x_1' - 2x_1'' + x_2 + x_3 + a_1 = 12$

$3x_1' - 3x_1'' + 4x_2 + a_2 = 11$

$x_1', x_1'', x_2, x_3, a_1, a_2 \geq 0$

		C_j →	3	-3	2	1	-M	-M	
Basic Variables	C_B	X_B	X_1'	X_1''	X_2	X_3	A_1	A_2	Min ratio X_B / X_k
a_1	-M	12	2	-2	1	1	1	0	12 /2 = 6
a_2	-M	11	$\boxed{3}$	-3	4	0	0	1	11/3 =3.6→
		Z = -23M	↑ -5M-3	5M+3	-5M-2	-M-1	0	0	←Δ_j
a_1	-M	14/3	0	0	-5/3	$\boxed{1}$	1	X	14/3/1 = 14/3→
x_1'	3	11/3	1	-1	4/3	0	0	X	-
		$Z = -\frac{14M}{3}+11$	0	-6	5/3M+1	↑ -M-1	0	X	←Δ_j
x_3	1	14/3	0	0	-5/3	1	X	X	
x_1'	3	11/3	1	-1	4/3	0	X	X	
		Z = 47/3	0	0	1/3	0	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

$x_1' = 11/3$, $x_1'' = 0$

$x_1 = x_1' - x_1'' = 11/3 – 0 = 11/3$

Therefore the solution is Max Z = 47/3, $x_1 = 11/3$, $x_2 = 0$, $x_3 = 14/3$

Example 5

Max Z = $8x_2$

Subject to

$x_1 - x_2 \geq 0$

$2x_1 + 3x_2 \leq -6$

and x_1 , x_2 unrestricted

Solution

SLPP

Max Z = $8 (x_2' – x_2'') + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$(x_1' - x_1'') - (x_2' – x_2'') – s_1 + a_1 = 0$

$-2(x_1' - x_1'') - 3(x_2' – x_2'') - s_2 + a_2 = 6$

$x_1', x_1'', x_2', x_2'', s_1, a_1, a_2 \geq 0$

Max Z = $8x_2' – 8x_2'' + 0s_1 + 0s_2 - M a_1 - M a_2$

Subject to

$x_1' - x_1'' - x_2' + x_2'' – s_1 + a_1 = 0$

$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - s_2 + a_2 = 6$

$x_1', x_1'', x_2', x_2'', s_1, a_1, a_2 \geq 0$

	$C_j \rightarrow$		0	0	8	-8	0	0	-M	-M	
Basic Variables	C_B	X_B	X_1'	X_1''	X_2'	X_2''	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-M	0	1	-1	-1	$\boxed{1}$	-1	0	1	0	0→

a_2	-M	6	-2	2	-3	3	0	-1	0	1	2
	Z = -6M		M	-M	4M-8	↑ -4M+8	M	M	0	0	←Δ_j
x_2''	-8	0	1	-1	-1	1	-1	0	X	0	-
a_2	-M	6	-5	$\boxed{5}$	0	0	3	-1	X	1	6/5→
	Z = -6M		5M-8	↑ -5M+8	0	0	-3M+8	M	X	0	←Δ_j
x_2''	-8	6/5	0	0	-1	1	-2/5	-1/5	X	X	
x_1''	0	6/5	-1	1	0	0	3/5	-1/5	X	X	
	Z = -48/5		0	0	0	0	16/5	8/5	X	X	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

$x_1' = 0$, $x_1'' = 6/5$

$x_1 = x_1' - x_1'' = 0 – 6/5 = -6/5$

$x_2' = 0$, $x_2'' = 6/5$

$x_2 = x_2' – x_2'' = 0 – 6/5 = -6/5$

Therefore the solution is Max Z = -48/5, $x_1 = -6/5$, $x_2 = -6/5$

2.3 Steps for Two-Phase Method

The process of eliminating artificial variables is performed in **phase-I** of the solution and **phase-II** is used to get an optimal solution. Since the solution of LPP is computed in two phases, it is called as **Two-Phase Simplex Method**.

Phase I – In this phase, the simplex method is applied to a specially constructed **auxiliary linear programming problem** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 – Assign a cost -1 to each artificial variable and a cost 0 to all other variables in the objective function.

Step 2 – Construct the Auxiliary LPP in which the new objective function Z* is to be maximized subject to the given set of constraints.

Step 3 – Solve the auxiliary problem by simplex method until either of the following three possibilities do arise

i. Max $Z^* < 0$ and atleast one artificial vector appear in the optimum basis at a positive level ($\Delta_j \geq 0$). In this case, given problem does not possess any feasible solution.

ii. Max $Z^* = 0$ and at least one artificial vector appears in the optimum basis at a zero level. In this case proceed to phase-II.

iii. Max $Z^* = 0$ and no one artificial vector appears in the optimum basis. In this case also proceed to phase-II.

Phase II – Now assign the actual cost to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints.

Simplex method is applied to the modified simplex table obtained at the end of phase-I, until an optimum basic feasible solution has been attained. The artificial variables which are non-basic at the end of phase-I are removed.

2.4 Worked Examples

Example 1

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 2$

$x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 3x_1 - x_2$

Subject to

$2x_1 + x_2 – s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1$

Subject to

$2x_1 + x_2 – s_1 + a_1 = 2$

$x_1 + 3x_2 + s_2 = 2$

$x_2 + s_3 = 4$

$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

Phase I

		C_j →	0	0	0	0	0	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min ratio X_B / X_k

a_1	-1	2	$\boxed{2}$	1	-1	0	0	1	1→
s_2	0	2	1	3	0	1	0	0	2
s_3	0	4	0	1	0	0	1	0	-
	$Z^* = -2$		↑ -2	-1	1	0	0	0	←Δ_j
x_1	0	1	1	1/2	-1/2	0	0	X	
s_2	0	1	0	5/2	1/2	1	0	X	
s_3	0	4	0	1	0	0	1	X	
	$Z^* = 0$		0	0	0	0	0	X	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

		C_j →	3	-1	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k
x_1	3	1	1	1/2	-1/2	0	0	-
s_2	0	1	0	5/2	$\boxed{1/2}$	1	0	2→
s_3	0	4	0	1	0	0	1	-
	$Z = 3$		0	5/2	↑ -3/2	0	0	←Δ_j

x_1	3	2	1	3	0	1	0	
s_1	0	2	0	5	1	2	0	
s_3	0	4	0	1	0	0	1	
	Z = 6		0	10	0	3	0	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = 6, $x_1 = 2$, $x_2 = 0$

Example 2

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 5x_1 + 8x_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - 1a_1 - 1a_2$

Subject to

$3x_1 + 2x_2 - s_1 + a_1 = 3$

$x_1 + 4x_2 - s_2 + a_2 = 4$

$x_1 + x_2 + s_3 = 5$
$x_1 , x_2 , s_1, s_2, s_3, a_1, a_2 \geq 0$

Phase I

	$C_j \rightarrow$		0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	3	2	-1	0	0	1	0	3/2
a_2	-1	4	1	$\boxed{4}$	0	-1	0	0	1	1→
s_3	0	5	1	1	0	0	1	0	0	5
	$Z^* = -7$		-4	↑ -6	1	1	0	0	0	←Δ_j
a_1	-1	1	$\boxed{5/2}$	0	-1	1/2	0	1	X	2/5→
x_2	0	1	1/4	1	0	-1/4	0	0	X	4
s_3	0	4	3/4	0	0	1/4	1	0	X	16/3
	$Z^* = -1$		↑ -5/2	0	1	-1/2	0	0	X	←Δ_j
x_1	0	2/5	1	0	-2/5	1/5	0	X	X	
x_2	0	9/10	0	1	1/10	-3/10	0	X	X	
s_3	0	37/10	0	0	3/10	1/10	1	X	X	
	$Z^* = 0$		0	0	0	0	0	X	X	←Δ_j

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		5	8	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	Min ratio X_B / X_k

x_1	5	2/5	1	0	-2/5	$\boxed{1/5}$	0	2→
x_2	8	9/10	0	1	1/10	-3/10	0	-
s_3	0	37/10	0	0	3/10	1/10	1	37
						↑		
	Z = 46/5		0	0	-6/5	-7/5	0	←Δ_j
s_2	0	2	5	0	-2	1	0	-
x_2	8	3/2	3/2	1	-1/2	0	0	-
s_3	0	7/2	-1/2	0	$\boxed{1/2}$	0	1	7→
					↑			
	Z = 12		7	0	-4	0	0	←Δ_j
s_2	0	16	3	0	0	1	2	
x_2	8	5	1	1	0	0	1/2	
s_1	0	7	-1	0	1	0	2	
	Z = 40		3	0	0	0	4	

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 40, $x_1 = 0$, $x_2 = 5$

Example 3

Max Z = $-4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 \geq 15$

$6x_1 + x_2 + 6x_3 \geq 12$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Solution

Standard LPP

Max Z = $-4x_1 - 3x_2 - 9x_3$

Subject to

$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$

$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 - 0x_2 - 0x_3 + 0s_1 + 0s_2 -1a_1 -1a_2$

Subject to

$2x_1 + 4x_2 + 6x_3 - s_1 + a_1 = 15$

$6x_1 + x_2 + 6x_3 - s_2 + a_2 = 12$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

Phase I

	$C_j \rightarrow$		0	0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-1	15	2	4	6	-1	0	1	0	15/6
a_2	-1	12	6	1	$\boxed{6}$	0	-1	0	1	2→
	$Z^* = -27$		-8	-5	↑ -12	1	1	0	0	$\leftarrow \Delta_j$
a_1	-1	3	-4	$\boxed{3}$	0	-1	1	1	X	1→
x_3	0	2	1	1/6	1	0	-1/6	0	X	12
	$Z^* = -3$		4	↑ -3	0	1	-1	0	X	$\leftarrow \Delta_j$
x_2	0	1	-4/3	1	0	-1/3	1/3	X	X	
x_3	0	11/6	22/18	0	1	1/18	-4/18	X	X	
	$Z^* = 0$		0	0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		-4	-3	-9	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min ratio X_B / X_k
x_2	-3	1	-4/3	1	0	-1/3	1/3	-
x_3	-9	11/6	$\boxed{22/18}$	0	1	1/18	-4/18	$3/2 \rightarrow$
	Z = -39/2		↑ -3	0	0	1/2	1	$\leftarrow \Delta_j$
x_2	-3	3	0	1	12/11	-3/11	1/11	
x_1	-4	3/2	1	0	18/22	1/22	-4/22	
	Z = -15		0	0	27/11	7/11	5/11	$\leftarrow \Delta_j$

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max Z = -15, $x_1 = 3/2$, $x_2 = 3$, $x_3 = 0$

Example 4

Min $Z = 4x_1 + x_2$

Subject to

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Standard LPP

Min Z = Max $Z' = - 4x_1 - x_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 – s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 – 0x_2 + 0s_1 + 0s_2 \; –1a_1 –1a_2$

Subject to

$3x_1 + x_2 + a_1 = 3$

$4x_1 + 3x_2 – s_1 + a_2 = 6$

$x_1 + 2x_2 + s_2 = 4$

$x_1 , x_2 , s_1, s_2, a_1, a_2 \geq 0$

Phase I

	C_j →		0	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	A_2	Min ratio X_B / X_k
a_1	-1	3	$\boxed{3}$	1	0	0	1	0	1→
a_2	-1	6	4	3	-1	0	0	1	6/4
s_2	0	4	1	2	0	1	0	0	4
			↑						
	$Z^* = -9$		-7	-4	1	0	0	0	
x_1	0	1	1	1/3	0	0	X	0	3
a_2	-1	2	0	$\boxed{5/3}$	-1	0	X	1	6/5→
s_2	0	3	0	5/3	0	1	X	0	9/5
				↑					
	$Z^* = -2$		0	-5/3	1	0	X	0	

x_1	0	3/5	1	0	1/5	0	X	X	
x_2	0	6/5	0	1	-3/5	0	X	X	
s_2	0	1	0	0	1	1	X	X	
	$Z^* = 0$		0	0	0	0	X	X	

Since all $\Delta_j \geq 0$, Max $Z^* = 0$ and no artificial vector appears in the basis, we proceed to phase II.

Phase II

	$C_j \rightarrow$		-4	-1	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min ratio X_B / X_k
x_1	-4	3/5	1	0	1/5	0	3
x_2	-1	6/5	0	1	-3/5	0	-
s_2	0	1	0	0	$\boxed{1}$	1	1→
	$Z' = -18/5$		0	0	↑ -1/5	0	←Δ_j
x_1	-4	2/5	1	0	0	-1/5	
x_2	-1	9/5	0	1	0	3/5	
s_1	0	1	0	0	1	1	
	$Z' = -17/5$		0	0	0	1/5	←Δ_j

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained

Therefore the solution is Max $Z' = -17/5$

Min $Z = 17/5$, $x_1 = 2/5$, $x_2 = 9/5$

Exercise

Solve by Big-M method

1. Min $Z = 4x_1 + 2x_2$

   Subject to

   $3x_1 + x_2 \geq 27$

   $x_1 + x_2 \geq 21$

   and $x_1 \geq 0, x_2 \geq 0$

   [Ans. Min $Z = 48$, $x_1 = 3$, $x_2 = 18$]

2. Min $Z = x_1 + x_2 + 3x_3$

   Subject to

   $3x_1 + 2x_2 + x_3 \leq 3$

   $2x_1 + x_2 + 2x_3 \geq 3$

   and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

   [Ans. Min $Z = 3$, $x_1 = 3/4$, $x_2 = 0$, $x_3 = 3/4$]

Solve by Two-phase method

1. Max $Z = 3x_1 - x_2$

   Subject to

   $2x_1 + x_2 \geq 2$

   $x_1 + 3x_2 \leq 2$

   $x_2 \leq 4$

   and $x_1 \geq 0, x_2 \geq 0$

   [Ans. Max $Z = 6$, $x_1 = 2$, $x_2 = 0$]

2. Max $Z = 5x_1 - 2x_2 + 3x_3$

   Subject to

   $2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

[Ans. Max Z = 85/3, x_1 = 23/3, x_2 =5, x_3 =0]

Unit 3

3.1 Special cases in Simplex Method

3.1.1 Degenaracy

3.1.2 Non-existing Feasible Solution

3.1.3 Unbounded Solution

3.1.4 Multiple Optimal Solutions

3.1.1 Degeneracy

The concept of obtaining a degenerate basic feasible solution in a LPP is known as degeneracy. The degeneracy in a LPP may arise

- At the initial stage when at least one basic variable is zero in the initial basic feasible solution.
- At any subsequent iteration when more than one basic variable is eligible to leave the basic and hence one or more variables becoming zero in the next iteration and the problem is said to degenerate. There is no assurance that the value of the objective function will improve, since the new solutions may remain degenerate. As a result, it is possible to repeat the same sequence of simplex iterations endlessly without improving the solutions. This concept is known as cycling or circling.

Rules to avoid cycling

- Divide each element in the tied rows by the positive coefficients of the key column in that row.
- Compare the resulting ratios, column by column, first in the identity and then in the body, from left to right.

- The row which first contains the smallest algebraic ratio contains the leaving variable.

Example 1

Max Z = $3x_1 + 9x_2$

Subject to

$x_1 + 4x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max Z = $3x_1 + 9x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + 4x_2 + s_1 = 8$

$x_1 + 2x_2 + s_2 = 4$

$x_1 , x_2 , s_1, s_2 \geq 0$

		$C_j \rightarrow$	3	9	0	0		
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	X_B / X_K	S_1 / X_2
s_1	0	8	1	4	1	0	$\left.\begin{matrix}2\\2\end{matrix}\right\}$	1/4
s_2	0	4	1	$\boxed{2}$	0	1		0/2→
				↑				
	Z = 0		-3	-9	0	0		$\leftarrow \Delta_j$

s_1 x_2	0 9	0 2	-1 1/2	0 1	1 0	-1 1/2		
	Z =18		3/2	0	0	9/2		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 18, $x_1 = 0$, $x_2 = 2$

Note – Since a tie in minimum ratio (degeneracy), we find minimum of s_1 / x_k for these rows for which the tie exists.

Example 2

Max $Z = 2x_1 + x_2$

Subject to

$4x_1 + 3x_2 \leq 12$

$4x_1 + x_2 \leq 8$

$4x_1 - x_2 \leq 8$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$4x_1 + 3x_2 + s_1 = 12$

$4x_1 + x_2 + s_2 = 8$

$4x_1 - x_2 + s_3 = 8$

$x_1 , x_2 , s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	2	1	0	0	0			
Basic Varibles	C_B	X_B	X_1	X_2	S_1	S_2	S_3	X_B / X_K	S_1 / X_1	S_2 / X_1
s_1	0	12	4	3	1	0	0	12/4=3		

s_2	0	8	4	1	0	1	0	8/4=2	4/0=0	1/4
s_3	0	8	$\boxed{4}$	-1	0	0	1	8/4=2	4/0=0	0/4=0→
	Z = 0		↑ -2	-1	0	0	0	←Δ_j		
s_1	0	4	0	4	1	0	-1	4/4=1		
s_2	0	0	0	$\boxed{2}$	0	1	-1	0→		
x_1	2	2	1	-1/4	0	0	1/4	-		
	Z = 4		0	↑ -3/2	0	0	1/2	←Δ_j		
s_1	0	4	0	0	1	-2	$\boxed{1}$	0→		
x_2	1	0	0	1	0	1/2	-1/2	-		
x_1	2	2	1	0	0	1/8	1/8	16		
	Z = 4		0	0	0	3/4	↑ -1/4	←Δ_j		
s_3	0	4	0	0	1	-2	1			
x_2	1	2	0	1	1/2	-1/2	0			
x_1	2 3/2		1	0	-1/8	3/8	0			
	Z = 5		0	0	1/4	1/4	0	←Δ_j		

Since all $\Delta_j \geq 0$, optimal basic feasible solution is obtained. Therefore the solution is Max Z = 5, $x_1 = 3/2$, $x_2 = 2$

3.1.2 Non-existing Feasible Solution

The feasible region is found to be empty which indicates that the problem has no feasible solution.

Example 1

Max Z = $3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max Z = $3x_1 + 2x_2 + 0s_1 + 0s_2 – Ma_1$

Subject to

$2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 - s_2 + a_1 = 12$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	3	2	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	2	2	$\boxed{1}$	1	0	0	2/1=2→
a_1	-M	12	3	4	0	-1	1	12/4=3
	Z= -12M		-3M-3	↑ -4M-2	0	M	0	←Δ_j
x_2	2	2	2	1	1	0	0	
a_1	-M	4	-5	0	-4	-1	1	
	Z= 4-4M		1+5M	0	2+4M	M	0	

$\Delta_j \geq 0$ so according to optimality condition the solution is optimal but the solution is called **pseudo optimal solution** since it does not satisfy all the constraints but satisfies the optimality condition. The artificial variable has a positive value which indicates there is no feasible solution.

Example 2

Min Z = $x_1 – 2x_2 – 3x_3$

Subject to

$–2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

Standard LPP

Min Z = Max $Z' = –x_1 + 2x_2 + 3x_3$

Subject to

$–2x_1 + x_2 + 3x_3 + a_1 = 2$

$2x_1 + 3x_2 + 4x_3 + a_2 = 1$

$x_1 , x_2 , a_1, a_2 \geq 0$

Auxiliary LPP

Max $Z^* = 0x_1 + 0x_2 + 0x_3 – 1a_1 – 1a_2$

Subject to

$–2x_1 + x_2 + 3x_3 + a_1 = 2$

$2x_1 + 3x_2 + 4x_3 + a_2 = 1$

$x_1 , x_2 , a_1, a_2 \geq 0$

Phase I

		C_j→	0	0	0	-1	-1	
Basic Variables	C_B	X_B	X_1	X_2	X_3	A_1	A_2	Min Ratio X_B / X_K
a_1	-1	2	-2	1	3	1	0	2/3
a_2	-1	1	2	3	$\boxed{4}$	0	1	1/4→
	Z* = -3		0	-4	↑ -7	0	0	←Δ_j
a_1	-1	5/4	-7/4	-5/4	0	1	X	
x_3	0	1/4	1/2	3/4	1	0	X	
	Z* = -5/4		7/4	5/4	0	1	X	←Δ_j

Since for all $\Delta_j \geq 0$, optimum level is achieved. At the end of phase-I Max $Z^* < 0$ and one of the artificial variable a_1 appears at the positive optimum level. Hence the given problem does not posses any feasible solution.

3.1.3 Unbounded Solution

In some cases if the value of a variable is increased indefinitely, the constraints are not violated. This indicates that the feasible region is unbounded at least in one direction. Therefore, the objective function value can be increased indefinitely. This means that the problem has been poorly formulated or conceived.

In simplex method, this can be noticed if Δ_j value is negative to a variable (entering) which is notified as key column and the ratio of solution value to key column value is either negative or infinity (both are to be ignored) to all the variables. This indicates that no variable is ready to leave the basis, though a variable is ready to enter. We cannot proceed further and the solution is unbounded or not finite.

Example 1

Max $Z = 6x_1 - 2x_2$

Subject to

$2x_1 - x_2 \leq 2$

$x_1 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 6x_1 - 2x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 - x_2 + s_1 = 2$

$x_1 + s_2 = 4$

$x_1 , x_2 , s_1, s_2 \geq 0$

		C_j→	6	-2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	2	$\boxed{2}$	-1	1	0	1→
s_2	0	4	1	0	0	1	4
	Z = 0		↑ -6	2	0	0	←Δ_j
x_1	6	1	1	-1/2	1/2	0	-
s_2	0	3	0	$\boxed{1/2}$	-1/2	1	6→
	Z = 6		0	↑ -1	3	0	←Δ_j
x_1	6	4	1	0	0	1	
x_2	-2	6	0	1	-1	2	
	Z = 12		0	0	2	2	←Δ_j

The optimal solution is $x_1 = 4$, $x_2 = 6$ and $Z = 12$

In the starting table, the elements of x_2 are negative and zero. This is an indication that the feasible region is not bounded. From this we conclude the problem has unbounded feasible region but still the optimal solution is bounded.

Example 2

$\text{Max } Z = -3x_1 + 2x_2$

Subject to

$x_1 \leq 3$

$x_1 - x_2 \leq 0$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

$\text{Max } Z = -3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$x_1 + s_1 = 3$

$x_1 - x_2 + s_2 = 0$

$x_1, x_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	-3	2	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	3	1	0	1	0	
s_2	0	0	1	-1	0	1	
	Z = 0		3	↑ -2	0	0	$\leftarrow \Delta_j$

Corresponding to the incoming vector (column x_2), all elements are negative or zero. So x_2 cannot enter the basis and the outgoing vector cannot be found. This is an indication that there exists unbounded solution for the given problem.

Example 3

Max $Z = 107x_1 + x_2 + 2x_3$

Subject to

$14/3x_1 + 1/3x_2 - 2x_3 \leq 7/3$

$16x_1 + 1/2x_2 - 6x_3 \leq 5$

$3x_1 - x_2 - x_3 \leq 0$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

Standard LPP

Max $Z = 107x_1 + x_2 + 2x_3$

Subject to

$14/3x_1 + 1/3x_2 - 2x_3 + s_1 = 7/3$

$16x_1 + 1/2x_2 - 6x_3 + s_2 = 5$

$3x_1 - x_2 - x_3 + s_3 = 0$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	107	1	2	0	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min Ratio X_B / X_K
s_1	0	7/3	14/3	1/3	-2	1	0	0	0.5
s_2	0	5	16	1/2	-6	0	1	0	0.8

s_3	0	0	$\boxed{3}$	-1	-1	0	0	1	0→
	Z = 0		↑ -107	-1	-2	0	0	0	←Δ_j
s_1	0	7/3	0	17/9	-4/9	1	0	-14/9	-
s_2	0	5	0	35/6	-2/3	0	1	-16/3	-
x_1	107	0	1	-1/3	-1/3	0	0	1/3	-
	Z = 0		0	-110/3	-113/3	0	0	107/3	←Δ_j

Corresponding to negative Δ_3, all the elements of x_3 column are negative. So x_3 cannot enter into the basis matrix. This is an indication that there exists an unbounded solution to the given problem.

3.1.4 Multiple Optimal Solution

When the objective function is parallel to one of the constraints, the multiple optimal solutions may exist. After reaching optimality, if at least one of the non-basic variables possess a zero value in Δ_j, the multiple optimal solution exist.

Example

Max $Z = 6x_1 + 4x_2$

Subject to

$2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Standard LPP

Max $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$

Subject to

$2x_1 + 3x_2 + s_1 = 30$

$3x_1 + 2x_2 + s_2 = 24$

$x_1 + x_2 - s_3 + a_1 = 3$

$x_1 , x_2 , s_1, s_2, s_3, a_1 \geq 0$

		$C_j \rightarrow$	6	4	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min Ratio X_B / X_K
s_1	0	30	2	3	1	0	0	0	15
s_2	0	24	3	2	0	1	0	0	8
a_1	-M	3	$\boxed{1}$	1	0	0	-1	1	3→
	Z = -3M		↑ -M-6	-M-4	0	0	M	0	←Δ_j
s_1	0	24	0	1	1	0	2	X	12
s_2	0	15	0	-1	0	1	$\boxed{3}$	X	5→
x_1	6	3	1	1	0	0	-1	X	-
	Z = 18		0	2	0	0	↑ -6	X	←Δ_j
s_1	0	14	0	$\boxed{5/3}$	1	-2/3	0	X	42/5→
s_3	0	5	0	-1/3	0	1/3	1	X	-
x_1	6	8	1	2/3	0	1/3	0	X	12
	Z = 48		0	↑ 0	0	2	0	X	←Δ_j

Since all $\Delta_j \geq 0$, optimum solution is obtained as $x_1 = 8$, $x_2 = 0$, Max Z = 48

Since Δ_2 corresponding to non-basic variable x_2 is obtained zero, this indicates that alternate solution or multiple optimal solution also exist. Therefore the solution as obtained above is not unique. Thus we can bring x_2 into the basis in place of s_1. The new optimum simplex table is obtained as follows

		$C_j \rightarrow$	6	4	0	0	0	-M	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	A_1	Min Ratio X_B / X_K
x_2	4	42/5	0	1	3/5	-2/5	0	X	
s_3	0	39/5	0	0	1/5	1/5	1	X	
x_1	6	12/5	1	0	-2/5	3/5	0	X	
	Z = 48		0	0	0	2	0	X	$\leftarrow \Delta_j$

Exercise

Solve

1. Max $Z = 3x_1 + 2.5x_2$

   Subject to

   $2x_1 + 4x_2 \geq 40$

   $3x_1 + 2x_2 \geq 50$

   and $x_1 \geq 0, x_2 \geq 0$

   [Ans. Unbounded solution]

2. Max $Z = 3x_1 + 2x_2$

   Subject to

   $2x_1 + x_2 \leq 2$

   $3x_1 + 4x_2 \geq 12$

   and $x_1 \geq 0, x_2 \geq 0$

[Ans. Pseudo-optimum solution]

3. Min $Z = x_1 - 2x_2 - 3x_3$

   Subject to

   $-2x_1 + x_2 + 3x_3 = 2$

   $2x_1 + 3x_2 + 4x_3 = 1$

   and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

[Ans. No feasible solution]

4. Max $Z = 3x_1 + 2x_2 + x_3 + 4x_4$

   Subject to

   $4x_1 + 5x_2 + x_3 - 3x_4 = 5$

   $2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$

   $x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6$

   and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

[Ans. No solution]

5. Max $Z = 3x_1 + 9x_2$

   Subject to

   $4x_1 + 4x_2 \geq 8$

   $x_1 + 2x_2 \geq 4$

   and $x_1 \geq 0, x_2 \geq 0$

[Degeneracy exists]

6. In the course of simplex table calculations, describe how u will detect a
    a. Degenerate
    b. An unbounded
    c. Non-existing feasible solution

7. What is degeneracy?

8. Write the role of pivot element in a simplex table.

Module 3

Unit 1

1.1 The Revised Simplex Method

1.2 Steps for solving Revised Simplex Method in Standard Form-I

1.3 Worked Examples

1.1 The Revised Simplex Method

While solving linear programming problem on a digital computer by regular simplex method, it requires storing the entire simplex table in the memory of the computer table, which may not be feasible for very large problem. But it is necessary to calculate each table during each iteration. The revised simplex method which is a modification of the original method is more economical on the computer, as it computes and stores only the relevant information needed currently for testing and / or improving the current solution. i.e. it needs only

- The net evaluation row Δ_j to determine the non-basic variable that enters the basis.
- The pivot column

- The current basis variables and their values (X_B column) to determine the minimum positive ratio and then identify the basis variable to leave the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basis matrix at any iteration.

There are two standard forms for revised simplex method

- **Standard form-I** – In this form, it is assumed that an identity matrix is obtained after introducing slack variables only.
- **Standard form-II** – If artificial variables are needed for an identity matrix, then two-phase method of ordinary simplex method is used in a slightly different way to handle artificial variables.

1.2 Steps for solving Revised Simplex Method in Standard Form-I

Solve by Revised simplex method

Max $Z = 2x_1 + x_2$

Subject to

$3 x_1 + 4 x_2 \leq 6$

$6 x_1 + x_2 \leq 3$

and $x_1, x_2 \geq 0$

SLPP

Max $Z = 2x_1 + x_2 + 0s_1 + 0s_2$

Subject to

$3 x_1 + 4 x_2 + s_1 = 6$

$6 x_1 + x_2 + s_2 = 3$

and $x_1, x_2, s_1, s_2 \geq 0$

Step 1 – Express the given problem in standard form – I

- Ensure all $b_i \geq 0$
- The objective function should be of maximization
- Use of non-negative slack variables to convert inequalities to equations

The objective function is also treated as first constraint equation

$$Z - 2x_1 - x_2 + 0s_1 + 0s_2 = 0$$

$$3\,x_1 + 4\,x_2 + s_1 + 0s_2 = 6 \qquad\qquad \text{-- (1)}$$

$$6\,x_1 + x_2 + 0s_1 + s_2 = 3$$

and $x_1, x_2, s_1, s_2 \geq 0$

Step 2 – Construct the starting table in the revised simplex form

Express (1) in the matrix form with suitable notation

$$\begin{array}{c} \begin{array}{ccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array} \\ \begin{bmatrix} 1 & -2 & -1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{array}{c} X_B \\ \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \end{array}$$

Column vector corresponding to Z is usually denoted by e_1. It is the first column of the basis matrix B_1, which is usually denoted as $B_1 = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)} \ldots \beta_n^{(1)}]$

Hence the column $\beta_0^{(1)}$, $\beta_1^{(1)}$, $\beta_2^{(1)}$ constitutes the basis matrix B_1 (whose inverse B_1^{-1} is also B_1)

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0		

$a_1^{(1)}$	$a_2^{(1)}$
-2	-1

s_1	0	1	0	6			3	4
s_2	0	0	1	3			6	1

Step 3 – Computation of Δ_j for $a_1^{(1)}$ and $a_2^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -2 + 0 * 3 + 0 *6 = -2$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -1 + 0 * 4 + 0 *1 = -1$

Step 4 – Apply the test of optimality

Both Δ_1 and Δ_2 are negative. So find the most negative value and determine the incoming vector.

Therefore most negative value is $\Delta_1 = -2$. This indicates $a_1^{(1)}$ (x_1) is incoming vector.

Step 5 – Compute the column vector X_k

$X_k = B_1^{-1} * a_1^{(1)}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$$

Stp 6 – Determine the outgoing vector. We are not supposed to calculate for Z row.

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0	-2	-
s_1	0	1	0	6	3	2

s_2	0	0	1	3	$\boxed{6}$ ↑ incoming	1/2→outgoing

Step 7 – Determination of improved solution

Column e_1 will never change, x_1 is incoming so place it outside the rectangular boundary

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	0	0	-2
R_2	1	0	6	3
R_3	0	1	3	$\boxed{6}$

Make the pivot element as 1 and the respective column elements to zero.

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_1
R_1	0	1/3	1	0
R_2	1	-1/2	9/2	0
R_3	0	1/6	1/2	1

Construct the table to start with second iteration

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1		
s_1	0	1	-1/2	9/2		
x_1	0	0	1/6	1/2		

$a_4^{(1)}$	$a_2^{(1)}$
0	-1
0	4
1	1

$\Delta_4 = 1 * 0 + 0 * 0 + 1/3 *1 = 1/3$

$\Delta_2 = 1 * -1 + 0 * 4 + 1/3 *1 = -2/3$

Δ_2 is most negative. Therefore $a_2^{(1)}$ is incoming vector.

Compute the column vector

$$\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{bmatrix} * \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/2 \\ 1/6 \end{bmatrix}$$

Determine the outgoing vector

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	1/3	1	-2/3	-
s_1	0	1	-1/2	9/2	$\boxed{7/2}$	9/7→outgoing
x_1	0	0	1/6	1/2	1/6 ↑ incoming	3

Determination of improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2
R_1	0	1/3	1	-2/3
R_2	1	-1/2	9/2	$\boxed{7/2}$
R_3	0	1/6	1/2	1/6

$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_2

R_1	4/21	5/21	13/7	0
R_2	2/7	-1/7	9/7	1
R_3	-1/21	8/42	2/7	0

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	4/21	5/21	13/7		
x_2	0	2/7	-1/7	9/7		
x_1	0	-1/21	8/42	2/7		

$a_4^{(1)}$	$a_3^{(1)}$
0	0
0	1
1	0

$\Delta_4 = 1 * 0 + 4/21 * 0 + 5/21 *1 = 5/21$

$\Delta_3 = 1 * 0 + 4/21 * 1 + 5/21 *0 = 4/21$

Δ_4 and Δ_3 are positive. Therefore optimal solution is Max Z = 13/7, x_1= 2/7, x_2 = 9/7

1.3 Worked Examples

Example 1

Max Z = $x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 3$

$x_1 + 2x_2 \leq 5$

$3x_1 + x_2 \leq 6$

and $x_1, x_2 \geq 0$

Solution

SLPP

Max Z = $x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + s_1 = 3$

$x_1 + 2x_2 + s_2 = 5$

$3x_1 + x_2 + s_3 = 6$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

Standard Form-I

$Z - x_1 - 2x_2 - 0s_1 - 0s_2 - 0s_3 = 0$

$x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 3$

$x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 5$

$3x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 6$

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

Matrix form

$$\begin{array}{cccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} & \beta_3^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Revised simplex table **Additional table**

Basic variables	B_1^{-1}				X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$			
Z	1	0	0	0	0		

$a_1^{(1)}$	$a_2^{(1)}$
-1	-2

s_1	0	1	0	0	3				1	1
s_2	0	0	1	0	5				1	2
s_3	0	0	0	1	6				3	1

Computation of Δ_j for $a_1{}^{(1)}$ and $a_2{}^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1{}^{(1)} = 1 * -1 + 0 * 1 + 0 *1 + 0 *3= -1$

Δ_2 = first row of $B_1^{-1} * a_2{}^{(1)} = 1 * -2 + 0 * 1 + 0 *2+ 0 *1 = -2$

$\Delta_2 = -2$ is most negative. So $a_2{}^{(1)}$ (x_2) is incoming vector.

Compute the column vector X_k

$$X_k = B_1^{-1} * a_2{}^{(1)}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

Basic variables	B_1^{-1}				X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$			
Z	1	0	0	0	0	-2	-
s_1	0	1	0	0	3	1	3
s_2	0	0	1	0	5	$\boxed{2}$	5/2→
s_3	0	0	0	1	6	1	6
						↑	

Improved Solution

$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k

R_1	0	0	0	0	-2
R_2	1	0	0	3	1
R_3	0	1	0	5	$\boxed{2}$
R_4	0	0	1	6	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X_B	X_k
R_1	0	1	0	5	0
R_2	1	-1/2	0	1/2	0
R_3	0	1/2	0	5/2	1
R_4	0	-1/2	1	7/2	0

Revised simplex table for II iteration

Basic variables	B_1^{-1}				X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$			
Z	1	0	1	0	5		
s_1	0	1	-1/2	0	1/2		
x_2	0	0	1/2	0	5/2		
s_3	0	0	-1/2	1	7/2		

$a_1^{(1)}$	$a_4^{(1)}$
-1	0
1	0
1	1
3	0

$\Delta_1 = 1 * -1 + 0 * 1 + 1 *1 + 0 *3= 0$

$\Delta_4 = 1 * 0 + 0 * 0 + 1 *1+ 0 *0 = 1$

Δ_1 and Δ_4 are positive. Therefore optimal solution is Max Z = 5, $x_1 = 0$, $x_2 = 5/2$

Example 2

Max $Z = 80x_1 + 55x_2$

Subject to

$4x_1 + 2x_2 \leq 40$

$2x_1 + 4x_2 \leq 32$

and $x_1, x_2 \geq 0$

Solution

Max $Z = 80x_1 + 55x_2$

Subject to

$2x_1 + x_2 \leq 20$ (divide by 2)

$x_1 + 2x_2 \leq 16$ (divide by 2)

and $x_1, x_2 \geq 0$

SLPP

Max $Z = 80x_1 + 55x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 + x_2 + s_1 = 20$

$x_1 + 2x_2 + s_2 = 16$

and $x_1, x_2, s_1, s_2 \geq 0$

Standard form-I

$Z - 80x_1 - 55x_2 - 0s_1 - 0s_2 = 0$

$2x_1 + x_2 + s_1 + 0s_2 = 20$

$x_1 + 2x_2 + 0s_1 + s_2 = 16$

and $x_1, x_2, s_1, s_2 \geq 0$

Matrix form

$$\begin{array}{ccccc} \beta_0^{(1)} & & & \beta_1^{(1)} & \beta_2^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -80 & -55 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \overset{X_B}{\begin{bmatrix} 0 \\ 20 \\ 16 \end{bmatrix}}$$

Revised simplex table **Additional table**

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0		
s_1	0	1	0	20		
s_2	0	0	1	16		

$a_1^{(1)}$	$a_2^{(1)}$
-80	-55
2	1
1	2

Computation of Δ_j for $a_1^{(1)}$ and $a_2^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -80 + 0 * 2 + 0 * 1 = -80$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -55 + 0 * 1 + 0 * 2 = -55$

$\Delta_1 = -80$ is most negative. So $a_1^{(1)}$, (x_1) is incoming vector.

Compute the column vector X_k

$X_k = B_1^{-1} * a_1^{(1)}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -80 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -80 \\ 2 \\ 1 \end{bmatrix}$$

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0	-80	-
s_1	0	1	0	20	$\boxed{2}$	10→
s_2	0	0	1	16	1	16
					↑	

Improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	0	0	0	-80
R_2	1	0	20	$\boxed{2}$
R_3	0	1	16	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	40	0	800	0
R_2	1/2	0	10	1
R_3	-1/2	1	6	0

Revised simplex table for II iteration

Basic	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$			

$a_3^{(1)}$	$a_2^{(1)}$

variables	(Z)					
Z	1	40	0	800		
x_1	0	1/2	0	10		
s_2	0	-1/2	1	6		

0	-55
1	1
0	2

Computation of Δ_j for $a_3^{(1)}$ and $a_2^{(1)}$

$$\Delta_3 = \text{first row of } B_1^{-1} * a_3^{(1)} = 1 * 0 + 40 * 1 + 0 * 0 = 40$$

$$\Delta_2 = \text{first row of } B_1^{-1} * a_2^{(1)} = 1 * -55 + 40 * 1 + 0 * 2 = -15$$

$\Delta_2 = -15$ is most negative. So $a_2^{(1)}$ (x_2) is incoming vector.

Compute the column vector X_k

$$\begin{bmatrix} 1 & 40 & 0 \\ 0 & 1/2 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} * \begin{bmatrix} -55 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -15 \\ 1/2 \\ 3/2 \end{bmatrix}$$

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	40	0	800	-15	-
x_1	0	1/2	0	10	1/2	20
s_2	0	-1/2	1	6	$\boxed{3/2}$ ↑	4→

Improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	40	0	800	-15
R_2	1/2	0	10	1/2
R_3	-1/2	1	6	$\boxed{3/2}$

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	35	10	860	0
R_2	2/3	-1/3	8	0
R_3	-1/3	2/3	4	1

Revised simplex table for III iteration

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	35	10	860		
x_1	0	2/3	-1/3	8		
x_2	0	-1/3	2/3	4		

$a_3^{(1)}$	$a_4^{(1)}$
0	0
1	0
0	1

Computation of Δ_3 and Δ_4

$\Delta_3 = 1 * 0 + 35 * 1 + 10 *0 = 35$

$\Delta_4 = 1 * 0 + 35 * 0 + 10 *1 = 10$

Δ_3 and Δ_4 are positive. Therefore optimal solution is Max Z = 860, $x_1 = 8$, $x_2 = 4$

Example 3

Max $Z = x_1 + x_2 + x_3$

Subject to

$4x_1 + 5x_2 + 3x_3 \leq 15$

$10x_1 + 7x_2 + x_3 \leq 12$

and $x_1, x_2, x_3 \geq 0$

Solution

SLPP

Max $Z = x_1 + x_2 + x_3 + 0s_1 + 0s_2$

Subject to

$$4x_1 + 5x_2 + 3x_3 + s_1 = 15$$

$$10x_1 + 7x_2 + x_3 + s_2 = 12$$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

Standard form-I

$$Z - x_1 - x_2 - x_3 - 0s_1 - 0s_2 = 0$$

$$4x_1 + 5x_2 + 3x_3 + s_1 + 0s_2 = 15$$

$$10x_1 + 7x_2 + x_3 + 0s_1 + s_2 = 12$$

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

Matrix form

$$\begin{array}{cccccc} \beta_0^{(1)} & & & & \beta_1^{(1)} & \beta_2^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 4 & 5 & 3 & 1 & 0 \\ 0 & 10 & 7 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix} = \begin{array}{c} X_B \\ \begin{bmatrix} 0 \\ 15 \\ 12 \end{bmatrix} \end{array}$$

Revised simplex table **Additional table**

Basic	B_1^{-1}			X_B	X_k	X_B /
	e_1	$\beta_1^{(1)}$	$\beta_2^{(1)}$			

$a_1^{(1)}$	$a_2^{(1)}$	$a_3^{(1)}$

variables	(Z)					X_k
Z	1	0	0	0		
s_1	0	1	0	15		
s_2	0	0	1	12		

-1	-1	-1
4	5	3
10	7	1

Computation of Δ_j for $a_1^{(1)}$, $a_2^{(1)}$ and $a_3^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -1 + 0 * 4 + 0 *10 = -1$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -1 + 0 * 5 + 0 *7 = -1$

Δ_3 = first row of $B_1^{-1} * a_3^{(1)} = 1 * -1 + 0 * 3 + 0 *1 = -1$

There is a tie, so perform the computation of Δ_j with second row

Δ_1 = second row of $B_1^{-1} * a_1^{(1)} = 0 * -1 + 1 * 4 + 0 *10 = 4$

Δ_2 = second row of $B_1^{-1} * a_2^{(1)} = 0 * -1 + 1 * 5 + 0 *7 = 5$

Δ_3 = second row of $B_1^{-1} * a_3^{(1)} = 0 * -1 + 1 * 3 + 0 *1 = 3$

Since $\Delta_j \geq 0$, we obtain pure optimum solution where Max Z = 0, $x_1= 0, x_2= 0$

Example 4

Max $Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5$

Subject to

$2x_1 + 3x_2 + 3x_3+ 2x_4 + 2x_5 \leq 20$

$3x_1 + 5x_2+ 4x_3 + 2x_4 + 4x_5 \leq 30$

and $x_1, x_2, x_3, x_4, x_5 \geq 0$

Solution

SLPP

Max $Z = 5x_1 + 8x_2 + 7x_3 + 4x_4 + 6x_5+ 0s_1+ 0s_2$

Subject to

$2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + s_1 = 20$

$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 + s_2 = 30$

and $x_1, x_2, x_3, x_4, x_5, s_1, s_2 \geq 0$

Standard form-I

$Z - 5x_1 - 8x_2 - 7x_3 - 4x_4 - 6x_5 - 0s_1 - 0s_2 = 0$

$2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 + s_1 + 0s_2 = 20$

$3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 + 0s_1 + s_2 = 30$

and $x_1, x_2, x_3, x_4, x_5, s_1, s_2 \geq 0$

Matrix form

$$\begin{array}{cccccccc} \beta_0^{(1)} & & & & & & \beta_1^{(1)} & \beta_2^{(1)} \\ e_1 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} & a_6^{(1)} & a_7^{(1)} \end{array}$$

$$\begin{bmatrix} 1 & -5 & -8 & -7 & -4 & -6 & 0 & 0 \\ 0 & 2 & 3 & 3 & 2 & 2 & 1 & 0 \\ 0 & 3 & 5 & 4 & 2 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ s_1 \\ s_2 \end{bmatrix} = \begin{array}{c} X_B \\ \begin{bmatrix} 0 \\ 20 \\ 30 \end{bmatrix} \end{array}$$

Revised simplex table

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0		
s_1	0	1	0	20		
s_2	0	0	1	30		

Additional table

$a_1^{(1)}$	$a_2^{(1)}$	$a_3^{(1)}$	$a_4^{(1)}$	$a_5^{(1)}$
-5	-8	-7	-4	-6
2	3	3	2	2
3	5	4	2	4

Computation of Δ_j for $a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, a_4^{(1)}, a_5^{(1)}$

Δ_1 = first row of $B_1^{-1} * a_1^{(1)} = 1 * -5 + 0 * 2 + 0 * 3 = -5$

Δ_2 = first row of $B_1^{-1} * a_2^{(1)} = 1 * -8 + 0 * 3 + 0 *5 = -8$

Δ_3 = first row of $B_1^{-1} * a_3^{(1)} = 1 * -7 + 0 * 3 + 0 *4 = -7$

Δ_4 = first row of $B_1^{-1} * a_4^{(1)} = 1 * -4 + 0 * 2 + 0 *2 = -4$

Δ_5 = first row of $B_1^{-1} * a_5^{(1)} = 1 * -6 + 0 * 2 + 0 *4 = -6$

$\Delta_2 = -8$ is most negative. So $a_2^{(1)}$, (x_2) is incoming vector.

Compute the column vector X_k

$X_k = B_1^{-1} * a_2^{(1)}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -8 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ 5 \end{bmatrix}$$

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	0	0	-8	-
s_1	0	1	0	20	3	20/3
s_2	0	0	1	30	$\boxed{5}$ ↑	6⟶

Improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	0	0	0	-8

$$\begin{array}{c|ccc|c} R_2 & 1 & 0 & 20 & 3 \\ R_3 & 0 & 1 & 30 & \boxed{5} \end{array}$$

$$\begin{array}{c|ccc|c} & \beta_1^{(1)} & \beta_2^{(1)} & X_B & X_k \\ \hline R_1 & 0 & 8/5 & 48 & 0 \\ R_2 & 1 & -3/5 & 2 & 0 \\ R_3 & 0 & 1/5 & 6 & 1 \end{array}$$

Revised simplex table for II iteration

Revised simplex table

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	8/5	48		
s_1	0	1	-3/5	2		
x_2	0	0	1/5	6		

Additional table

$a_1^{(1)}$	$a_7^{(1)}$	$a_3^{(1)}$	$a_4^{(1)}$	$a_5^{(1)}$
-5	0	-7	-4	-6
2	0	3	2	2
3	1	4	2	4

Computation of Δ_j for $a_1^{(1)}$, $a_2^{(1)}$, $a_3^{(1)}$, $a_4^{(1)}$, $a_5^{(1)}$

$$\Delta_1 = -1/5,\ \Delta_7 = 8/5,\ \Delta_3 = -3/5,\ \Delta_4 = -4/5,\ \Delta_5 = 2/5$$

$\Delta_4 = -4/5$ is most negative. So $a_4^{(1)}$, (x_4) is incoming vector.

Compute the column vector X_k

$X_k = B_1^{-1} * a_4^{(1)}$

$$\begin{bmatrix} 1 & 0 & 8/5 \\ 0 & 1 & -3/5 \\ 0 & 0 & 1/5 \end{bmatrix} * \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 4/5 \\ 2/5 \end{bmatrix}$$

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	0	8/5	48	-4/5	-
s_1	0	1	-3/5	2	$\boxed{4/5}$	10/4→
x_2	0	0	1/5	6	2/5	15
					↑	

Improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	0	8/5	48	-4/5
R_2	1	-3/5	2	$\boxed{4/5}$
R_3	0	1/5	6	2/5

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	1	1	50	0
R_2	5/4	-3/4	5/2	1
R_3	-1/2	1/2	5	0

Revised simplex table for III iteration

Revised simplex table

Basic variables	B_1^{-1}			X_B	X_k	X_B / X_k
	e_1 (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$			
Z	1	1	1	50		
x_4	0	5/4	-3/4	5/2		
x_2	0	-1/2	1/2	5		

Additional table

$a_1^{(1)}$	$a_7^{(1)}$	$a_3^{(1)}$	$a_6^{(1)}$	$a_5^{(1)}$
-5	0	-7	0	-6
2	0	3	1	2
3	1	4	0	4

Computation of Δ_j for $a_1^{(1)}$, $a_2^{(1)}$, $a_3^{(1)}$, $a_4^{(1)}$, $a_5^{(1)}$

$\Delta_1 = 0$, $\Delta_7 = 1$, $\Delta_3 = 0$, $\Delta_6 = 1$, $\Delta_5 = 0$

$\Delta_j \geq 0$, Therefore optimal solution is Max Z = 50, $x_1 = 0$, $x_2 = 5$, $x_3 = 0$, $x_4 = 5/2$, $x_5 = 0$

Exercise

Solve by Revised Simplex method

1. Max Z = $x_1 + x_2$
   Subject to
   $3x_1 + 3x_2 \leq 6$
   $x_1 + 4x_2 \leq 4$
   $x_1, x_2 \geq 0$
   [Ans. Max Z = 11/5, $x_1 = 8/5$, $x_2 = 3/5$]

2. Max Z = $x_1 + 2x_2$
   Subject to
   $x_1 + 2x_2 \leq 3$
   $x_1 + 3x_2 \leq 1$
   $x_1, x_2 \geq 0$
   [Ans. Max Z = 1, $x_1 = 1$, $x_2 = 0$]

3. Max Z = $5x_1 + 3x_2$
   Subject to
   $3x_1 + 5x_2 \leq 15$
   $3x_1 + 2x_2 \leq 10$
   $x_1, x_2 \geq 0$
   [Ans. Max Z = 285/19, $x_1 = 22/19$, $x_2 = 45/19$]

4. Max Z = $x_1 + x_2$
   Subject to
   $x_1 + 2x_2 \leq 2$
   $4x_1 + x_2 \leq 4$
   $x_1, x_2 \geq 0$
   [Ans. Max Z = 10/7, $x_1 = 6/7$, $x_2 = 4/7$]

5. Max Z = $3x_1 + x_2 + 2x_3 + 7x_4$
   Subject to
   $2x_1 + 3x_2 - x_3 + 4x_4 \leq 40$
   $-2x_1 + 2x_2 + 5x_3 - x_4 \leq 35$
   $x_1 + x_2 - 2x_3 + 3x_4 \leq 100$
   $x_1, x_2, x_3, x_4 \geq 0$

   [Ans. Max Z = 445/4, $x_1 = 71/4$, $x_2 = 1$, $x_3 = 29/2$, $x_4 = 4$]

Unit 2

2.1 Computational Procedure of Revised Simplex Table in Standard Form-II

2.2 Worked Examples

2.3 Advantages and Disadvantages

2.1 Computational Procedure of Revised Simplex Table in Standard Form-II

Phase I – When the artificial variables are present in the initial solution with positive values

Step 1 – First construct the simplex table in the following form

Variables in the basis	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	…	$\beta_m^{(2)}$	$X_B^{(2)}$	$X_k^{(2)}$
x_0	1	0	0	0	…	0		
x'_{n+1}	0	1	0	0	…	0		
x_{n+1}	0	0	1	0	…	0		
x_{n+2}	0	0	0	1	…	0		
.	.	.	.	.		.		
.	.	.	.	.		.		
x_{n+m}	0	0	0	0	…	1		

Step 2 – If $x'_{n+1} < 0$, compute Δ_j = second row of $B_2^{-1} * a_j^{(2)}$ and continue to step 3. If max $x'_{n+1} = 0$ then go to phase II.

Step 3 – To find the vector to be introduced into the basis

- If $\Delta_j \geq 0$, x'_{n+1} is at its maximum and hence no feasible solution exists for the problem
- If at least one $\Delta_j < 0$, the vector to be introduced in the basis, $X_k^{(2)}$, corresponds to such value of k which is obtained by $\Delta_k = \min \Delta_j$
- If more than one value of Δ_j are equal to the maximum, we select Δ_k such that k is the smallest index.

Step 4 – To compute $X_k^{(2)}$ by using the formula $X_k^{(2)} = B_2^{-1} a_k^{(2)}$

Step 5 – To find the vector to be removed from the basis.

The vector to be removed from the basis is obtained by using the minimum ratio rule.

Step 6 – After determining the incoming and outgoing vector, next revised simplex table can be easily obtained

Repeat the procedure of phase I to get max $x'_{n+1} = 0$ or all Δ_j for phase I are ≥ 0.

If max x'_{n+1} comes out of zero in phase I, all artificial variables must have the value zero. It should be noted carefully that max x'_{n+1} will always come out to be zero at the end of phase I if the feasible solution to the problem exists.

Proceed to phase II

Phase II - x'_{n+1} is considered like any other artificial variable; it can be removed from the basic solution. Only x_0 must always remain in the basic solution. However there will always be at least one artificial vector in B_2, otherwise it is not possible to have an m+2 dimensional bases. The procedure in phase II will be the same as described in standard form-I

2.1 Worked Examples

Solve by revised simplex method

Example 1

Min Z = $x_1 + 2x_2$

Subject to

$2x_1 + 5x_2 \geq 6$

$x_1 + x_2 \geq 2$

and $x_1, x_2 \geq 0$

Solution

SLPP

Min Z = Max Z' = $-x_1 - 2x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 + 5x_2 - s_1 + a_1 = 6$

$x_1 + x_2 - s_2 + a_2 = 2$

and $x_1, x_2, s_1, s_2 \geq 0$

Standard form-II

$Z' + x_1 + 2x_2 = 0$

$-3x_1 - 6x_2 + s_1 + s_2 + a_v = -8$ where $a_v = -(a_1 + a_2)$

$2x_1 + 5x_2 - s_1 + a_1 = 6$

$x_1 + x_2 - s_2 + a_2 = 2$

and $x_1, x_2, s_1, s_2 \geq 0$

The second constraint equation is formed by taking the negative sum of two constraints.

Matrix form

$$\begin{array}{c}\begin{array}{cccccccc} & & & & & & \beta_1^{(2)} & \beta_2^{(2)} \\ e_1 & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & e_2 & a_5^{(2)} & a_6^{(2)} \\ (z') & x_1 & x_2 & s_1 & s_2 & a_v & a_1 & a_2 \end{array} \\ \left[\begin{array}{cccccccc} 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -6 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]\end{array} \begin{bmatrix} z' \\ x_1 \\ x_2 \\ s_1 \\ s_2 \\ a_v \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \\ 6 \\ 2 \end{bmatrix}$$

Phase -I

I Iteration

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
e_1	1	0	0	0	0		
a_v	0	1	0	0	-8		
a_1	0	0	1	0	6		
a_2	0	0	0	1	2		

$a_1^{(2)}$	$a_2^{(2)}$	$a_3^{(2)}$	$a_4^{(2)}$
1	2	0	0
-3	-6	1	1
2	5	-1	0
1	1	0	-1

Calculation of Δ_j

Δ_1 = second row of $B_2^{-1} * a_1^{(2)} = -3$

Δ_2 = second row of $B_2^{-1} * a_2^{(2)} = -6$

Δ_3 = second row of $B_2^{-1} * a_3^{(2)} = 1$

Δ_4 = second row of $B_2^{-1} * a_4^{(2)} = 1$

Δ_2 is most negative. Therefore $a_2^{(2)}$ (x_2) is incoming vector

Compute the column vector X_k

$X_k = B_2^{-1} * a_2^{(2)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 \\ -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 5 \\ 1 \end{bmatrix}$$

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
e_1	1	0	0	0	0	2	
a_v	0	1	0	0	-8	-6	
a_1	0	0	1	0	6	$\boxed{5}$	6/5→
a_2	0	0	0	1	2	1	2
						↑	

Improved Solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	0	0	0	2
R_2	0	0	-8	-6
R_3	1	0	6	$\boxed{5}$
R_4	0	1	2	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	-2/5	0	-12/5	0
R_2	6/5	0	-4/5	0
R_3	1/5	0	6/5	1
R_4	-1/5	1	4/5	0

II iteration

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
z'	1	0	-2/5	0	-12/5		
a_v	0	1	6/5	0	-4/5		
x_2	0	0	1/5	0	6/5		
a_2	0	0	-1/5	1	4/5		

$a_1^{(2)}$	$a_5^{(2)}$	$a_3^{(2)}$	$a_4^{(2)}$
1	0	0	0
-3	0	1	1
2	1	-1	0
1	0	0	-1

Calculation of Δ_j

$\Delta_1 = -3/5$, $\Delta_5 = 6/5$, $\Delta_3 = -1/5$, $\Delta_4 = 1$

Δ_1 is most negative. Therefore $a_1^{(2)}$ (x_1) is incoming vector

Compute the column vector X_k

$X_k = B_2^{-1} * a_1^{(2)}$

$$\begin{bmatrix} 1 & 0 & -2/5 & 0 \\ 0 & 1 & 6/5 & 0 \\ 0 & 0 & 1/5 & 0 \\ 0 & 0 & -1/5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -3/5 \\ 2/5 \\ 3/5 \end{bmatrix}$$

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
z'	1	0	-2/5	0	-12/5	1/5	
a_v	0	1	6/5	0	-4/5	-3/5	
x_2	0	0	1/5	0	6/5	2/5	3
a_2	0	0	-1/5	1	4/5	$\boxed{3/5}$	4/3→
						↑	

Improved Solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	-2/5	0	-12/5	1/5
R_2	6/5	0	-4/5	-3/5
R_3	1/5	0	6/5	2/5
R_4	-1/5	1	4/5	$\boxed{3/5}$

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X_B	X_k
R_1	-1/3	-1/3	-8/3	0
R_2	1	1	0	0
R_3	1/3	-2/3	2/3	0
R_4	-1/3	5/3	4/3	1

III iteration

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
z'	1	0	-1/3	-1/3	-8/3		
a_v	0	1	1	1	0		
x_2	0	0	1/3	-2/3	2/3		
x_1	0	0	-1/3	5/3	4/3		

$a_6^{(2)}$	$a_5^{(2)}$	$a_3^{(2)}$	$a_4^{(2)}$
0	0	0	0
0	0	1	1
0	1	-1	0
1	0	0	-1

Since a_v =0 in X_B column. We proceed to phase II

Phase II

Basic variables	B_2^{-1}				X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$			
z'	1	0	-1/3	-1/3	-8/3		

$a_3^{(2)}$	$a_4^{(2)}$
0	0

a_v	0	1	1	1	0		
x_2	0	0	1/3	-2/3	2/3		
x_1	0	0	-1/3	5/3	4/3		

1	1
-1	0
0	-1

Δ_3 = first row of $B_2^{-1} * a_3^{(2)} = 1/3$

Δ_4 = first row of $B_2^{-1} * a_4^{(2)} = 1/3$

Δ_3 and Δ_4 are positive. Therefore optimal solution is $Z' = -8/3 \rightarrow Z = 8/3$, $x_1 = 4/3$, $x_2 = 2/3$

Example 2

Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to

$x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

and $x_1, x_2, x_3 \geq 0$

Solution

SLPP

Max $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to

$x_1 + 2x_2 + 3x_3 + a_1 = 15$

$2x_1 + x_2 + 5x_3 + a_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 + a_3 = 10$

and $x_1, x_2, a_1, a_2 \geq 0$

Standard form-II

$Z - x_1 - 2x_2 - 3x_3 + x_4 = 0$

$-4x_1 - 5x_2 - 9x_3 - x_4 + a_v = -45$ where $a_v = -(a_1 + a_2 + a_3)$

$x_1 + 2x_2 + 3x_3 + a_1 = 15$

$2x_1 + x_2 + 5x_3 + a_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 + a_3 = 10$

$x_1, x_2, a_1, a_2, a_3 \geq 0$

Matrix form

$$\begin{array}{ccccccccc} & & & & & & \beta_1^{(2)} & \beta_2^{(2)} & \beta_3^{(2)} \\ e_1 & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & e_2 & a_5^{(2)} & a_6^{(2)} & a_7^{(2)} \\ (z) & x_1 & x_2 & x_3 & x_4 & a_v & a_1 & a_2 & a_3 \\ 1 & -1 & -2 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -4 & -5 & -9 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 5 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_v \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -45 \\ 15 \\ 20 \\ 10 \end{bmatrix}$$

Phase I

I Iteration

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
e_1	1	0	0	0	0	0		
a_v	0	1	0	0	0	-45		
a_1	0	0	1	0	0	15		
a_2	0	0	0	1	0	20		
a_3	0	0	0	0	1	10		

$a_1^{(2)}$	$a_2^{(2)}$	$a_3^{(2)}$	$a_4^{(2)}$
-1	-2	-3	1
-4	-5	-9	-1
1	2	3	0
2	1	5	0
1	2	1	1

Calculation of Δ_j

Δ_1 = second row of $B_2^{-1} * a_1^{(2)} = -4$

Δ_2 = second row of $B_2^{-1} * a_2^{(2)} = -5$

Δ_3 = second row of $B_2^{-1} * a_3^{(2)} = -9$

Δ_4 = second row of $B_2^{-1} * a_4^{(2)} = -1$

Δ_3 is most negative. Therefore $a_3^{(2)}$ (x_3) is incoming vector

Compute the column vector X_k

$X_k = B_2^{-1} * a_3^{(2)}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} -3 \\ -9 \\ 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ 3 \\ 5 \\ 1 \end{bmatrix}$$

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
e_1	1	0	0	0	0	0	-3	
a_v	0	1	0	0	0	-45	-9	
a_1	0	0	1	0	0	15	3	5
a_2	0	0	0	1	0	20	$\boxed{5}$	4→
a_3	0	0	0	0	1	10	1	10
							↑	

Improved Solution

	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k
R_1	0	0	0	0	-3

R_2	0	0	0	-45	-9
R_3	1	0	0	15	3
R_4	0	1	0	20	$\boxed{5}$
R_5	0	0	1	10	1

	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k
R_1	0	3/5	0	12	0
R_2	0	9/5	0	-9	0
R_3	1	-3/5	0	3	0
R_4	0	1/5	0	4	1
R_5	0	-1/5	1	6	0

II Iteration

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
z	1	0	0	3/5	0	12		
a_v	0	1	0	9/5	0	-9		
a_1	0	0	1	-3/5	0	3		
x_3	0	0	0	1/5	0	4		
a_3	0	0	0	-1/5	1	6		

$a_1^{(2)}$	$a_2^{(2)}$	$a_6^{(2)}$	$a_4^{(2)}$
-1	-2	0	1
-4	-5	0	-1
1	2	0	0
2	1	1	0
1	2	0	1

Calculation of Δ_j

$\Delta_1 = -2/5$, $\Delta_2 = -16/5$, $\Delta_6 = 9/5$, $\Delta_4 = -1$

Δ_4 is most negative. Therefore $a_4^{(2)}$ (x_4) is incoming vector

Compute the column vector X_k

$$\begin{bmatrix} 1 & 0 & 0 & 3/5 & 0 \\ 0 & 1 & 0 & -9/5 & 0 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & -1/5 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
Z	1	0	0	3/5	0	12	1	
a_v	0	1	0	9/5	0	-9	-1	
a_1	0	0	1	-3/5	0	3	0	
x_3	0	0	0	1/5	0	4	0	
a_3	0	0	0	-1/5	1	6	$\boxed{1}$	6→
							↑	

Improved Solution

	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k
R_1	0	3/5	0	12	1
R_2	0	9/5	0	-9	-1
R_3	1	-3/5	0	3	0
R_4	0	1/5	0	4	0
R_5	0	-1/5	1	6	$\boxed{1}$

$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k

R_1	0	4/5	-1	6	0
R_2	0	8/5	1	-3	0
R_3	1	-3/5	0	3	0
R_4	0	1/5	0	4	0
R_5	0	-1/5	1	6	1

III Iteration

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
Z	1	0	0	4/5	-1	6		
a_v	0	1	0	8/5	1	-3		
a_1	0	0	1	-3/5	0	3		
x_3	0	0	0	1/5	0	4		
x_4	0	0	0	-1/5	1	6		

$a_1^{(2)}$	$a_2^{(2)}$	$a_6^{(2)}$	$a_7^{(2)}$
-1	-2	0	0
-4	-5	0	0
1	2	0	0
2	1	1	0
1	2	0	1

Calculation of Δ_j

$\Delta_1 = 1/5$, $\Delta_2 = -7/5$, $\Delta_6 = 8/5$, $\Delta_7 = 1$

Δ_2 is most negative. Therefore $a_2^{(2)}$ (x_2) is incoming vector

Compute the column vector X_k

$$\begin{bmatrix} 1 & 0 & 0 & 4/5 & -1 \\ 0 & 1 & 0 & 8/5 & 1 \\ 0 & 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 1/5 & 0 \\ 0 & 0 & 0 & -1/5 & 1 \end{bmatrix} * \begin{bmatrix} -2 \\ -5 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -16/5 \\ -7/5 \\ 7/5 \\ 1/5 \\ 9/5 \end{bmatrix}$$

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			

z	1	0	0	4/5	-1	6	-16/5	
a_v	0	1	0	8/5	1	-3	-7/5	
a_1	0	0	1	-3/5	0	3	$\boxed{7/5}$	15/7→
x_3	0	0	0	1/5	0	4	1/5	20
x_4	0	0	0	-1/5	1	6	9/5	30/9
							↑	

Improved Solution

	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k
R_1	0	4/5	-1	6	-16/5
R_2	0	8/5	1	-3	-7/5
R_3	1	-3/5	0	3	$\boxed{7/5}$
R_4	0	1/5	0	4	1/5
R_5	0	-1/5	1	6	9/5

	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k
R_1	16/7	4/7	-1	90/7	0
R_2	1	1	1	0	0
R_3	5/7	-3/7	0	15/7	1
R_4	-1/7	2/7	0	25/7	0
R_5	-9/7	4/7	1	15/7	0

IV Iteration

Basic	B_2^{-1}					$a_1^{(2)}$	$a_5^{(2)}$	$a_6^{(2)}$	$a_7^{(2)}$

variables	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$	X_B	X_k	X_B/X_k
z	1	0	16/7	4/7	-1	90/7		
a_v	0	1	1	1	1	0		
x_2	0	0	5/7	-3/7	0	15/7		
x_3	0	0	-1/7	2/7	0	25/7		
x_4	0	0	-9/7	4/7	1	15/7		

-1	0	0	0
-4	0	0	0
1	1	0	0
2	0	1	0
1	0	0	1

Since a_v =0 in X_B column. We proceed to phase II

Phase II

Basic variables	B_2^{-1}					X_B	X_k	X_B/X_k
	e_1	e_2	$\beta_1^{(2)}$	$\beta_2^{(2)}$	$\beta_3^{(2)}$			
z	1	0	16/7	4/7	-1	90/7		
a_v	0	1	1	1	1	0		
x_2	0	0	5/7	-3/7	0	15/7		
x_3	0	0	-1/7	2/7	0	25/7		
x_4	0	0	-9/7	4/7	1	15/7		

$a_1^{(2)}$
-1
-4
1
2
1

$\Delta_1 = 0$, Δ_1 is positive. Therefore optimal solution is Z =90/7, x_1= 0, x_2 = 15/7, x_3 = 25/7, x_4 = 15/7

2.3 Advantages and Disadvantages

Advantages

- The method automatically generates the inverse of the current basis matrix and the new basic feasible solution as well.
- It provides more information at lesser computational effort
- It requires lesser computations than the ordinary simplex method
- A less number of entries are needed in each table of revised simplex table
- The control of rounding-off-errors occurs when a digital computer is used

Disadvantages

In solving the numerical problems side computations are also requires, therefore more computational mistakes may occur in comparison to original simplex method.

Exercise

Solve by revised simplex method

1. Max $Z = 3x_1 + 5x_2$

   Subject to

   $x_1 \leq 4$

   $x_2 \leq 6$

   $3x_1 + 2x_2 \leq 18$

   $x_{1,} x_2 \geq 0$

   [Ans. Max $Z = 36$, $x_1 = 2$, $x_2 = 6$]

2. Max $Z = 5x_1 + 3x_2$

   Subject to

   $4x_1 + 5x_2 \geq 10$

   $5x_1 + 2x_2 \leq 10$

   $3x_1 + 8x_2 \leq 12$

   $x_{1,} x_2 \geq 0$

   [Ans. Max $Z = 185/17$, $x_1 = 28/17$, $x_2 = 15/17$]

3. Max $Z = x_1 + x_2 + 3x_3$

   Subject to

   $3x_1 + 2x_2 + x_3 \leq 3$

   $2x_1 + x_2 + 2x_3 \leq 2$

   $x_{1,} x_{2,} x_3 \geq 0$

[Ans. Max Z = 3, $x_1 = 0$, $x_2 = 0$, $x_3 = 1$]

4. Min Z = $3x_1 + x_2$

   Subject to

   $x_1 + x_2 \geq 1$

   $2x_1 + x_2 \geq 0$

   $x_1, x_2 \geq 0$

   [Ans. Min Z = 1, $x_1 = 0$, $x_2 = 1$]

5. Min Z = $4x_1 + 2x_2 + 3x_3$

   Subject to

   $2x_1 + 4x_3 \geq 5$

   $2x_1 + 4x_2 + x_3 \geq 4$

   $x_1, x_2, x_3 \geq 0$

   [Ans. Min Z = 67/12, $x_1 = 0$, $x_2 = 11/12$, $x_3 = 5/4$]

Unit 3

3.1 Duality in LPP

Every LPP called the **primal** is associated with another LPP called **dual**. Either of the problems is primal with the other one as dual. The optimal solution of either problem reveals the information about the optimal solution of the other.

Let the primal problem be

Max $Z_x = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

.

.

.

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_n$

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

The corresponding dual is defined as

Min $Z_w = b_1w_1 + b_2w_2 + \ldots + b_mw_m$

Subject to restrictions

$a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m \geq c_1$

$a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m \geq c_2$

.

.

.

$a_{1n}w_1 + a_{2n}w_2 + \ldots\ldots\ldots + a_{mn}w_m \geq c_n$

and

$w_1, w_2, \ldots, w_m \geq 0$

Matrix Notation

Primal

Max $Z_x = CX$

Subject to

$AX \leq b$ and $X \geq 0$

Dual

Min $Z_w = b^T W$

Subject to

$A^T W \geq C^T$ and $W \geq 0$

3.2 Important characteristics of Duality

1. Dual of dual is primal
2. If either the primal or dual problem has a solution then the other also has a solution and their optimum values are equal.
3. If any of the two problems has an infeasible solution, then the value of the objective function of the other is unbounded.
4. The value of the objective function for any feasible solution of the primal is less than the value of the objective function for any feasible solution of the dual.
5. If either the primal or dual has an unbounded solution, then the solution to the other problem is infeasible.
6. If the primal has a feasible solution, but the dual does not have then the primal will not have a finite optimum solution and vice versa.

3.3 Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly when the number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision in the activities being programmed.
3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling the managers to take further decisions.

3.4 Steps for a Standard Primal Form

Step 1 – Change the objective function to Maximization form

Step 2 – If the constraints have an inequality sign '≥' then multiply both sides by -1 and convert the inequality sign to '≤'.

Step 3 – If the constraint has an '=' sign then replace it by two constraints involving the inequalities going in opposite directions. For example $x_1+ 2x_2 = 4$ is written as

$x_1+2x_2 \leq 4$

$x_1+2x_2 \geq 4$ (using step2) → $- x_1-2x_2 \leq - 4$

Step 4 – Every unrestricted variable is replaced by the difference of two non-negative variables.

Step5 – We get the standard primal form of the given LPP in which.

- All constraints have '$\leq$' sign, where the objective function is of maximization form.
- All constraints have '$\geq$' sign, where the objective function is of minimization from.

3.5 Rules for Converting any Primal into its Dual

1. Transpose the rows and columns of the constraint co-efficient.
2. Transpose the co-efficient ($c_1, c_2, \ldots c_n$) of the objective function and the right side constants ($b_1, b_2, \ldots b_n$)
3. Change the inequalities from '$\leq$' to '$\geq$' sign.
4. Minimize the objective function instead of maximizing it.

3.6 Example Problems

Write the dual of the given problems

Example 1

Min $Z_x = 2x_2 + 5x_3$

Subject to

$x_1+x_2 \geq 2$

$2x_1+x_2+6x_3 \leq 6$

$x_1 - x_2 +3x_3 = 4$

$x_1, x_2, x_3 \geq 0$

Solution

Primal

$\text{Max } Z_x' = -2x_2 - 5x_3$

Subject to

$-x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 \leq 4$

$-x_1 + x_2 - 3x_3 \leq -4$

$x_1, x_2, x_3 \geq 0$

Dual

$\text{Min } Z_w = -2w_1 + 6w_2 + 4w_3 - 4w_4$

Subject to

$-w_1 + 2w_2 + w_3 - w_4 \geq 0$

$-w_1 + w_2 - w_3 + w_4 \geq -2$

$6w_2 + 3w_3 - 3w_4 \geq -5$

$w_1, w_2, w_3, w_4 \geq 0$

Example 2

$\text{Min } Z_x = 3x_1 - 2x_2 + 4x_3$

Subject to

$3x_1 + 5x_2 + 4x_3 \geq 7$

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 \geq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$

Solution

Primal

$\text{Max } Z_x' = -3x_1 + 2x_2 - 4x_3$

Subject to

$-3x_1 - 5x_2 - 4x_3 \leq -7$

$-6x_1 - x_2 - 3x_3 \leq -4$

$-7x_1 + 2x_2 + x_3 \leq - 10$

$-x_1 + 2x_2 - 5x_3 \leq - 3$

$-4x_1 - 7x_2 + 2x_3 \leq - 2$

$x_1, x_2 , x_3 \geq 0$

Dual

$\text{Min } Z_w = -7w_1 - 4w_2 - 10w_3 – 3w_4 -2w_5$

Subject to

$-3w_1 - 6w_2 - 7w_3 –w_4 – 4w_5 \geq -3$

$-5w_1 - w_2 + 2w_3 + 2w_4 – 7w_5 \geq 2$

$-4w_1 - 3w_2 + w_3 - 5w_4 + 2w_5 \geq -4$

$w_1, w_2, w_3, w_4, w_5 \geq 0$

Example 3

$\text{Max } Z = 2x_1+ 3x_2 + x_3$

Subject to

$4x_1+ 3x_2 + x_3 = 6$

$x_1+ 2x_2 + 5x_3 = 4$

$x_1, x_2 \geq 0$

Solution

Primal

Max $Z_x = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq -4$

$x_1, x_2 \geq 0$

Dual

Min $Z_w = 6w_1 - 6w_2 + 4w_3 - 4w_4$

Subject to

$4w_1 - 4w_2 + w_3 - w_4 \geq 2$

$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$

$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$

$w_1, w_2, w_3, w_4 \geq 0$

Example 4

Min $Z_x = x_1 + x_2 + x_3$

Subject to

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$,x_3 is unrestricted in sign

Solution

Primal

Max $Z' = -x_1 - x_2 - x_3' + x_3''$

Subject to

$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$

$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$

$x_1 - 2x_2 \leq 3$

$-2x_2 + x_3' - x_3'' \leq -4$

$x_1, x_2, x_3', x_3'' \geq 0$

Dual

Min $Z_w = 5w_1 - 5w_2 + 3w_3 - 4w_4$

Subject to

$w_1 - w_2 + w_3 \geq -1$

$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$

$4w_1 - 4w_2 + w_4 \geq -1$

$-4w_1 + 4w_2 - w_4 \geq 1$

$w_1, w_2, w_3, w_4, \geq 0$

Example 5

Max $Z = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Solution

Primal

Max $Z_x = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Dual

Min $Z_w = 10w_1 + 2w_2 + 6w_3$

Subject to

$w_1 + 2w_2 + 2w_3 \geq 1$

$w_1 - 2w_3 \geq -1$

$w_1 - w_2 + 3w_3 \geq 3$

$w_1, w_2, w_3 \geq 0$

3.7 Primal –Dual Relationship

Weak duality property

If x is any feasible solution to the primal problem and w is any feasible solution to the dual problem then $CX \leq b^T W$. i.e. $Z_X \leq Z_W$

Strong duality property

If x* is an optimal solution for the primal problem and w* is the optimal solution for the dual problem then $CX^* = b^T W^*$ i.e. $Z_X = Z_W$

Complementary optimal solutions property

At the final iteration, the simplex method simultaneously identifies an optimal solution x* for primal problem and a complementary optimal solution w* for the dual problem where $Z_X = Z_{W.}$

Symmetry property

For any primal problem and its dual problem, all relationships between them must be symmetric because dual of dual is primal.

Fundamental duality theorem

- If one problem has feasible solution and a bounded objective function (optimal solution) then the other problem has a finite optimal solution.
- If one problem has feasible solution and an unbounded optimal solution then the other problem has no feasible solution
- If one problem has no feasible solution then the other problem has either no feasible solution or an unbounded solution.

If kth constraint of primal is equality then the dual variable w_k is unrestricted in sign

If pth variable of primal is unrestricted in sign then pth constraint of dual is an equality.

Complementary basic solutions property

Each basic solution in the primal problem has a complementary basic solution in the dual problem where $Z_X = Z_W$.

Complementary slackness property

The variables in the primal basic solution and the complementary dual basic solution satisfy the complementary slackness relationship as shown in the table.

Primal variable	**Associated dual variable**
Decision variable (x_j)	$Z_j - C_j$ (surplus variable) j = 1, 2, ..n
Slack variable (S_i)	W_i (decision variable) i = 1, 2, .. n

3.8 Duality and Simplex Method

1. Solve the given primal problem using simplex method. Hence write the solution of its dual

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7$

$x_1 \geq 0, x_2 \geq 0$

Solution

Primal form

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7$

$x_1 \geq 0, x_2 \geq 0$

SLPP

Max $Z = 30x_1 + 23x_2 + 29x_3 + 0s_1 + 0s_2$

Subject to

$6x_1 + 5x_2 + 3x_3 + s_1 = 26$

$4x_1 + 2x_2 + 6x_3 + s_2 = 7$

$x_1, x_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	30	23	29	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min Ratio X_B / X_K
s_1	0	26	6	5	3	1	0	26/6
s_2	0	7	$\boxed{4}$	2	6	0	1	$7/4 \rightarrow$
			↑					
	Z = 0		-30	-23	-29	0	0	$\leftarrow \Delta_j$
s_1	0	31/2	0	2	-6	1	-3/2	31/4
x_1	30	7/4	1	$\boxed{1/2}$	3/2	0	1/4	$7/2 \rightarrow$
				↑				
	Z = 105/2		0	-8	16	0	15/2	$\leftarrow \Delta_j$

s_1	0	17/2	-4	0	-12	1	-5/2	
x_2	23	7/2	2	1	3	0	1/2	
	Z =161/2		16	0	40	0	23/2	$\leftarrow \Delta_j$

$\Delta_j \geq 0$ so the optimal solution is Z = 161/2, $x_1 = 0$, $x_2 = 7/2$, $x_3 = 0$.

The optimal solution to the dual of the above problem will be

$Z_w^* = 161/2$, $w_1 = \Delta_4 = 0$, $w_2 = \Delta_5 = 23/2$

In this way we can find the solution to the dual without actually solving it.

2. Use duality to solve the given problem. Also read the solution of its primal.

Min $Z = 3x_1 + x_2$
Subject to
$x_1 + x_2 \geq 1$
$2x_1 + 3x_2 \geq 2$
$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max $Z' = -3x_1 - x_2$

Subject to

$- x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq - 2$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

Min $Z_w = -w_1 - 2w_2$

Subject to

$-w_1 - 2w_2 \geq -3$

$-w_1 - 3w_2 \geq -1$

$w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w' = w_1 + 2w_2 + 0s_1 + 0s_2$

Subject to

$w_1 + 2w_2 + s_1 = 3$

$w_1 + 3w_2 + s_2 = 1$

$w_1, w_2, s_1, s_2 \geq 0$

		$C_j \rightarrow$	1	2	0	0	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	Min Ratio W_B / W_K
s_1	0	3	1	2	1	0	3/2
s_2	0	1	1	$\boxed{3}$	0	1	1/3←
	$Z_w' = 0$		-1	↑ -2	0	0	$\leftarrow \Delta_j$
s_1	0	7/3	1/3	0	1	-2/3	7
w_2	2	1/3	$\boxed{1/3}$	1	0	1/3	1→
	$Z_w' = 2/3$		↑ -1/3	0	0	2/3	$\leftarrow \Delta_j$
s_1	0	2	0	-1	1	-1	
w_1	1	1	1	3	0	1	
	$Z_w' = 1$		0	1	0	1	$\leftarrow \Delta_j$

$\Delta_j \geq 0$ so the optimal solution is $Z_w' = 1$, $w_1 = 1$, $w_2 = 0$

The optimal solution to the primal of the above problem will be

$Z_x^* = 1$, $x_1 = \Delta_3 = 0$, $x_2 = \Delta_4 = 1$

3. Write down the dual of the problem and solve it.

Max $Z = 4x_1 + 2x_2$

Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$

$x_1 \geq 0, x_2 \geq 0$

Solution

Primal

Max $Z = 4x_1 + 2x_2$

Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$

$x_1 \geq 0, x_2 \geq 0$

Dual

Min $Z_w = -3w_1 - 2w_2$

Subject to

$-w_1 - w_2 \geq 4$

$-w_1 + w_2 \geq 2$

$w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w{}' = 3w_1 + 2w_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

Subject to

$-w_1 - w_2 - s_1 + a_1 = 4$

$-w_1 + w_2 - s_2 + a_2 = 2$

$w_{1,} w_2, s_1, s_{2,} a_1, a_2 \geq 0$

$C_j \rightarrow$	3	2	0	0	-M	-M

Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	A_2	Min Ratio W_B / W_K
a_1	-M	4	-1	-1	-1	0	1	0	-
a_2	-M	2	-1	$\boxed{1}$	0	-1	0	1	2→
	$Z_w' = -6M$		2M - 3	↑ -2	M	M	0	0	←Δ_j
a_1	-M	6	-2	0	-1	-1	1	X	
w_2	2	2	-1	1	0	-1	0	X	
	$Z_w' = -6M+4$		2M-5	0	M	M-2	0	X	←Δ_j

$\Delta_j \geq 0$ and at the positive level an artificial vector (a_1) appears in the basis. Therefore the dual problem does not posses any optimal solution. Consequently there exists no finite optimum solution to the given problem.

4. Use duality to solve the given problem.

Min $Z = x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max $Z' = -x_1 + x_2$

Subject to

$- 2x_1 - x_2 \leq -2$

$x_1 + x_2 \leq -1$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

Min $Z_w = -2w_1 - w_2$

Subject to

$-2w_1 + w_2 \geq -1$

$-w_1 + w_2 \geq 1$

$w_1, w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w' = 2w_1 + w_2 + 0s_1 + 0s_2 - Ma_1$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_1, w_2, s_1, s_2 \geq 0$

Auxiliary LPP

Max $Z_w' = 0w_1 + 0w_2 + 0s_1 + 0s_2 - 1a_1$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_1, w_2, s_1, s_2, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	-1	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	1	2	-1	1	0	0	-
a_1	-1	1	-1	$\boxed{1}$	0	-1	1	1→
				↑				
	$Z_w' = -1$		1	-1	0	1	0	←Δ_j

s_1	0	2	1	0	1	-1	x	
w_2	0	1	-1	1	0	-1	x	
	$Z_w' = 0$		0	0	0	0	x	←Δ_j

$\Delta_j \geq 0$ and no artificial vector appear at the positive level of the basis. Hence proceed to phase II

Phase II

		C_j→	2	1	0	0	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	2	$\boxed{1}$	0	1	-1	2→
w_2	1	1	-1	1	0	-1	-
	$Z_w' = 1$		↑ -3	0	0	-1	←Δ_j
w_1	2	2	1	0	1	-1	-
w_2	1	3	0	1	1	-2	-
	$Z_w' = 7$		0	0	3	↑ -4	←Δ_j

$\Delta_j = -4$ and all the elements of s_2 are negative; hence we cannot find the outgoing vector. This indicates there is an unbounded solution. Consequently by duality theorem the original primal problem will have no feasible solution.

5. Solve the given primal problem using simplex method. Hence write the solution of its dual

Max $Z = 40x_1 + 50x_2$
Subject to
$2x_1 + 3x_2 \leq 3$
$8x_1 + 4x_2 \leq 5$
$x_1 \geq 0, x_2 \geq 0$

Solution

Primal form

Max $Z = 40x_1 + 50x_2$
Subject to
$2x_1 + 3x_2 \leq 3$
$8x_1 + 4x_2 \leq 5$
$x_1 \geq 0, x_2 \geq 0$

SLPP

Max $Z_x = 40x_1 + 50x_2 + 0s_1 + 0s_2$

Subject to

$2x_1 + 3x_2 + s_1 = 3$

$8x_1 + 4x_2 + s_2 = 5$

$x_1, x_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$		40	50	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	3	2	$\boxed{3}$	1	0	1→
s_2	0	5	8	4	0	1	5/4
	$Z_x = 0$		-40	↑ -50	0	0	$\leftarrow \Delta_j$
x_2	50	1	2/3	1	1/3	0	3/2
s_2	0	1	$\boxed{16/3}$	0	-4/3	1	3/16→
	$Z_x = 50$		↑ -20/3	0	50/3	0	$\leftarrow \Delta_j$

x_2	50	7/8	0	1	1/2	-1/8	
x_1	40	3/16	1	0	-1/4	3/16	
	$Z_x = 205/4$		0	0	15	5/4	$\leftarrow\Delta_j$

$\Delta_j \geq 0$ so the optimal solution is $Z = 205/4$, $x_1 = 3/16$, $x_2 = 7/8$

The optimal solution to the dual of the above problem will be

$Z_w^* = 205/4$, $w_1 = \Delta_4 = 15$, $w_2 = \Delta_5 = 5/4$

Exercise

1. Explain the concept of duality in LPP.
2. Explain the characteristics of duality.
3. Mention the advantages and application of duality
4. Write the steps for converting LPP into its dual.
5. Explain the concept of primal- dual relationship

Obtain the dual of the following linear programming problems

1. Max $Z = 3x_1 + 4x_2$

   Subject to

   $2x_1 + 6x_2 \leq 16$

   $5x_1 + 2x_2 \geq 20$

   $x_1 \geq 0, x_2 \geq 0$

2. Min $Z = 7x_1 + 3x_2 + 8x_3$

   Subject to

$8x_1 + 2x_2 + x_3 \geq 3$

$3x_1 + 6x_2 + 4x_3 \geq 4$

$4x_1 + x_2 + 5x_3 \geq 1$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

3. Max $Z = 6x_1 + 4x_2 + 6x_3 + x_4$

   Subject to

   $4x_1 + 4x_2 + 4x_3 + 8x_4 = 21$

   $3x_1 + 17x_2 + 80x_3 + 2x_4 \leq 48$

   $x_1 \geq 0, x_2 \geq 0, x_3, x_4$ are unrestricted

Use duality to solve the following LPP

1. Max $Z = 3x_1 + 2x_2$

   Subject to

   $2x_1 + x_2 \leq 5$

   $x_1 + x_2 \leq 3$

   $x_1 \geq 0, x_2 \geq 0$

[Ans. Max $Z = 8$, $x_1 = 2$, $x_2 = 1$, $w_1 = 1$, $w_2 = 2$, Min $Z_w = 8$]

2. Min $Z = 2x_1 + 2x_2$

   Subject to

   $2x_1 + 4x_2 \geq 1$

   $x_1 + 2x_2 \geq 1$

   $2x_1 + x_2 \geq 1$

   $x_1 \geq 0, x_2 \geq 0$

[Ans. Max $Z = 4/3$, $x_1 = 1/3$, $x_2 = 1/3$]

3. Max $Z = 8x_1 + 6x_2$

   Subject to

   $x_1 - x_2 \leq 3/5$

   $x_1 - x_2 \geq 2$

   $x_1 \geq 0, x_2 \geq 0$

   [Ans. Dual problem does not possess feasible solution]

Module 4

Unit 1

1.1 Introduction

Any LPP for which it is possible to find infeasible but better than optimal initial basic solution can be solved by using dual simplex method. Such a situation can be recognized by first expressing the constraints in '≤' form and the objective function in the maximization form. After adding slack variables, if any right hand side element is negative and the optimality condition is satisfied then the problem can be solved by dual simplex method.

Negative element on the right hand side suggests that the corresponding slack variable is negative. This means that the problem starts with optimal but infeasible basic solution and we proceed towards its feasibility.

The dual simplex method is similar to the standard simplex method except that in the latter the starting initial basic solution is feasible but not optimum while in the former it is infeasible but optimum or better than optimum. The dual simplex method works towards feasibility while simplex method works towards optimality.

1.2 Computational Procedure of Dual Simplex Method

The iterative procedure is as follows

Step 1 - First convert the minimization LPP into maximization form, if it is given in the minimization form.

Step 2 - Convert the '≥' type inequalities of given LPP, if any, into those of '≤' type by multiplying the corresponding constraints by -1.

Step 3 – Introduce slack variables in the constraints of the given problem and obtain an initial basic solution.

Step 4 – Test the nature of Δ_j in the starting table

- If all Δ_j and X_B are non-negative, then an optimum basic feasible solution has been attained.
- If all Δ_j are non-negative and at least one basic variable X_B is negative, then go to step 5.
- If at least Δ_j one is negative, the method is not appropriate.

Step 5 – Select the most negative X_B. The corresponding basis vector then leaves the basis set B. Let X_r be the most negative basic variable.

Step 6 – Test the nature of X_r

- If all X_r are non-negative, then there does not exist any feasible solution to the given problem.
- If at least one X_r is negative, then compute Max (Δ_j / X_r) and determine the least negative for incoming vector.

Step 7 – Test the new iterated dual simplex table for optimality.

Repeat the entire procedure until either an optimum feasible solution has been attained in a finite number of steps.

1.3 Worked Examples

Example 1

Minimize $Z = 2x_1 + x_2$

Subject to

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Step 1 – Rewrite the given problem in the form

Maximize $Z' = -2x_1 - x_2$

Subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 \;\; - 2x_2 \leq -3$

$x_1, x_2 \geq 0$

Step 2 – Adding slack variables to each constraint

Maximize $Z' = -2x_1 - x_2$

Subject to

$-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_2 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

$x_1, x_{2,} s_1, s_2, s_3 \geq 0$

Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-3	-3	-1	1	0	0	
s_2	0	-6	-4	$\boxed{-3}$	0	1	0	→ outgoing
s_3	0	-3	-1	-2	0	0	1	
	$Z' = 0$		2	↑ 1	0	0	0	←Δ_j

Step 4 – To find the leaving vector

Min (-3, -6, -3) = -6. Hence s_2 is outgoing vector

Step 5 – To find the incoming vector

Max $(\Delta_1 / x_{21}, \Delta_2 / x_{22}) = (2/-4, 1/-3) = -1/3$. So x_2 is incoming vector

Step 6 –The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-1	$\boxed{-5/3}$	0	1	-1/3	0	→ outgoing
x_2	-1	2	4/3	1	0	-1/3	0	
s_3	0	1	5/3	0	0	-2/3	1	
	$Z' = -2$		↑ 2/3	0	0	1/3	0	←Δ_j

Step 7 – To find the leaving vector

Min (-1, 2, 1) = -1. Hence s_1 is outgoing vector

Step 8 – To find the incoming vector

Max (Δ_1 / x_{11}, Δ_4 / x_{14}) = (-2/5, -1) = -2/5. So x_1 is incoming vector

Step 9 –The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x_1	-2	3/5	1	0	-3/5	1/5	0	
x_2	-1	6/5	0	1	4/5	-3/5	0	
s_3	0	0	0	0	1	-1	1	
	$Z' = -12/5$		0	0	2/5	1/5	0	←Δ_j

Step 10 – $\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max $Z' = -12/5$, $Z = 12/5$, and $x_1=3/5$, $x_2 = 6/5$

Example 2

Minimize $Z = 3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1 \geq 0, x_2 \geq 0$

Solution

Maximize $Z' = -3x_1 - x_2$

Subject to

$-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

$x_1, x_2 \geq 0$

SLPP

Maximize $Z' = -3x_1 - x_2$

Subject to

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	
s_1	0	-1	-1	-1	1	0	
s_2	0	-2	-2	$\boxed{-3}$	0	1	→
	$Z' = 0$		3	↑ 1	0	0	$\leftarrow \Delta_j$

s_1	0	-1/3	-1/3	0	1	$\boxed{-1/3}$	→
x_2	-1	2/3	2/3	1	0	-1/3	
	$Z' = -2/3$		7/3	0	0	↑ 1/3	←Δ_j
s_2	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	$Z' = -1$		2	0	1	0	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max $Z' = -1$, $Z = 1$, and $x_1 = 0$, $x_2 = 1$

Example 3

Max $Z = -2x_1 - x_3$

Subject to

$x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 \leq -5$

$-x_1 + 2x_2 - 4x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

SLPP

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 + s_1 = -5$

$-x_1 + 2x_2 - 4x_3 + s_2 = -8$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-2	0	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	
s_1	0	-5	-1	-1	1	1	0	
s_2	0	-8	-1	2	$\boxed{-4}$	0	1	→
					↑			
	Z = 0		2	0	1	0	0	←Δ_j
s_1	0	-7	-5/4	$\boxed{-1/2}$	0	1	1/4	→
x_3	-1	2	1/4	-1/2	1	0	-1/4	
				↑				
	Z = -2		7/4	1/2	0	0	1/4	←Δ_j
x_2	0	14	5/2	1	0	-2	-1/2	
x_3	-1	9	3/2	0	1	-1	-1/2	
	Z = -9		1/2	0	0	1	1/2	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Z = -9, and $x_1 = 0$, $x_2 = 14$, $x_3 = 9$

Example 4

Find the optimum solution of the given problem without using artificial variable.

Max $Z = -4x_1 - 6x_2 - 18x_3$

Subject to

$x_1 + 3x_3 \geq 3$

$x_2 + 2x_3 \geq 5$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Solution

Max $Z = -4x_1 - 6x_2 - 18x_3$

Subject to

$-x_1 - 3x_3 \leq -3$

$-x_2 - 2x_3 \leq -5$

$x_1, x_2, x_3 \geq 0$

SLPP

Max $Z = -4x_1 - 6x_2 - 18x_3$

Subject to

$-x_1 - 3x_3 + s_1 = -3$

$-x_2 - 2x_3 + s_2 = -5$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-4	-6	-18	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	
s_1	0	-3	-1	0	-3	1	0	
s_2	0	-5	0	$\boxed{-1}$	-2	0	1	→
	Z = 0		4	↑ 6	18	0	0	←Δ_j
s_1	0	-3	-1	0	$\boxed{-3}$	1	0	→
x_2	-6	5	0	1	2	0	-1	
	Z = -30		4	0	↑ 6	0	6	←Δ_j
x_3	-18	1	1/3	0	1	-1/3	0	
x_2	-6	3	-2/3	1	0	2/3	-1	
	Z = -36		2	0	0	2	6	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Z = -36, and $x_1 = 0$, $x_2 = 3$, $x_3 = 1$

Example 5

Min $Z = 6x_1 + 7x_2 + 3x_3 + 5x_4$

Subject to

$5x_1 + 6x_2 - 3x_3 + 4x_4 \geq 12$

$x_2 + 5x_3 - 6x_4 \geq 10$

$2x_1 + 5x_2 + x_3 + x_4 \geq 8$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

Solution

Max $Z' = -6x_1 - 7x_2 - 3x_3 - 5x_4$

Subject to

$-5x_1 - 6x_2 + 3x_3 - 4x_4 \leq -12$

$-x_2 - 5x_3 + 6x_4 \leq -10$

$-2x_1 - 5x_2 - x_3 - x_4 \leq -8$

$x_1, x_2, x_3, x_4 \geq 0$

SLPP

Max $Z' = -6x_1 - 7x_2 - 3x_3 - 5x_4$

Subject to

$-5x_1 - 6x_2 + 3x_3 - 4x_4 + s_1 = -12$

$-x_2 - 5x_3 + 6x_4 + s_2 = -10$

$-2x_1 - 5x_2 - x_3 - x_4 + s_3 = -8$

$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$

	$C_j \rightarrow$		-6	-7	-3	-5	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	X_4	S_1	S_2	S_3	
s_1	0	-12	-5	$\boxed{-6}$	3	-4	1	0	0	→
s_2	0	-10	0	-1	-5	6	0	1	0	

s_3	0	-8	-2	-5	-1	1	0	0	1	
	Z'= 0		6	↑ 7	3	5	0	0	0	
x_2	-7	2	5/6	1	-1/2	2/3	-1/6	0	0	
s_2	0	-8	5/6	0	$\boxed{-11/2}$	20/3	-1/6	1	0	→
s_3	0	2	13/6	0	-7/2	7/3	-5/6	0	1	
	Z'= -14		1/6	0	↑ 13/2	1/3	7/6	0	0	
x_2	-7	30/11	25/33	1	0	2/33	-5/33	-1/11	0	
x_3	-3	16/11	-5/33	0	1	-40/33	1/33	-2/11	0	
s_3	0	78/11	18/11	0	0	-21/11	-8/11	-7/11	1	
	Z'= -258/11		38/33	0	0	271/33	32/33	13/11	0	

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Z= 258/11 and x_1= 0, x_2 = 30/11, x_3 = 16/11, x_4= 0

1.4 Advantage of Dual Simplex over Simplex Method

The main advantage of dual simplex over the usual simplex method is that we do not require any **artificial variables** in the dual simplex method. Hence a lot of labor is saved whenever this method is applicable.

1.5 Introduction to Transportation Problem

The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

It can also be defined as to ship goods from various origins to various destinations in such a manner that the transportation cost is a minimum.

The availability as well as the requirements is finite. It is assumed that the cost of shipping is linear.

1.6 Mathematical Formulation

Let there be m origins, i^{th} origin possessing a_i units of a certain product

Let there be n destinations, with destination j requiring b_j units of a certain product

Let c_{ij} be the cost of shipping one unit from i^{th} source to j^{th} destination

Let x_{ij} be the amount to be shipped from i^{th} source to j^{th} destination

It is assumed that the total availabilities Σa_i satisfy the total requirements Σb_j i.e.

$\Sigma a_i = \Sigma b_j$ ($i = 1, 2, 3 \ldots m$ and $j = 1, 2, 3 \ldots n$)

The problem now, is to determine non-negative of x_{ij} satisfying both the availability constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, .., m$$

as well as requirement constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{for } j = 1, 2, .., n$$

and the minimizing cost of transportation (shipping)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij} \qquad \text{(objective function)}$$

This special type of LPP is called as **Transportation Problem**.

1.7 Tabular Representation

Let ‘m’ denote number of factories ($F_1, F_2 \ldots F_m$)

Let ‘n’ denote number of warehouse ($W_1, W_2 \ldots W_n$)

W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	c_{11}	c_{12}	…	c_{1n}	a_1
F_2	c_{21}	c_{22}	…	c_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	c_{m1}	c_{m2}	…	c_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$
W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	x_{11}	x_{12}	…	x_{1n}	a_1
F_2	x_{21}	x_{22}	…	x_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	x_{m1}	x_{m2}	…	x_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$

In general these two tables are combined by inserting each unit cost c_{ij} with the corresponding amount x_{ij} in the cell (i, j). The product c_{ij} x_{ij} gives the net cost of shipping units from the factory F_i to warehouse W_j.

1.8 Some Basic Definitions

- **Feasible Solution**

  A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called as feasible solution.

- **Basic Feasible Solution**

  A feasible solution to ‘m’ origin, ‘n’ destination problem is said to be basic if the number of positive allocations are m+n-1. If the number of allocations is less than m+n-1 then it is called as **Degenerate Basic Feasible Solution**. Otherwise it is called as Non- Degenerate Basic Feasible Solution.

- **Optimum Solution**

  A feasible solution is said to be optimal if it minimizes the total transportation cost.

Exercise

Solve by dual simplex method

1. Max Z = $-3x_1 - 2x_2$

   Subject to

   $x_1 + x_2 \geq 1$

   $x_1 + x_2 \leq 7$

$x_1 + 2x_2 \leq 10$

$x_2 \leq 3$

and $x_1 \geq 0, x_2 \geq 0$

[Ans. Max Z = -2, $x_1 = 0$, $x_2 = 1$]

2. Max $Z = -2x_1 - 2x_2 - 4x_3$

   Subject to

   $2x_1 + 3x_2 + 5x_3 \geq 2$

   $3x_1 + x_2 + 7x_3 \leq 3$

   and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

   [Ans. Max Z = 4/3, $x_1 = 0$, $x_2 = 2/3$, $x_3 = 0$]

3. Min $Z = x_1 + 2x_2 + 3x_3$

   Subject to

   $2x_1 - x_2 + x_3 \geq 4$

   $x_1 + x_2 + 2x_3 \geq 8$

   $x_2 - x_3 \geq 2$

   and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

   [Ans. Min Z = 10, $x_1 = 6$, $x_2 = 2$, $x_3 = 0$]

4. Min $Z = 3x_1 + 2x_2 + x_3 + 4x_4$

   Subject to

   $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10$

   $3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$

$5x_1 + 2x_2 + x_3 + 6x_4 \geq 15$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$

[Ans. Min Z = 215/23, $x_1 = 65/23$, $x_2 = 0$, $x_3 = 20$, $x_4 = 0$]

5. Min Z = $x_1 + x_2$

Subject to

$2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

and $x_1 \geq 0, x_2 \geq 0$

[Ans. Pseudo Optimum basic feasible solution]

Unit 2

2.1 Methods for Initial Basic Feasible Solution

2.1.1 North-West Corner Rule

2.1.2 Row Minima Method

2.1.3 Column Minima Method

2.1.4 Lowest Cost Entry Method (Matrix Minima Method)

2.1.5 Vogel’s Approximation Method (Unit Cost Penalty Method)

2.1 Methods for Initial Basic Feasible Solution

Some simple methods to obtain the initial basic feasible solution are

1. North-West Corner Rule
2. Row Minima Method
3. Column Minima Method
4. Lowest Cost Entry Method (Matrix Minima Method)
5. Vogel’s Approximation Method (Unit Cost Penalty Method)

2.1.1 North-West Corner Rule

Step 1

- The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the table.
- The maximum possible amount is allocated here i.e. $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1,1) of the transportation table.

Step 2

i. If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

ii. If $b_1 < a_1$, move horizontally right side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

iii. If $b_1 = a_1$, there is tie for the second allocation. One can make a second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2)$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1)$ in the cell (2, 1)

Step 3

Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

Find the initial basic feasible solution by using North-West Corner Rule

1.

W→ F ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution

	W_1	W_2	W_3	W_5	Availability
F_1	5 (19)	2 (30)	▨	▨	7 2 0
F_2	▨	6 (30)	3 (40)	▨	9 3 0
F_3	▨	▨	4 (70)	14 (20)	18 14 0
Requirement	5 0	8 6 0	7 4 0	14 0	

Initial Basic Feasible Solution

$x_{11} = 5, x_{12} = 2, x_{22} = 6, x_{23} = 3, x_{33} = 4, x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

	D_1	D_2	D_3	D_4	Supply
O_1	1	5	3	3	34
O_2	3	3	1	2	15
O_3	0	2	2	3	12
O_4	2	7	2	4	19
Demand	21	25	17	17	80

Solution

	D_1	D_2	D_3	D_4	Supply
O_1	21 (1)	13 (5)	▨	▨	34 13 0
O_2	▨	12 (3)	3 (1)	▨	15 3 0
O_3	▨	▨	12 (2)	▨	12 0
O_4	▨	▨	2 (2)	17 (4)	19 17
Demand	21 0	25 12 0	17 14 2 0	17 0	

Initial Basic Feasible Solution

$x_{11} = 21$, $x_{12} = 13$, $x_{22} = 12$, $x_{23} = 3$, $x_{33} = 12$, $x_{43} = 2$, $x_{44} = 17$

The transportation cost is 21 (1) + 13 (5) + 12 (3) + 3 (1) + 12 (2) + 2 (2) + 17 (4) = Rs. 221

3.

From To Supply

	2	11	10	3	7	4
	1	4	7	2	1	8
	3	1	4	8	12	9
Demand	3	3	4	5	6	

Solution

From	To					Supply
	3 (2)	1 (11)	▨	▨	▨	4 1 0
	▨	2 (4)	4 (7)	2 (2)	▨	8 6 2 0
	▨	▨	▨	3 (8)	6 (12)	9 6 0
Demand	3 0	3 2 0	4 0	5 3 0	6 0	

Initial Basic Feasible Solution

$x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$

The transportation cost is 3 (2) + 1 (11) + 2 (4) + 4 (7) + 2 (2) + 3 (8) + 6 (12) = Rs. 153

2.1.2 Row Minima Method

Step 1

- The smallest cost in the first row of the transportation table is determined.
- Allocate as much as possible amount $x_{ij} = \min(a_1, b_j)$ in the cell (1, j) so that the capacity of the origin or the destination is satisfied.

Step 2

- If $x_{1j} = a_1$, so that the availability at origin O_1 is completely exhausted, cross out the first row of the table and move to second row.
- If $x_{1j} = b_j$, so that the requirement at destination D_j is satisfied, cross out the jth column and reconsider the first row with the remaining availability of origin O_1.
- If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is satisfied. An arbitrary tie-breaking choice is made. Cross out the jth column and make the second allocation $x_{1k} = 0$ in the cell (1, k) with c_{1k} being the new minimum cost in the first row. Cross out the first row and move to second row.

Step 3

Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied

Determine the initial basic feasible solution using Row Minima Method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3		(80)	(70)	(20)	18

(40)			
5	8	7	7

	W₁	W₂	W₃	W₄	
F₁	(19)	(30)	(50)	7 (10)	X
F₂	(70)	8 (30)	(40)	(60)	1
F₃	(40)	(80)	(70)	(20)	18
	5	X	7	7	

	W₁	W₂	W₃	W₄	
F₁	(19)	(30)	(50)	7 (10)	X
F₂	(70)	8 (30)	1 (40)	(60)	X
F₃	(40)	(80)	(70)	(20)	18
	5	X	6	7	

	W₁	W₂	W₃	W₄	
F₁	(19)	(30)	(50)	7 (10)	X
F₂	(70)	8 (30)	1 (40)	(60)	X
F₃	5 (40)	(80)	6 (70)	7 (20)	X

X X X X

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{22} = 8$, $x_{23} = 1$, $x_{31} = 5$, $x_{33} = 6$, $x_{34} = 7$

The transportation cost is 7 (10) + 8 (30) + 1 (40) + 5 (40) + 6 (70) + 7 (20) = Rs. 1110

2.

	A	B	C	Availability
I	50	30	220	1
II	90	45	170	4
III	250	200	50	4
Requirement	4	2	3	

Solution

	A	B	C	Availability
I		1 (30)		1 0
II	3 (90)	1 (45)		4 3 0
III	1 (250)		3 (50)	4 1 0
Requirement	4 1 0	2 1 0	3 0	

Initial Basic Feasible Solution

$x_{12} = 1$, $x_{21} = 3$, $x_{22} = 1$, $x_{31} = 1$, $x_{33} = 3$

The transportation cost is 1 (30) + 3 (90) + 1 (45) + 1 (250) + 3 (50) = Rs. 745

2.1.3 Column Minima Method

Step 1

Determine the smallest cost in the first column of the transportation table. Allocate x_{i1} = min (a_i, b_1) in the cell (i, 1).

Step 2

- If $x_{i1} = b_1$, cross out the first column of the table and move towards right to the second column
- If $x_{i1} = a_i$, cross out the ith row of the table and reconsider the first column with the remaining demand.
- If $x_{i1} = b_1 = a_i$, cross out the ith row and make the second allocation $x_{k1} = 0$ in the cell (k, 1) with c_{k1} being the new minimum cost in the first column, cross out the column and move towards right to the second column.

Step 3

Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

Use Column Minima method to determine an initial basic feasible solution

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	5 (19)	(30)	(50)	(10)	2

F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	8	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	6	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	(40)	(60)	3
F_3	(40)	(80)	(70)	(20)	18
	X	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X

F_3	(40)	(80)	(70)	(20)	18
	X	X	4	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	4 (70)	(20)	14
	X	X	X	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X

F_3	(40)	(80)	4 (70)	14 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

	D_1	D_2	D_3	D_4	Availability
S_1	11	13	17	14	250
S_2	16	18	14	10	300
S_3	21	24	13	10	400
Requirement	200	225	275	250	

Solution

	D_1	D_2	D_3	D_4	
S_1	200 (11)	50 (13)			250 50 0
S_2		175 (18)		125 (10)	300 125 0
S_3			275 (13)	125 (10)	400 125 0
	200	225	275	250	

0	175	0	0
	0		

Initial Basic Feasible Solution

$x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$

The transportation cost is

200 (11) + 50 (13) + 175 (18) + 125 (10) + 275 (13) + 125 (10) = Rs. 12075

2.1.4 Lowest Cost Entry Method (Matrix Minima Method)

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j)

Step 2

- If $x_{ij} = a_i$, cross out the i^{th} row of the table and decrease b_j by a_i. Go to step 3.
- If $x_{ij} = b_j$, cross out the j^{th} column of the table and decrease a_i by b_j. Go to step 3.
- If $x_{ij} = a_i = b_j$, cross out the i^{th} row or j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Find the initial basic feasible solution using Matrix Minima method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	7	

	W1	W2	W3	W4	
F1	(19)	(30)	(50)	7 (10)	X
F2	(70)	(30)	(40)	(60)	9
F3	(40)	8 (8)	(70)	7 (20)	3
	5	X	7	X	

	W1	W2	W3	W4	
F1	(19)	(30)	(50)	7 (10)	X
F2	(70)	(30)	(40)	(60)	9
F3	3 (40)	8 (8)	(70)	7 (20)	X
	2	X	7	X	

	W1	W2	W3	W4	
F1	(19)	(30)	(50)	7 (10)	X
F2	2 (70)	(30)	7 (40)	(60)	X
F3	3 (40)	8 (8)	(70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$

The transportation cost is 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20) = Rs. 814

2.

	To					Availability
	2	11	10	3	7	4
From	1	4	7	2	1	8
	3	9	4	8	12	9
Requirement	3	3	4	5	6	

Solution

To

				4 (3)		4 0
	3 (1)				5 (1)	8 5 0
From		3 (9)	4 (4)	1 (8)	1 (12)	9 5 4 1 0
	3	3	4	5	6	
	0	0	0	1	1	
				0	0	

Initial Basic Feasible Solution

$x_{14} = 4$, $x_{21} = 3$, $x_{25} = 5$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 1$, $x_{35} = 1$

The transportation cost is 4 (3) + 3 (1) + 5(1) + 3 (9) + 4 (4) + 1 (8) + 1 (12) = Rs. 78

2.1.5 Vogel’s Approximation Method (Unit Cost Penalty Method)

Step1

For each row of the table, identify the **smallest** and the **next to smallest cost**. Determine the difference between them for each row. These are called **penalties**. Put them aside by enclosing them in the parenthesis against the respective rows. Similarly compute penalties for each column.

Step 2

Identify the row or column with the largest penalty. If a tie occurs then use an arbitrary choice. Let the largest penalty corresponding to the ith row have the cost c_{ij}. Allocate the largest possible amount x_{ij} = min (a_i, b_j) in the cell (i, j) and cross out either ith row or jth column in the usual manner.

Step 3

Again compute the row and column penalties for the reduced table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Find the initial basic feasible solution using vogel's approximation method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	19	30	50	10	7	19-10=9
F_2	70	30	40	60	9	40-30=10
F_3	40	8	70	20	18	20-8=12
Requirement	5	8	7	14		
Penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	(19)	(30)	(50)	(10)	7	9
F_2	(70)	(30)	(40)	(60)	9	10
F_3	(40)	**8**(8)	(70)	(20)	18/10	12
Requirement	5	8/0	7	14		
Penalty	21	22	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	9
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	(20)	18/10	20
Requirement	5/0	X	7	14		
Penalty	21	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	18/10/0	50
Requirement	X	X	7	14/4		
Penalty	X	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	7/2/0	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	7	14/4/2		
Penalty	X	X	10	50		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

The transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2.

		Stores				Availability
		I	II	III	IV	
Warehouse	A	21	16	15	13	11
	B	17	18	14	23	13
	C	32	27	18	41	19
Requirement		6	10	12	15	

Solution

		Stores				Availability	Penalty
		I	II	III	IV		
Warehouse	A	(21)	(16)	(15)	(13)	11	2
	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15		

Penalty		4	2	1	10		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	11/0	2
Warehouse	B	(17)	(18)	(14)	(23)	13	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4		
Penalty		4	2	1	10		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	X	X
Warehouse	B	(17)	(18)	(14)	**4**(23)	13/9	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6	10	12	15/4/0		
Penalty		15	9	4	18		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	X	X
Warehouse	B	**6**(17)	(18)	(14)	**4**(23)	13/9/3	3
	C	(32)	(27)	(18)	(41)	19	9
Requirement		6/0	10	12	X		
Penalty		15	9	4	X		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	X	X
Warehouse	B	**6**(17)	**3**(18)	(14)	**4**(23)	13/9/3/0	4
	C	(32)	(27)	(18)	(41)	19	9
Requirement		X	10/7	12	X		
Penalty		X	9	4	X		

		Stores				Availability	Penalty
		I	II	III	IV		
	A	(21)	(16)	(15)	**11**(13)	X	X
Warehouse	B	**6**(17)	**3**(18)	(14)	**4**(23)	X	X
	C	(32)	**7**(27)	**12**(18)	(41)	X	X
Requirement		X	X	X	X		
Penalty		X	X	X	X		

Initial Basic Feasible Solution

$x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 12$

The transportation cost is 11 (13) + 6 (17) + 3 (18) + 4 (23) + 7 (27) + 12 (18) = Rs. 796

Exercise

1. Determine an initial basic feasible solution to the following transportation problem using north-west corner rule.

		I	II	III	IV	Supply
From	A	13	11	15	20	2000
	B	17	14	12	13	6000
	C	18	18	15	12	7000
	Demand	3000	3000	4000	5000	

[Ans. $x_{11} = 2$, $x_{21} = 1$, $x_{22} = 3$, $x_{23} = 2$, $x_{34} = 5$]

2. Determine an initial basic feasible solution to the following transportation problem using row/column minima method.

	To				Supply
From	6	3	5	4	22
	5	9	2	7	15
	5	7	8	6	8
Demand	7	12	17	9	

[Ans. $x_{12} = 12$, $x_{13} = 1$, $x_{14} = 9$, $x_{23} = 15$, $x_{31} = 7$, $x_{33} = 1$]

3. Obtain an initial basic feasible solution to the following transportation problem using matrix minima method.

		D1	D2	D3	D4	Capacity
From	O1	1	2	3	4	6
	O2	4	3	2	0	8
	O3	0	2	2	1	10
	Demand	4	6	8	6	

[Ans. $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = 0$, $x_{33} = 6$]

4. Determine the minimum cost to the following transportation problem using Vogel's method.

		D1	D2	D3	D4	D5	Capacity
	O1	2	11	10	3	7	4
From	O2	1	4	7	2	1	8
	O3	3	9	4	8	12	9
	Demand	3	3	4	5	6	21

[Ans. Min cost = Rs 68]

5. Determine the minimum cost to the following transportation problem using matrix minima method and vogel's method

		D1	D2	D3	D4	Capacity
	O1	1	2	1	4	30
From	O2	3	3	2	1	50
	O3	4	2	5	9	20
	Demand	20	40	30	10	

[Ans. Min cost = Rs 180]

Unit 3

3.1 Examining the Initial Basic Feasible Solution for Non-Degeneracy

Examine the initial basic feasible solution for non-degeneracy. If it is said to be non-degenerate then it has the following two properties

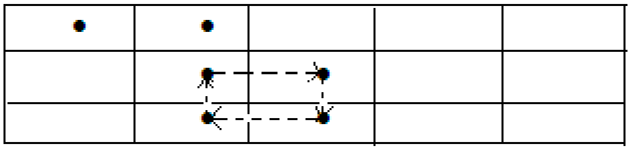
- Initial basic feasible solution must contain exactly m + n – 1 number of individual allocations.
- These allocations must be in independent positions

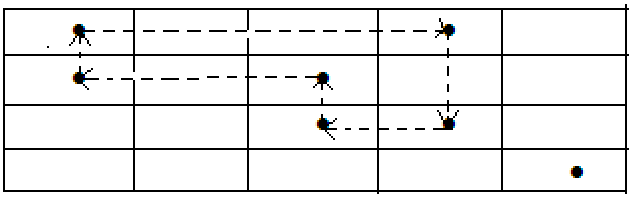
Independent Positions

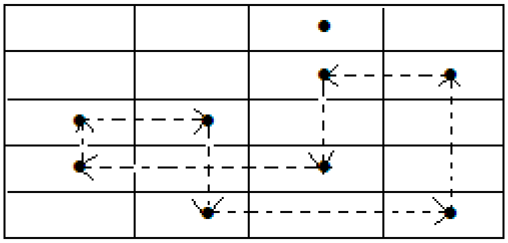
•	•	•		
		•	•	•
	•			•

•				•
			•	•
		•		•

Non-Independent Positions







3.2 Transportation Algorithm for Minimization Problem (MODI Method)

Step 1

Construct the transportation table entering the origin capacities a_i, the destination requirement b_j and the cost c_{ij}

Step 2

Find an initial basic feasible solution by vogel's method or by any of the given method.

Step 3

For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$, calculate the values of u_i and v_j on the transportation table

Step 4

Compute the cost differences $d_{ij} = c_{ij} – (u_i + v_j)$ for all the non-basic cells

Step 5

Apply optimality test by examining the sign of each d_{ij}

- If all $d_{ij} \geq 0$, the current basic feasible solution is optimal
- If at least one $d_{ij} < 0$, select the variable x_{rs} (most negative) to enter the basis.
- Solution under test is not optimal if any d_{ij} is negative and further improvement is required by repeating the above process.

Step 6

Let the variable x_{rs} enter the basis. Allocate an unknown quantity Θ to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount Θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7

Assign the largest possible value to the Θ in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step 8

Now, return to step 3 and repeat the process until an optimal solution is obtained.

3.3 Worked Examples

Example 1

Find an optimal solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

1. **Applying vogel's approximation method for finding the initial basic feasible solution**

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Minimum transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2. **Check for Non-degeneracy**

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

3. **Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$**

					u_i
	• (19)			• (10)	u_1= -10
			• (40)	• (60)	$u_2 = 40$
		• (8)		• (20)	$u_3 = 0$
v_j	$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$	

Assign a 'u' value to zero. (Convenient rule is to select the u_i, which has the largest number of allocations in its row)

Let $u_3 = 0$, then

$u_3 + v_4$= 20 which implies $0 + v_4 = 20$, so $v_4 = 20$

$u_2 + v_4$= 60 which implies $u_2 + 20 = 60$, so $u_2 = 40$

$u_1 + v_4$= 10 which implies $u_1 + 20 = 10$, so $u_1 = -10$

$u_2 + v_3$= 40 which implies $40 + v_3 = 40$, so $v_3 = 0$

$u_3 + v_2$= 8 which implies $0 + v_2 = 8$, so $v_2 = 8$

$u_1 + v_1$= 19 which implies $-10 + v_1$= 19, so $v_1 = 29$

4. **Calculation of cost differences for non basic cells $d_{ij} = c_{ij} - (u_i + v_j)$**

c_{ij}

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

$u_i + v_j$

•	-2	-10	•
69	48	•	•
29	•	0	•

$d_{ij} = c_{ij} - (u_i + v_j)$

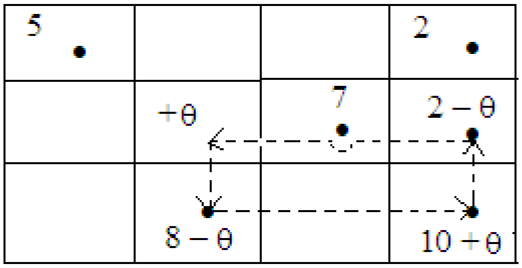
•	32	60	•
1	**-18**	•	•
11	•	70	•

5. **Optimality test**

$d_{ij} < 0$ i.e. $d_{22} = -18$

so x_{22} is entering the basis

6. **Construction of loop and allocation of unknown quantity Θ**



We allocate Θ to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Θ in the marked cell. The value of Θ is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Θ, 2-Θ) = 0 which gives Θ = 2. Therefore x_{24} is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

7. **Improved Solution**

				u_i
• (19)			• (10)	$u_1 = -10$
	• (30)	• (40)		$u_2 = 22$
	• (8)		• (20)	$u_3 = 0$

v_j $v_1 = 29$ $v_2 = 8$ $v_3 = 18$ $v_4 = 20$

c_{ij}

•	(30)	(50)	•
(70)	•	•	(60)
(40)	•	(70)	•

$u_i + v_j$

•	-2	8	•
51	•	•	42
29	•	18	•

$d_{ij} = c_{ij} – (u_i + v_j)$

•	32	42	•
19	•	•	18
11	•	52	•

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.743

Example 2

Solve by lowest cost entry method and obtain an optimal solution for the following problem

				Available
	50	30	220	1
From	90	45	170	3
	250	200	50	4
Required	4	2	2	

Solution

By lowest cost entry method

				Available
		1(30)		1/0
From	2(90)	1(45)		3/2/0
	2(250)		2(50)	4/2/0
Required	4/2/2	2/1/0	2/0	

Minimum transportation cost is 1 (30) + 2 (90) + 1 (45) + 2 (250) + 2 (50) = Rs. 855

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 3 – 1 = 5 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		• (30)		u_1= -15
	• (90)	• (45)		u_2 = 0
	• (250)		• (50)	u_3 = 160
v_j	v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

50	•	220
•	•	170
•	200	•

$u_i + v_j$

75	•	-125
•	•	-110
•	205	•

$d_{ij} = c_{ij} – (u_i + v_j)$

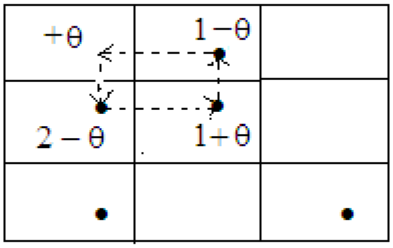
-25	●	345
●	●	280
●	-5	●

Optimality test

$d_{ij} < 0$ i.e. $d_{11} = -25$ is most negative

So x_{11} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(2-Θ, 1-Θ) = 0 which gives $\Theta = 1$. Therefore x_{12} is outgoing as it becomes zero.

1(50)		
1(90)	2(45)	
2(250)		2(50)

Minimum transportation cost is 1 (50) + 1 (90) + 2 (45) + 2 (250) + 2 (50) = Rs. 830

II Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	● (50)			$u_1 = -40$
	● (90)	● (45)		$u_2 = 0$
	● (250)		● (50)	$u_3 = 160$
v_j	$v_1 = 90$	$v_2 = 45$	$v_3 = -110$	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

●	30	220
●	●	170
●	200	●

$u_i + v_j$

●	5	-150
●	●	-110
●	205	●

$d_{ij} = c_{ij} – (u_i + v_j)$

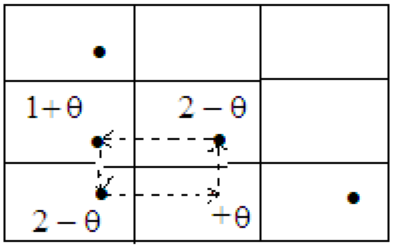
●	25	370
●	●	280
●	-5	●

Optimality test

$d_{ij} < 0$ i.e. $d_{32} = -5$

So x_{32} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



2 – ϴ = 0 which gives ϴ = 2. Therefore x_{22} and x_{31} is outgoing as it becomes zero.

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	• (50)			u_1= -40
	• (90)	• (45)		u_2 = 0
		• (200)	• (50)	u_3 = 155
v_j	v_1 = 90	v_2 = 45	v_3 = -105	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

•	30	220
•	•	170
250	•	•

$u_i + v_j$

•	5	-145
•	•	-105
245	•	•

$d_{ij} = c_{ij} – (u_i + v_j)$

•	25	365
•	•	275
5	•	•

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.820

Example 3

Is $x_{13} = 50$, $x_{14} = 20$, $x_{21} = 55$, $x_{31} = 30$, $x_{32} = 35$, $x_{34} = 25$ an optimal solution to the transportation problem.

					Available
	6	1	9	3	70
From	11	5	2	8	55
	10	12	4	7	90
Required	85	35	50	45	

Solution

					Available
From			50(9)	20(3)	X

	55(11)				X
	30(10)	35(12)		25(7)	X
Required	X	X	X	X	

Minimum transportation cost is 50 (9) + 20 (3) + 55 (11) + 30 (10) + 35 (12) + 25 (7) = Rs. 2010

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independent positions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
			• (9)	• (3)	u_1= -4
	• (11)				u_2 = 1
	• (10)	• (12)		• (7)	u_3 = 0
v_j	v_1 = 10	v_2 = 12	v_3 = 13	v_4 = 7	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	1	•	•
•	5	2	8
•	•	4	•

$u_i + v_j$

6	8	•	•
•	13	14	8
•	•	13	•

$d_{ij} = c_{ij} – (u_i + v_j)$

0	-7	•	•

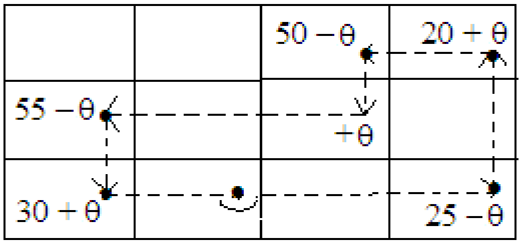
●	-8	-12	0
●	●	-9	●

Optimality test

$d_{ij} < 0$ i.e. $d_{23} = -12$ is most negative

So x_{23} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(50-ϴ, 55-ϴ, 25-ϴ) = 25 which gives ϴ = 25. Therefore x_{34} is outgoing as it becomes zero.

		25(9)	45(3)
30(11)		25(2)	
55(10)	35(12)		

Minimum transportation cost is 25 (9) + 45 (3) + 30 (11) + 25 (2) + 55 (10) + 35 (12) = Rs. 1710

II iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
		● (9)	● (3)	u_1= 8

	• (11)		• (2)		$u_2 = 1$
	• (10)	• (12)			$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 12$	$v_3 = 1$	$v_4 = -5$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	1	•	•
•	5	•	8
•	•	4	7

$u_i + v_j$

18	20	•	•
•	13	•	-4
•	•	1	-5

$d_{ij} = c_{ij} – (u_i + v_j)$

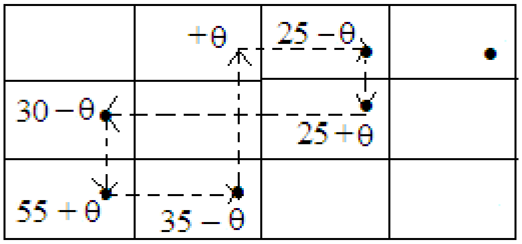
-12	-19	•	•
•	-8	•	12
•	•	3	12

Optimality test

$d_{ij} < 0$ i.e. $d_{12} = -19$ is most negative

So x_{12} is entering the basis

Construction of loop and allocation of unknown quantity Θ



min(25-Ɵ, 30-Ɵ, 35-Ɵ) = 25 which gives Ɵ = 25. Therefore x_{13} is outgoing as it becomes zero.

	25(1)		45(3)
5(11)		50(2)	
80(10)	10(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (11) + 50 (2) + 80 (10) + 10 (12) = Rs. 1235

III Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		• (1)		• (3)	$u_1 = -11$
	• (11)		• (2)		$u_2 = 1$
	• (10)	• (12)			$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 12$	$v_3 = 1$	$v_4 = 14$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

6	•	9	•
•	5	•	8
•	•	4	7

$u_i + v_j$

-1	•	-10	•
•	13	•	15
•	•	1	14

$d_{ij} = c_{ij} - (u_i + v_j)$

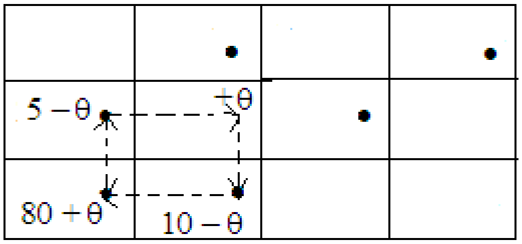
7	●	19	●
●	-8	●	-7
●	●	3	-7

Optimality test

$d_{ij} < 0$ i.e. $d_{22} = -8$ is most negative

So x_{22} is entering the basis

Construction of loop and allocation of unknown quantity Ө



min(5-Ө, 10-Ө) = 5 which gives Ө = 5. Therefore x_{21} is outgoing as it becomes zero.

	25(1)		45(3)
	5(5)	50(2)	
85(10)	5(12)		

Minimum transportation cost is 25 (1) + 45 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (12) = Rs. 1195

IV Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		● (1)		● (3)	u_1= -11
		● (5)	● (2)		u_2 = -7
	● (10)	● (12)			u_3 = 0
v_j	v_1 = 10	v_2 = 12	v_3 = 9	v_4 = 14	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

6	●	9	●
11	●	●	8
●	●	4	7

$u_i + v_j$

-1	●	-2	●
3	●	●	7
●	●	9	14

$d_{ij} = c_{ij} - (u_i + v_j)$

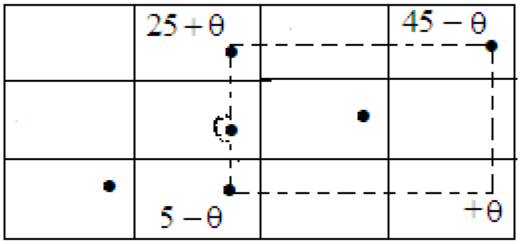
7	●	11	●
8	●	●	1
●	●	-5	-7

Optimality test

$d_{ij} < 0$ i.e. d_{34} = -7 is most negative

So x_{34} is entering the basis

Construction of loop and allocation of unknown quantity Ө



min(5-ϴ, 45-ϴ) = 5 which gives ϴ = 5. Therefore x_{32} is outgoing as it becomes zero.

	30(1)		40(3)
	5(5)	50(2)	
85(10)			5(7)

Minimum transportation cost is 30 (1) + 40 (3) + 5 (5) + 50 (2) + 85 (10) + 5 (7) = Rs. 1160

V Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

					u_i
		• (1)		• (3)	u_1= -4
		• (5)	• (2)		$u_2 = 0$
	• (10)			• (7)	$u_3 = 0$
v_j	$v_1 = 10$	$v_2 = 5$	$v_3 = 2$	$v_4 = 7$	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

6	•	9	•
11	•	•	8
•	12	4	•

$u_i + v_j$

6	•	-2	•
10	•	•	7
•	5	2	•

$d_{ij} = c_{ij} – (u_i + v_j)$

0	•	11	•
1	•	•	1

•	7	2	•

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.1160. Further more $d_{11} = 0$ which indicates that alternative optimal solution also exists.

Exercise

1. Determine the optimal solution of the given transportation problem

	To				Supply
	2	3	11	7	6
From	1	0	6	1	1
	5	8	15	10	10
Demand	7	5	3	2	17

[Ans. $x_{12} = 5$, $x_{13} = 1$, $x_{24} = 1$, $x_{31} = 7$, $x_{33} = 2$, $x_{34} = 1$ Min cost = Rs 102]

2. Using North-West Corner rule for initial basic feasible solution, obtain an optimum basic feasible solution to the following problem

	To			Available
	7	3	4	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

[Ans. $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$ Min cost = Rs 33]

3. Determine the optimal solution of the given transportation problem

	To				Supply
	10	7	3	6	3

From	1	6	7	3	5
	7	4	5	3	7
Demand	3	2	6	4	

[Ans. $x_{13} = 3$, $x_{21} = 3$, $x_{24} = 2$, $x_{32} = 2$, $x_{33} = 3$, $x_{34} = 2$, Min cost = Rs 47]

Module 5

Unit 1

1.1 Introduction to Assignment Problem

1.2 Algorithm for Assignment Problem

1.3 Worked Examples

1.4 Unbalanced Assignment Problem

1.5 Maximal Assignment Problem

1.1 Introduction to Assignment Problem

In assignment problems, the objective is to assign a number of jobs to the equal number of persons at a minimum cost of maximum profit.

Suppose there are ‘n’ jobs to be performed and ‘n’ persons are available for doing these jobs. Assume each person can do each job at a time with a varying degree of efficiency. Let c_{ij} be the cost of i^{th} person assigned to j^{th} job. Then the problem is to find an assignment so that the total cost for performing all jobs is minimum. Such problems are known as **assignment problems.**

These problems may consist of assigning men to offices, classes to the rooms or problems to the research team etc.

Mathematical formulation

Cost matrix: $c_{ij}= \begin{array}{lllll} c_{11} & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2n} \\ . & & & & \end{array}$

.

.

c_{n1} c_{n2} c_{n3} … c_{nn}

$$\text{Minimize cost : } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n$$

Subject to restrictions of the form

$$x_{ij} = \begin{cases} 1 \text{ if ith person is assigned jth job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad (\text{one job is done by the ith person, } i = 1,2,\ldots n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (\text{only one person should be assigned the jth job, } j = 1,2,\ldots n)$$

Where x_{ij} denotes that j^{th} job is to be assigned to the i^{th} person.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

1.2 Algorithm for Assignment Problem (Hungarian Method)

Step 1

Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows (Row reduced matrix).

Step 2

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus first modified matrix is obtained.

Step 3

Draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities

- If N = n, the number of rows (columns) of the given matrix then an optimal assignment can be made. So make the zero assignment to get the required solution.
- If N < n then proceed to step 4

Step 4

Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 5

Repeat step 3 and step 4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N = n.

Step 6

To make zero assignment - examine the rows successively until a row-wise exactly single zero is found; mark this zero by ‘□‘to make the assignment. Then, mark a ‘X’ over all zeroes if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Step 7

Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If there lies more than one of the unmarked zeroes in any column or row, then mark ‘□‘one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8

Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment.

1.3 Worked Examples

Example 1

A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

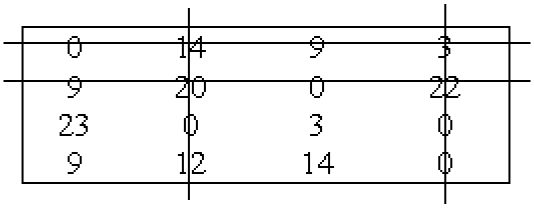
		Subordinates			
		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution

Row Reduced Matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

I Modified Matrix



$$\begin{matrix} 0 & 14 & 9 & 3 \\ 9 & 20 & 0 & 22 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{matrix}$$

N = 4, n = 4

Since N = n, we move on to zero assignment

Zero assignment

$$\begin{matrix} \boxed{0} & 14 & 9 & 3 \\ 9 & 20 & \boxed{0} & 22 \\ 23 & \boxed{0} & 3 & \cancel{0} \\ 9 & 12 & 14 & \boxed{0} \end{matrix}$$

Optimal assignment	A – I	B – III	C – II	D – IV
Man-hours	8	4	19	10

Total man-hours = 8 + 4 + 19 + 10 = 41 hours

Example 2

A car hire company has one car at each of five depots a, b, c, d and e. a customer requires a car in each town namely A, B, C, D and E. Distance (kms) between depots (origins) and towns (destinations) are given in the following distance matrix

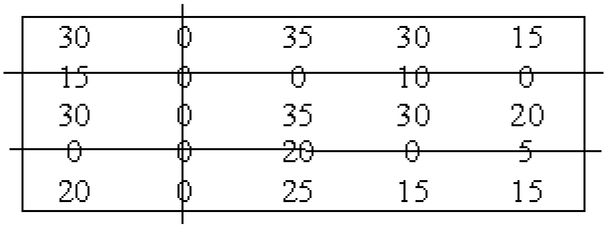
$$\begin{array}{c|ccccc|} & a & b & c & d & e \\ \hline A & 160 & 130 & 175 & 190 & 200 \\ B & 135 & 120 & 130 & 160 & 175 \\ C & 140 & 110 & 155 & 170 & 185 \\ D & 50 & 50 & 80 & 80 & 110 \\ E & 55 & 35 & 70 & 80 & 105 \\ \hline \end{array}$$

Solution

Row Reduced Matrix

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

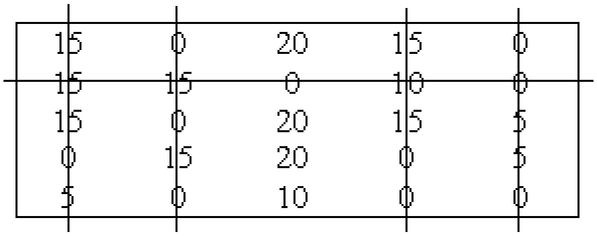
I Modified Matrix



30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

N < n i.e. 3 < 5, so move to next modified matrix

II Modified Matrix

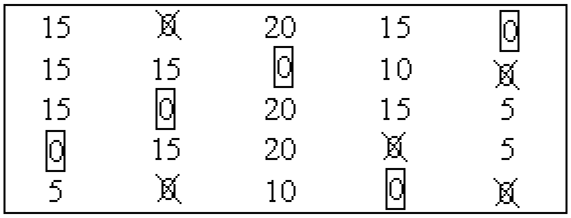


15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

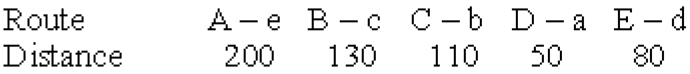
N = 5, n = 5

Since N = n, we move on to zero assignment

Zero assignment



$$\begin{array}{ccccc}
15 & \cancel{0} & 20 & 15 & \boxed{0} \\
15 & 15 & \boxed{0} & 10 & \cancel{0} \\
15 & \boxed{0} & 20 & 15 & 5 \\
\boxed{0} & 15 & 20 & \cancel{0} & 5 \\
5 & \cancel{0} & 10 & \boxed{0} & \cancel{0}
\end{array}$$



Route	A – e	B – c	C – b	D – a	E – d
Distance	200	130	110	50	80

Minimum distance travelled = 200 + 130 + 110 + 50 + 80 = 570 kms

Example 3

Solve the assignment problem whose effectiveness matrix is given in the table

$$\begin{array}{c|cccc|}
 & 1 & 2 & 3 & 4 \\
\hline
A & 49 & 60 & 45 & 61 \\
B & 55 & 63 & 45 & 69 \\
C & 52 & 62 & 49 & 68 \\
D & 55 & 64 & 48 & 66 \\
\hline
\end{array}$$

Solution

Row-Reduced Matrix

$$\begin{array}{|cccc|}
\hline
4 & 15 & 0 & 16 \\
10 & 18 & 0 & 24 \\
3 & 13 & 0 & 19 \\
7 & 16 & 0 & 18 \\
\hline
\end{array}$$

I Modified Matrix

$$\begin{array}{|cccc|}
\hline
1 & 2 & 0 & 0 \\
7 & 5 & 0 & 8 \\
0 & 0 & 0 & 3 \\
4 & 3 & 0 & 2 \\
\hline
\end{array}$$

N < n i.e 3 < 4, so II modified matrix

II Modified Matrix

$$\begin{array}{|cccc|}
\hline
1 & 2 & 2 & 0 \\
5 & 3 & 0 & 6 \\
0 & 0 & 2 & 3 \\
2 & 1 & 0 & 0 \\
\hline
\end{array}$$

N < n i.e 3 < 4

III Modified matrix

$$\begin{array}{|cccc|}
\hline
0 & 1 & 2 & 0 \\
4 & 2 & 0 & 6 \\
0 & 0 & 3 & 4 \\
1 & 0 & 0 & 0 \\
\hline
\end{array}$$

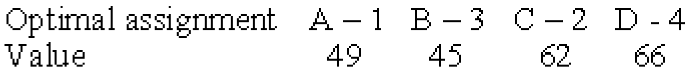
Since N = n, we move on to zero assignment

Zero assignment

Multiple optimal assignments exists

Solution - I

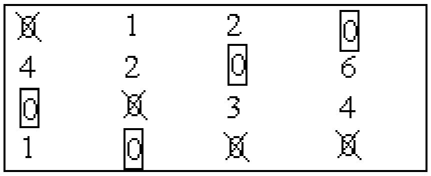
$$\begin{array}{|cccc|}
\hline
\boxed{0} & 1 & 2 & \xcancel{0} \\
4 & 2 & \boxed{0} & 6 \\
\xcancel{0} & \boxed{0} & 3 & 4 \\
1 & \xcancel{0} & \xcancel{0} & \boxed{0} \\
\hline
\end{array}$$



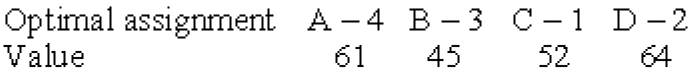
Optimal assignment	A – 1	B – 3	C – 2	D - 4
Value	49	45	62	66

Total cost = 49 + 45 + 62 + 66 = 222 units

Solution – II



$\cancel{0}$	1	2	$\boxed{0}$
4	2	$\boxed{0}$	6
$\boxed{0}$	$\cancel{0}$	3	4
1	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$



Optimal assignment	A – 4	B – 3	C – 1	D – 2
Value	61	45	52	64

Minimum cost = 61 + 45 + 52 + 64 = 222 units

Example 4

Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table. Assuming that each mechanic can be assigned only one job, determine the minimum time assignment.

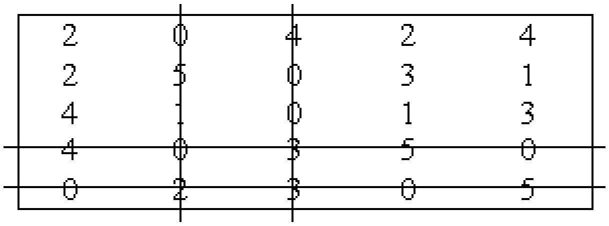
	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Solution

Row Reduced Matrix

2	0	4	3	6
2	5	0	4	3
4	1	0	2	5
4	0	3	6	2
0	2	3	1	7

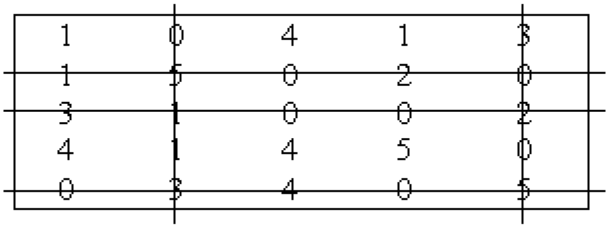
I Modified Matrix



2	0	4	2	4
2	5	0	3	1
4	1	0	1	3
4	0	3	5	0
0	2	3	0	5

N < n

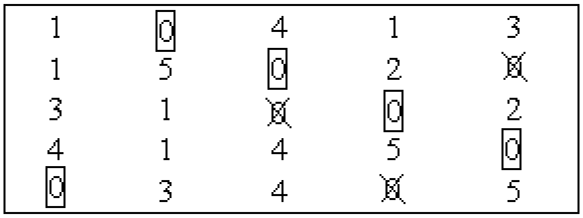
II Modified Matrix



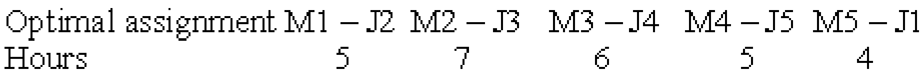
1	0	4	1	3
1	5	0	2	0
3	1	0	0	2
4	1	4	5	0
0	3	4	0	5

N = n

Zero assignment



1	$\boxed{0}$	4	1	3
1	5	$\boxed{0}$	2	$\cancel{0}$
3	1	$\cancel{0}$	$\boxed{0}$	2
4	1	4	5	$\boxed{0}$
$\boxed{0}$	3	4	$\cancel{0}$	5



Optimal assignment	M1 – J2	M2 – J3	M3 – J4	M4 – J5	M5 – J1
Hours	5	7	6	5	4

Minimum time = 5 + 7 + 6 + 5 + 4 = 27 hours

1.4 Unbalanced Assignment Problems

If the number of rows and columns are not equal then such type of problems are called as unbalanced assignment problems.

Example 1

A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

		Machines			
		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

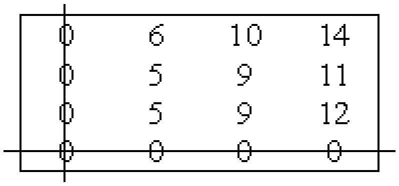
Solution

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Row Reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

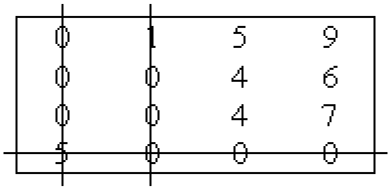
I Modified Matrix



0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

N < n i.e. 2 < 4

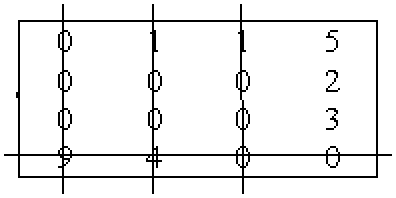
II Modified Matrix



0	1	5	9
0	0	4	6
0	0	4	7
5	0	0	0

N < n i.e. 3 < 4

III Modified Matrix



0	1	1	5
0	0	0	2
0	0	0	3
9	4	0	0

N = n

Zero assignment

Multiple assignments exists

Solution -I

$$\begin{array}{|cccc|}\hline \boxed{0} & 1 & 1 & 5 \\ \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 \\ 9 & 4 & \cancel{0} & \boxed{0} \\ \hline\end{array}$$

Optimal assignment	W – A	X – B	Y – C
Cost	18	13	19

Minimum cost = 18 + 13 + 19 = Rs 50

Solution -II

$$\begin{array}{|cccc|}\hline \boxed{0} & 1 & 1 & 5 \\ \cancel{0} & \cancel{0} & \boxed{0} & 2 \\ \cancel{0} & \boxed{0} & \cancel{0} & 3 \\ 9 & 4 & \cancel{0} & \boxed{0} \\ \hline\end{array}$$

Optimal assignment	W – A	X – C	Y – B
Cost	18	17	15

Minimum cost = 18 + 17 + 15 = Rs 50

Example 2

Solve the assignment problem whose effectiveness matrix is given in the table

	R1	R2	R3	R4
C1	9	14	19	15
C2	7	17	20	19
C3	9	18	21	18
C4	10	12	18	19

C5	10	15	21	16

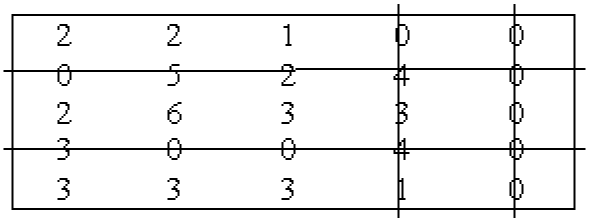
Solution

9	14	19	15	0
7	17	20	19	0
9	18	21	18	0
10	12	18	19	0
10	15	21	16	0

Row Reduced Matrix

9	14	19	15	0
7	17	20	19	0
9	18	21	18	0
10	12	18	19	0
10	15	21	16	0

I Modified Matrix



2	2	1	0	0
0	5	2	4	0
2	6	3	3	0
3	0	0	4	0
3	3	3	1	0

N < n i.e. 4 < 5

II Modified Matrix

$$\begin{array}{|ccccc|}\hline 1 & 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 5 & 1 \\ 1 & 5 & 2 & 3 & 0 \\ 3 & 0 & 0 & 5 & 1 \\ 2 & 2 & 2 & 1 & 0 \\ \hline\end{array}$$

N < n i.e. 4 < 5

III Modified Matrix

$$\begin{array}{|ccccc|}\hline 2 & 1 & 0 & 0 & 1 \\ 0 & 4 & 1 & 4 & 1 \\ 1 & 4 & 1 & 2 & 0 \\ 4 & 0 & 0 & 5 & 2 \\ 2 & 1 & 1 & 0 & 0 \\ \hline\end{array}$$

N = n

Zero assignment

$$\begin{array}{|ccccc|}\hline 2 & 1 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 4 & 1 & 4 & 1 \\ 1 & 4 & 1 & 2 & \boxed{0} \\ 4 & \boxed{0} & \cancel{0} & 5 & 2 \\ 2 & 1 & 1 & \boxed{0} & \cancel{0} \\ \hline\end{array}$$

Optimal assignment	C1 – R3	C2 – R1	C4 – R2	C5 – R4
Units	19	7	12	16

Minimum cost = 19 + 7 + 12 + 16 = 54 units

1.5 Maximal Assignment Problem

Example 1

A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution

Subtract all the elements from the highest element

Highest element = 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row Reduced matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8

8	0	10	3	7
5	3	4	0	7

I Modified Matrix

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

N < n i.e. 3 < 5

II Modified Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	4	6
0	2	3	0	5

N < n i.e. 4 < 5

III Modified Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

N = n

Zero assignment

1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
$\cancel{0}$	3	$\cancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

Optimal assignment 1 – C   2 – E   3 – D   4 – B   5 – A

Maximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50

Exercise

What is assignment problem? Give any two areas of its applications

Find the optimal solution for the assignment problem with the following cost matrix

	I	II	III	IV
A	5	3	1	8
B	7	9	2	6
C	6	4	5	7

D	5	7	7	6

[Ans. A - III, B - IV, C - II, D – I, Min cost = Rs.16]

Solve the following assignment problem

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[Ans. A - 2, B - 3, C - 4, D – 1, Min cost = Rs.38]

The jobs A, B, C are to be assigned to three machines X, Y, Z. The processing costs (Rs.) are as given in the matrix below. Find the allocation which will minimize the overall processing cost.

	X	Y	Z
A	19	28	31
B	11	17	16
C	12	15	13

[Ans. A – X, B - Y, C – Z]

A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only).the profits are estimated as follows

	A	B	C	D
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7

5	5	2	4	3
6	5	7	6	4

[Ans. 2 - A, 3 - B, 4 - D, 6 – C, Max profit = Rs.28]

Unit 2

2.1 Introduction to Game Theory

2.2 Properties of a Game

2.3 Characteristics of Game Theory

2.4 Classification of Games

2.5 Limitations of Game Theory

2.5 Solving Two-Person and Zero-Sum Game

2.1 Introduction to Game Theory

Game theory is a distinct and interdisciplinary approach to the study of human behavior. The disciplines most involved in game theory are mathematics, economics and the other social and behavioral sciences. Game theory (like computational theory and so many other contributions) was founded by the great mathematician John von Neumann.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcome results. Game theory does not insist on how a game should be played but tells the procedure and principles by which action should be selected. Thus it is a decision theory useful in competitive situations.

Game is defined as an activity between two or more persons according to a set of rules at the end of which each person receives some benefit or suffers loss. The set of rules defines the **game**. Going through the set of rules once by the participants defines a **play**.

A Scientific Metaphor

Since the work of John von Neumann, "games" have been a scientific metaphor for a much wider range of human interactions in which the outcomes depend on the interactive strategies of two or more persons, who have opposed or at best mixed motives. Among the issues discussed in game theory are

1) What does it mean to choose strategies "rationally" when outcomes depend on the strategies chosen by others and when information is incomplete?

2) In "games" that allow mutual gain (or mutual loss) is it "rational" to cooperate to realize the mutual gain (or avoid the mutual loss) or is it "rational" to act aggressively in seeking individual gain regardless of mutual gain or loss?

3) If the answers to 2) are "sometimes," in what circumstances is aggression rational and in what circumstances is cooperation rational?

4) In particular, do ongoing relationships differ from one-off encounters in this connection?

5) Can moral rules of cooperation emerge spontaneously from the interactions of rational egoists?

6) How does real human behavior correspond to "rational" behavior in these cases?

7) If it differs, in what direction? Are people more cooperative than would be "rational?" More aggressive? Both?

Thus, among the "games" studied by game theory are

- Bankruptcy
- Barbarians at the Gate
- Battle of the Networks
- Caveat Emptor
- Conscription
- Coordination
- Escape and Evasion
- Frogs Call for Mates
- Hawk versus Dove
- Mutually Assured Destruction
- Majority Rule
- Market Niche
- Mutual Defense

- Prisoner's Dilemma
- Subsidized Small Business
- Tragedy of the Commons
- Ultimatum
- Video System Coordination

Why Do Economists Study Games?

- Games are a convenient way in which to model the strategic interactions among economic agents.
- Many economic issues involve strategic interaction.
  - Behavior in imperfectly competitive markets, e.g. Coca-Cola versus Pepsi.
  - Behavior in auctions, e.g. Investment banks bidding on U.S. Treasury bills.
  - Behavior in economic negotiations, e.g. trade.
- Game theory is not limited to Economics

2.2 Properties of a Game

1. There are finite numbers of competitors called ‘players’
2. Each player has a finite number of possible courses of action called ‘strategies’
3. All the strategies and their effects are known to the players but player does not know which strategy is to be chosen.
4. A game is played when each player chooses one of his strategies. The strategies are assumed to be made simultaneously with an outcome such that no player knows his opponents strategy until he decides his own strategy.
5. The game is a combination of the strategies and in certain units which determines the gain or loss.
6. The figures shown as the outcomes of strategies in a matrix form are called ‘pay-off matrix’.

7. The player playing the game always tries to choose the best course of action which results in optimal pay off called ‘optimal strategy’.
8. The expected pay off when all the players of the game follow their optimal strategies is known as ‘value of the game’. The main objective of a problem of a game is to find the value of the game.
9. The game is said to be ‘fair’ game if the value of the game is zero otherwise it s known as ‘unfair’.

2.3 Characteristics of Game Theory

1. Competitive game

A competitive situation is called a **competitive game** if it has the following four properties

1. There are finite number of competitors such that n ≥ 2. In case n = 2, it is called a **two-person game** and in case n > 2, it is referred as **n-person game**.
2. Each player has a list of finite number of possible activities.
3. A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously i.e. no player knows the choice of the other until he has decided on his own.
4. Every combination of activities determines an outcome which results in a gain of payments to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

2. Strategy

The strategy of a player is the predetermined rule by which player decides his course of action from his own list during the game. The two types of strategy are

1. Pure strategy
2. Mixed strategy

Pure Strategy

If a player knows exactly what the other player is going to do, a deterministic situation is obtained and objective function is to maximize the gain. Therefore, the pure strategy is a decision rule always to select a particular course of action.

Mixed Strategy

If a player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained and objective function is to maximize the expected gain. Thus the mixed strategy is a selection among pure strategies with fixed probabilities.

Repeated Game Strategies

- In repeated games, the sequential nature of the relationship allows for the adoption of strategies that are contingent on the actions chosen in previous plays of the game.
- Most contingent strategies are of the type known as "trigger" strategies.
- Example trigger strategies
  - In prisoners' dilemma: Initially play doesn’t confess. If your opponent plays Confess, then play Confess in the next round. If your opponent plays don’t confess, then play doesn’t confess in the next round. This is known as the "tit for tat" strategy.
  - In the investment game, if you are the sender: Initially play Send. Play Send as long as the receiver plays Return. If the receiver plays keep, never play Send again. This is known as the "grim trigger" strategy.

3. Number of persons

A game is called ‘n’ person game if the number of persons playing is ‘n’. The person means an individual or a group aiming at a particular objective.

Two-person, zero-sum game

A game with only two players (player A and player B) is called a ‘two-person, zero-sum game’, if the losses of one player are equivalent to the gains of the other so that the sum of their net gains is zero.

Two-person, zero-sum games are also called rectangular games as these are usually represented by a payoff matrix in a rectangular form.

4. Number of activities

The activities may be finite or infinite.

5. Payoff

The quantitative measure of satisfaction a person gets at the end of each play is called a payoff

6. Payoff matrix

Suppose the player A has ‘m’ activities and the player B has ‘n’ activities. Then a payoff matrix can be formed by adopting the following rules

- Row designations for each matrix are the activities available to player A
- Column designations for each matrix are the activities available to player B
- Cell entry V_{ij} is the payment to player A in A’s payoff matrix when A chooses the activity i and B chooses the activity j.
- With a zero-sum, two-person game, the cell entry in the player B’s payoff matrix will be negative of the corresponding cell entry V_{ij} in the player A’s payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

7. Value of the game

Value of the game is the maximum guaranteed game to player A (maximizing player) if both the players uses their best strategies. It is generally denoted by ‘V’ and it is unique.

2.4 Classification of Games

Simultaneous v. Sequential Move Games

- Games where players choose actions simultaneously are simultaneous move games.
  - Examples: Prisoners' Dilemma, Sealed-Bid Auctions.
  - Must anticipate what your opponent will do right now, recognizing that your opponent is doing the same.
- Games where players choose actions in a particular sequence are sequential move games.
  - Examples: Chess, Bargaining/Negotiations.
  - Must look ahead in order to know what action to choose now.
  - Many sequential move games have deadlines/ time limits on moves.
- Many strategic situations involve both sequential and simultaneous moves.

One-Shot versus Repeated Games

- One-shot: play of the game occurs once.
  - Players likely to not know much about one another.
  - Example - tipping on your vacation
- Repeated: play of the game is repeated with the same players.
  - Indefinitely versus finitely repeated games
  - Reputational concerns matter; opportunities for cooperative behavior may arise.
- Advise: If you plan to pursue an *aggressive* strategy, ask yourself whether you are in a one-shot or in a repeated game. If a repeated game, *think again*.

Generally games are classified into

- Pure strategy games
- Mixed strategy games

The method for solving these two types varies. By solving a game, we need to find best strategies for both the players and also to find the value of the game.

Pure strategy games can be solved by **saddle point method.**

The different methods for solving a mixed strategy game are

- Analytical method
- Graphical method
- Dominance rule
- Simplex method

2.5 Limitations of game theory

The major limitations are

- The assumption that the players have the knowledge about their own payoffs and others is rather unrealistic.
- As the number of players increase in the game, the analysis of the gaming strategies become increasingly complex and difficult.
- The assumptions of maximin and minimax show that the players are risk-averse and have complete knowledge of the strategies. It doesn't seem practical.
- Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out collusion. Then the mixed strategies are not very useful.

2.6 Solving Two-Person and Zero-Sum Game

Two-person zero-sum games may be deterministic or probabilistic. The deterministic games will have saddle points and pure strategies exist in such games. In contrast, the probabilistic games will have no saddle points and mixed strategies are taken with the help of probabilities.

Definition of saddle point

A saddle point of a matrix is the position of such an element in the payoff matrix, which is minimum in its row and the maximum in its column.

Procedure to find the saddle point

- Select the minimum element of each row of the payoff matrix and mark them with circles.
- Select the maximum element of each column of the payoff matrix and mark them with squares.
- If their appears an element in the payoff matrix with a circle and a square together then that position is called saddle point and the element is the value of the game.

Solution of games with saddle point

To obtain a solution of a game with a saddle point, it is feasible to find out

- Best strategy for player A
- Best strategy for player B
- The value of the game

The best strategies for player A and B will be those which correspond to the row and column respectively through the saddle point.

Examples

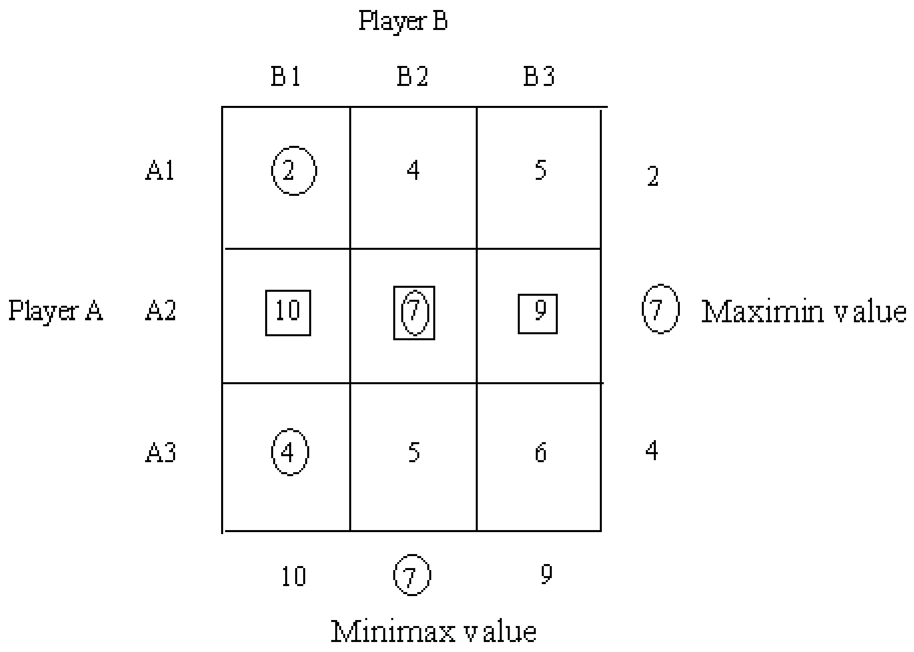
Solve the payoff matrix

1.

	Player B		
Player A	B1	B2	B3

A1	2	4	5
A2	10	7	9
A3	4	5	6

Solution



		Player B			
		B1	B2	B3	
	A1	②	4	5	2
Player A	A2	$\boxed{10}$	$\boxed{⑦}$	$\boxed{9}$	⑦ Maximin value
	A3	④	5	6	4
		10	⑦	9	
			Minimax value		

Strategy of player A – A2

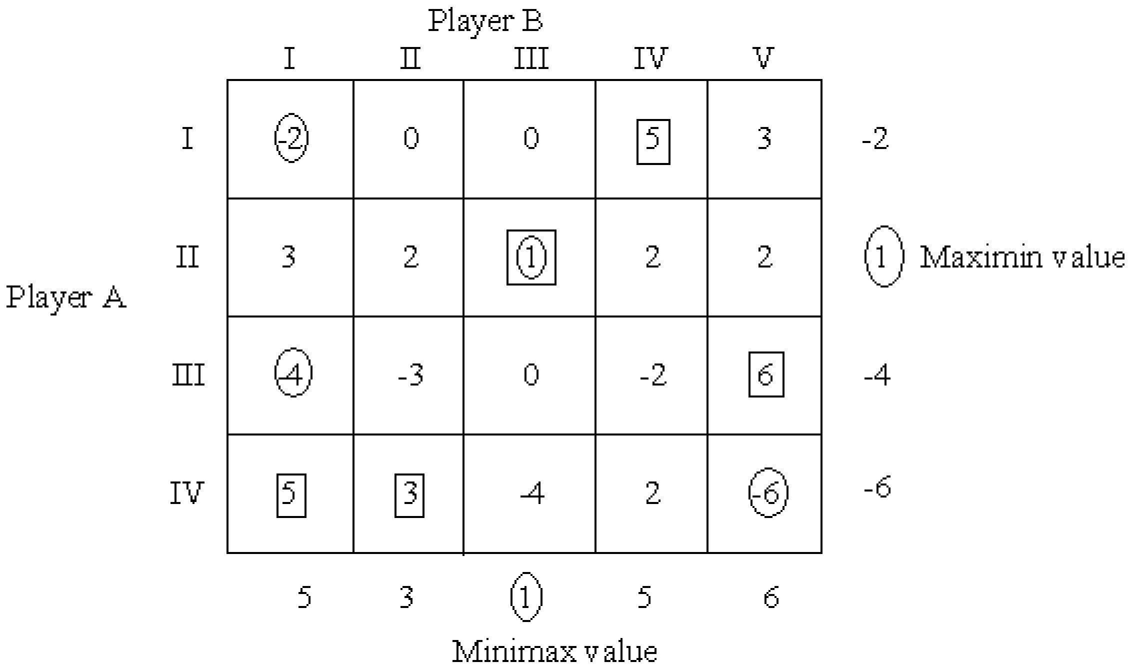
Strategy of player B – B2

Value of the game = 7

2.

		Player B				
		I	II	III	IV	V
	I	-2	0	0	5	3
Player A	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Solution



Player B

Player A	I	II	III	IV	V	
I	-2	0	0	$\boxed{5}$	3	-2
II	3	2	$\boxed{1}$	2	2	① Maximin value
III	-4	-3	0	-2	$\boxed{6}$	-4
IV	$\boxed{5}$	$\boxed{3}$	-4	2	-6	-6
	5	3	①	5	6	

Minimax value

Strategy of player A – II

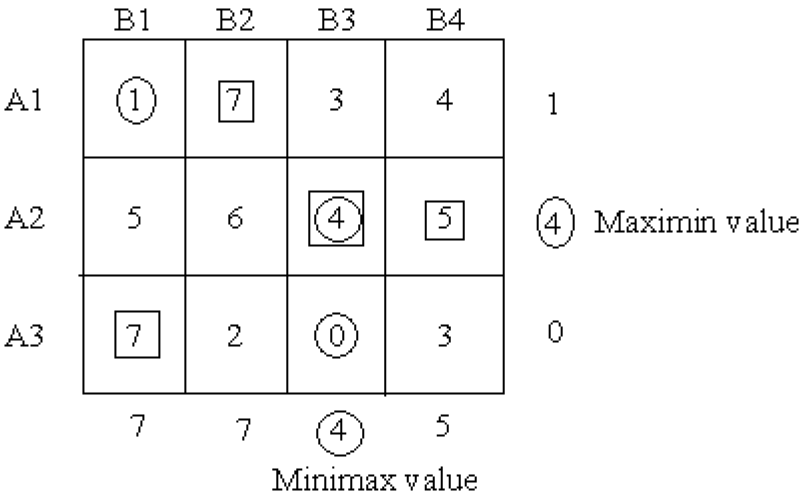
Strategy of player B - III

Value of the game = 1

3..

	B1	B2	B3	B4
A1	1	7	3	4
A2	5	6	4	5
A3	7	2	0	3

Solution



	B1	B2	B3	B4	
A1	①	$\boxed{7}$	3	4	1
A2	5	6	$\boxed{④}$	$\boxed{5}$	④ Maximin value
A3	$\boxed{7}$	2	⓪	3	0
	7	7	④	5	
			Minimax value		

Strategy of player A – A2

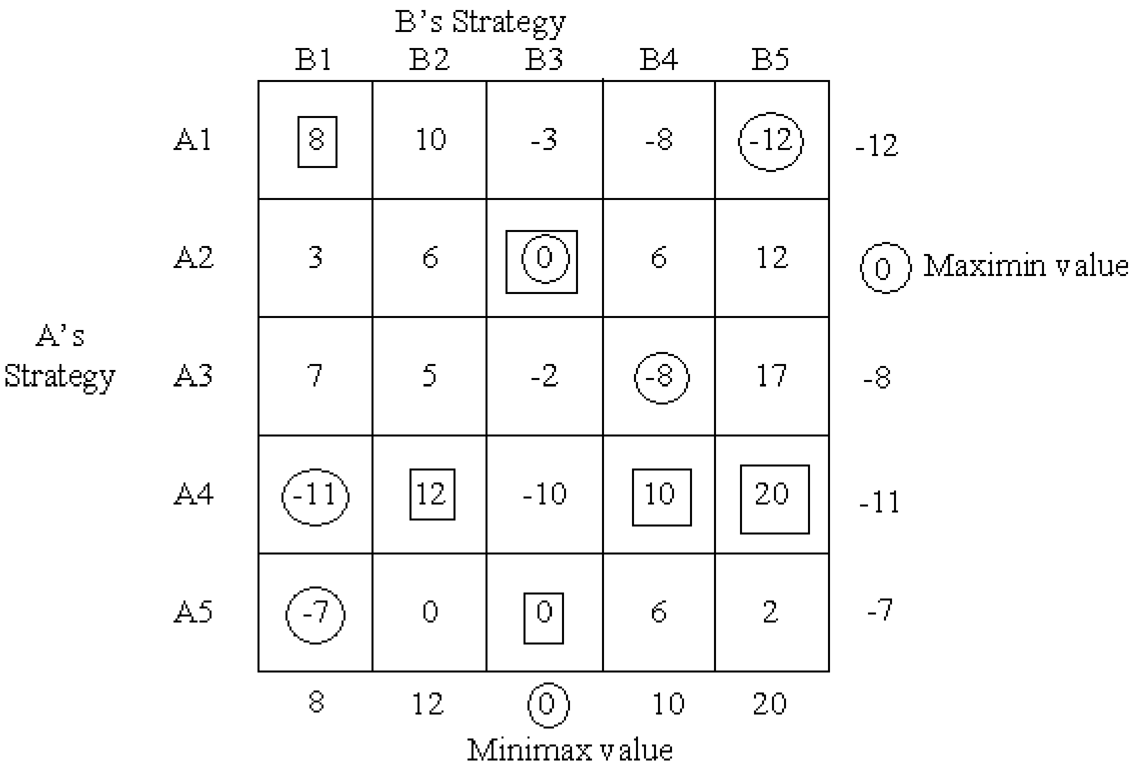
Strategy of player B – B3

Value of the game = 4

4.

		B's Strategy				
		B1	B2	B3	B4	B5
	A1	8	10	-3	-8	-12
A's	A2	3	6	0	6	12
Strategy	A3	7	5	-2	-8	17
	A4	-11	12	-10	10	20
	A5	-7	0	0	6	2

Solution



B's Strategy

A's Strategy	B1	B2	B3	B4	B5	
A1	8	10	-3	-8	-12	-12
A2	3	6	0	6	12	0 Maximin value
A3	7	5	-2	-8	17	-8
A4	-11	12	-10	10	20	-11
A5	-7	0	0	6	2	-7
	8	12	0	10	20	

Minimax value

Strategy of player A – A2

Strategy of player B – B3

Value of the game = 0

5.

$$\begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix}$$

Solution

$\boxed{9}$	3	1	$\boxed{8}$	⓪	0	
6	5	$\boxed{④}$	6	7	④	Maximin value
②	4	3	3	$\boxed{8}$	2	
5	$\boxed{6}$	2	2	①	1	
9	6	④	8	8		

Minimax value

Value of the game = 4

Exercise

1. Explain the concept of game theory.
2. What is a rectangular game?
3. What is a saddle point?
4. Define pure and mixed strategy in a game.
5. What are the characteristics of game theory?
6. Explain two-person zero-sum game giving suitable examples.
7. What are the limitations of game theory?
8. Explain the following terms
    a. Competitive Game
    b. Strategy
    c. Value of the game
    d. Pay-off-matrix
    e. Optimal strategy
9. Explain Maximin and Minimax used in game theory
10. For the game with payoff matrix

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

Determine the best strategies for player A and B and also the value of the game.

Unit 3

3.1 Games with Mixed Strategies

In certain cases, no pure strategy solutions exist for the game. In other words, saddle point does not exist. In all such game, both players may adopt an optimal blend of the strategies called **Mixed Strategy** to find a saddle point. The optimal mix for each player may be determined by assigning each strategy a probability of it being chosen. Thus these mixed strategies are probabilistic combinations of available better strategies and these games hence called **Probabilistic games**.

The probabilistic mixed strategy games without saddle points are commonly solved by any of the following methods

Sl. No.	**Method**	**Applicable to**
1	Analytical Method	2x2 games
2	Graphical Method	2x2, mx2 and 2xn games
3	Simplex Method	2x2, mx2, 2xn and mxn games

3.1.1 Analytical Method

A 2 x 2 payoff matrix where there is no saddle point can be solved by analytical method. Given the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Value of the game is

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

With the coordinates

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} \quad , \quad x_2 = \frac{a_{11} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} \quad , \quad y_2 = \frac{a_{11} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

Alternative procedure to solve the strategy

- Find the difference of two numbers in column 1 and enter the resultant under column 2. Neglect the negative sign if it occurs.
- Find the difference of two numbers in column 2 and enter the resultant under column 1. Neglect the negative sign if it occurs.
- Repeat the same procedure for the two rows.

1. Solve

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Solution

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} & \begin{array}{c} 1 \\ 4 \end{array} \\ & \begin{array}{cc} 3 & 2 \end{array} & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{20 - 3}{9 - 4}$$

$V = 17 / 5$

$S_A = (x_1, x_2) = (1/5, 4/5)$

$S_B = (y_1, y_2) = (3/5, 2/5)$

2. Solve the given matrix

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \end{array} \\ & \begin{array}{cc} 1 & 3 \end{array} & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - 1}{2 + 2}$$

$V = - 1 / 4$

$S_A = (x_1, x_2) = (1/4, 3/4)$

$S_B = (y_1, y_2) = (1/4, 3/4)$

3.1.2 Graphical method

The graphical method is used to solve the games whose payoff matrix has

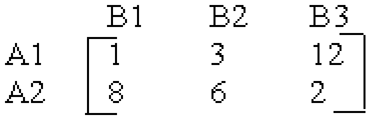
- Two rows and n columns (2 x n)
- m rows and two columns (m x 2)

Algorithm for solving 2 x n matrix games

- Draw two vertical axes 1 unit apart. The two lines are $x_1 = 0$, $x_1 = 1$
- Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.
- The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the highest point of the lower envelope obtained. This will be the **maximin point**.
- The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

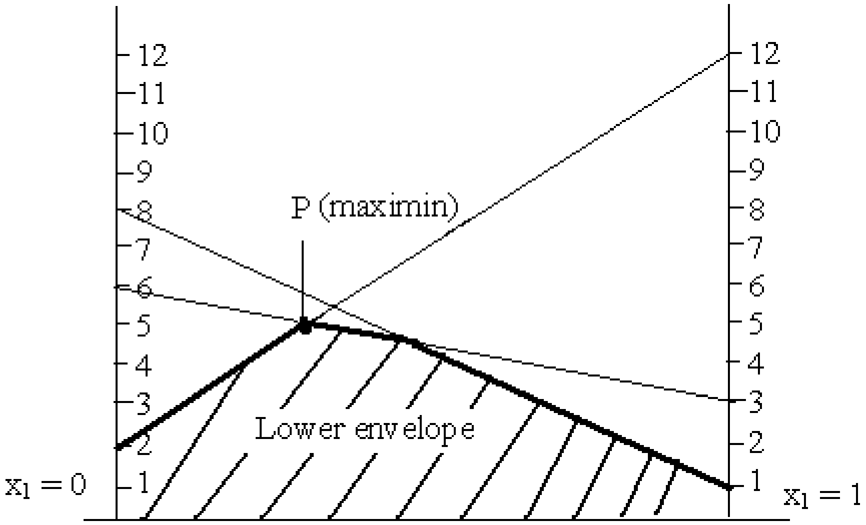
Example 1

Solve by graphical method



$$\begin{array}{c|ccc|} & B1 & B2 & B3 \\ A1 & 1 & 3 & 12 \\ A2 & 8 & 6 & 2 \end{array}$$

Solution



$$\begin{array}{cccc} & B2 & B3 & \\ A1 & 3 & 12 & 4 \\ A2 & 6 & 2 & 9 \\ & 10 & 3 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 72}{5 - 18}$$

V = 66/13

S_A = (4/13, 9 /13)

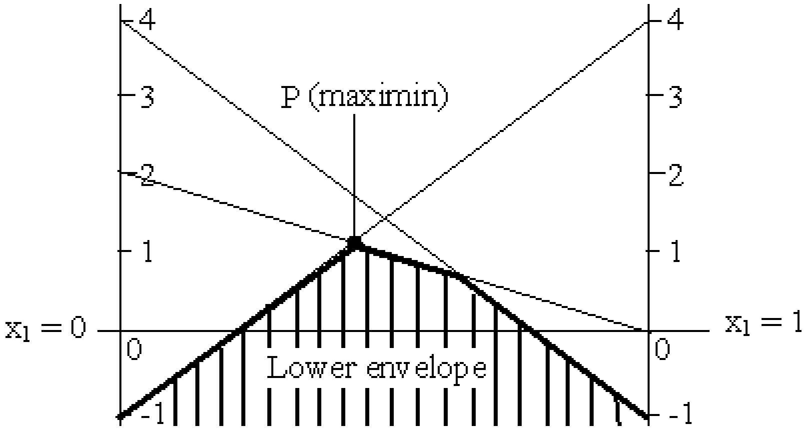
S_B = (0, 10/13, 3 /13)

Example 2

Solve by graphical method

$$\begin{array}{c|ccc} & B1 & B2 & B3 \\ A1 & 4 & -1 & 0 \\ A2 & -1 & 4 & 2 \end{array}$$

Solution



$$\begin{array}{cccc} & B1 & B3 & \\ A1 & 4 & 0 & 3 \\ A2 & -1 & 2 & 4 \\ & 2 & 5 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{8 - 0}{6 + 1}$$

V = 8/7

S_A = (3/7, 4 /7)

S_B = (2/7, 0, 5 /7)

Algorithm for solving m x 2 matrix games

- Draw two vertical axes 1 unit apart. The two lines are x_1 =0, x_1 = 1
- Take the points of the first row in the payoff matrix on the vertical line x_1 = 1 and the points of the second row in the payoff matrix on the vertical line x_1 = 0.
- The point a_{1j} on axis x_1 = 1 is then joined to the point a_{2j} on the axis x_1 = 0 to give a straight line. Draw 'n' straight lines for j=1, 2… n and determine the lowest point of the upper envelope obtained. This will be the **minimax point**.

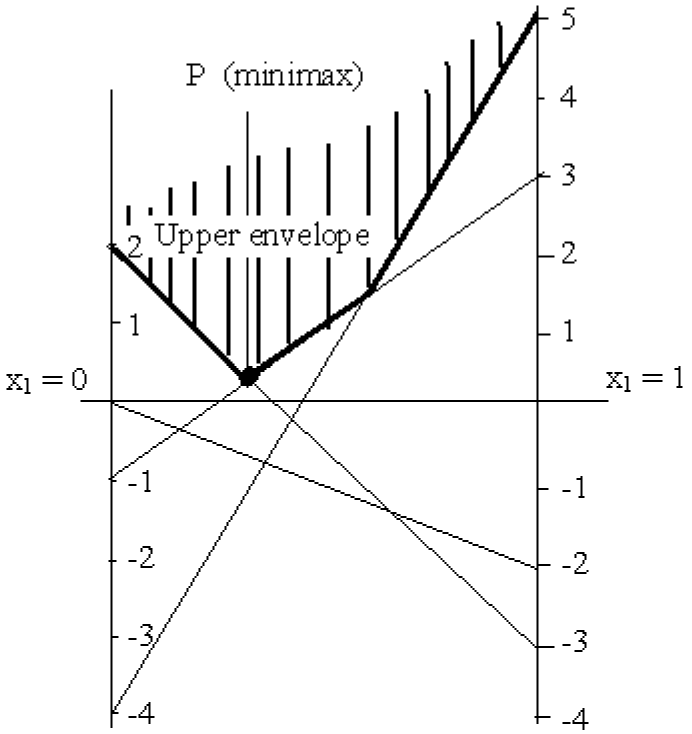
- The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$\begin{array}{c c} & \begin{array}{cc} B1 & B2 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A4 \end{array} & \begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{c c c} & \begin{array}{cc} B1 & B2 \end{array} & \\ \begin{array}{c} A2 \\ A3 \end{array} & \begin{bmatrix} 3 & -1 \\ -3 & 2 \end{bmatrix} & \begin{array}{c} 5 \\ 4 \end{array} \\ & \begin{array}{cc} 3 & 6 \end{array} & \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 3}{5 + 4}$$

V = 3/9 = 1/3

S_A = (0, 5 /9, 4/9, 0)

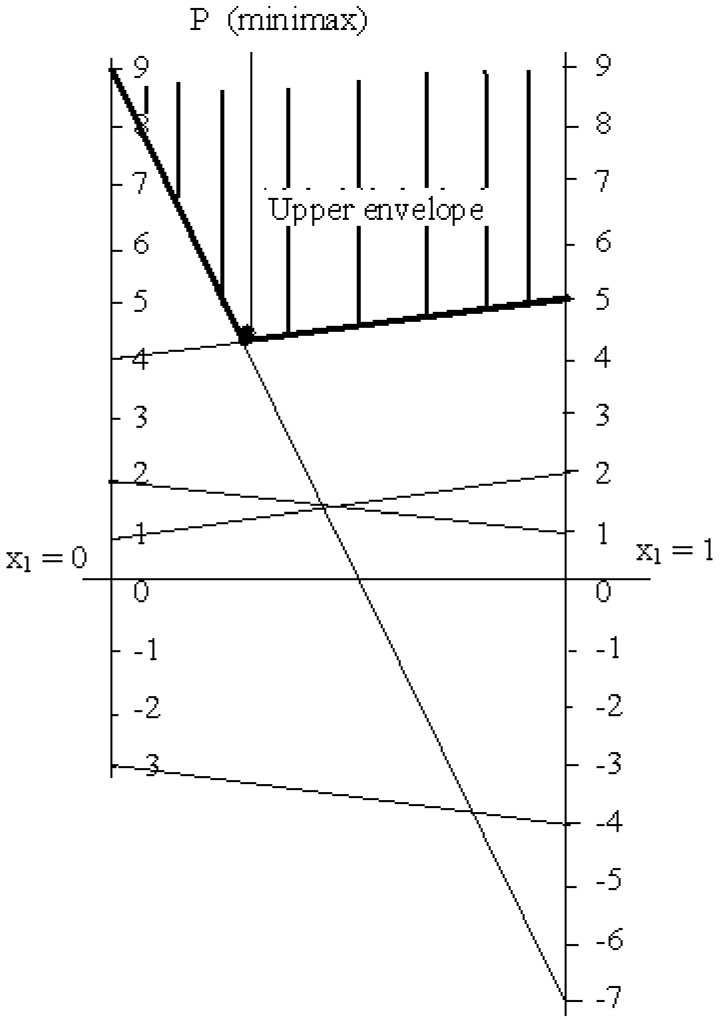
S_B = (3/9, 6 /9)

Example 2

Solve by graphical method

$$\begin{array}{c|cc|} & B1 & B2 \\ A1 & 1 & 2 \\ A2 & 5 & 4 \\ A3 & -7 & 9 \\ A4 & -4 & -3 \\ A5 & 2 & 1 \end{array}$$

Solution



$$\begin{array}{cccc} & B1 & B2 & \\ A2 & 5 & 4 & 16 \\ A3 & -7 & 9 & 1 \\ & 5 & 12 & \end{array}$$

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17

S_A = (0, 16/17, 1/17, 0, 0)

S_B = (5/17, 12 /17)

3.1.3 Simplex Method

Let us consider the 3 x 3 matrix

$$\begin{array}{cccc} & B1 & B2 & B3 \\ A1 & a_{11} & a_{12} & a_{13} \\ A2 & a_{21} & a_{22} & a_{23} \\ A3 & a_{31} & a_{32} & a_{33} \end{array}$$

As per the assumptions, A always attempts to choose the set of strategies with the non-zero probabilities say p_1, p_2, p_3 where $p_1 + p_2 + p_3 = 1$ that maximizes his minimum expected gain.

Similarly B would choose the set of strategies with the non-zero probabilities say q_1, q_2, q_3 where $q_1 + q_2 + q_3 = 1$ that minimizes his maximum expected loss.

Step 1

Find the minimax and maximin value from the given matrix

Step 2

The objective of A is to maximize the value, which is equivalent to minimizing the value 1/V. The LPP is written as

$$\text{Min } 1/V = p_1/V + p_2/V + p_3/V$$

and constraints ≥ 1

It is written as

$$\text{Min } 1/V = x_1 + x_2 + x_3$$

and constraints ≥ 1

Similarly for B, we get the LPP as the dual of the above LPP

$$\text{Max } 1/V = Y_1 + Y_2 + Y_3$$

and constraints ≤ 1

Where $Y_1 = q_1/V$, $Y_2 = q_2/V$, $Y_3 = q_3/V$

Step 3

Solve the LPP by using simplex table and obtain the best strategy for the players

Example 1

Solve by Simplex method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{ccl} & B & \\ A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} & \begin{array}{l} -2 \\ -1 \\ \textcircled{2}\ \text{Maximin} \end{array} \\ & \begin{array}{ccc} \textcircled{3} & 4 & 6 \\ \text{Minimax} & & \end{array} & \end{array}$$

We can infer that $2 \leq V \leq 3$. Hence it can be concluded that the value of the game lies between 2 and 3 and the $V > 0$.

LPP

Max $1/V = Y_1 + Y_2 + Y_3$

Subject to

$3Y_1 – 2Y_2 + 4Y_3 \leq 1$

$-1Y_1 + 4Y_2 + 2Y_3 \leq 1$

$2Y_1 + 2Y_2 + 6Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

SLPP

Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$3Y_1 – 2Y_2 + 4Y_3 + s_1 = 1$

$-1Y_1 + 4Y_2 + 2Y_3 + s_2 =1$

$2Y_1 + 2Y_2 + 6Y_3 + s_3 = 1$

$Y_1, Y_2, Y_3, s_1, s_2, s_3 \geq 0$

		$C_j \rightarrow$	1	1	1	0	0	0	
Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
S_1	0	1	$\boxed{3}$	-2	4	1	0	0	1/3→
S_2	0	1	-1	4	2	0	1	0	-
S_3	0	1	2	2	6	0	0	1	1/2
			↑						
	1/V = 0		-1	-1	-1	0	0	0	
Y_1	1	1/3	1	-2/3	4/3	1/3	0	0	-
S_2	0	4/3	0	10/3	10/3	1/3	1	0	2/5
S_3	0	1/3	0	$\boxed{10/3}$	10/3	-2/3	0	1	1/10→
				↑					
	1/V =1/3		0	-5/3	1/3	1/3	0	0	
Y_1	1	2/5	1	0	2	1/5	0	1/5	
S_2	0	1	0	0	0	1	1	-1	
Y_2	1	1/10	0	1	1	-1/5	0	3/10	
	1/V = 1/2		0	0	2	0	0	1/2	

1/V =1/2

V = 2

$y_1 = 2/5 * 2 = 4/5$

$y_2 = 1/10 * 2 = 1/5$

$y_3 = 0 * 2 = 0$

$x_1 = 0*2 = 0$

$x_2 = 0*2 = 0$

$x_3 = 1/2*2 = 1$

$S_A = (0, 0, 1)$

$S_B = (4/5, 1/5, 0)$

Value = 2

Example 2

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \end{array}$$

Solution

$$\begin{array}{ccc} & B & \\ A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} & \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \\ & \begin{matrix} 1 & \quad 2 & \quad 3 \end{matrix} & \end{array}$$

Maximin = -1

Minimax = 1

We can infer that $-1 \leq V \leq 1$

Since maximin value is -1, it is possible that value of the game may be negative or zero, thus the constant ‘C’ is added to all the elements of matrix which is at least equal to the negative of maximin.

Let C = 1, add this value to all the elements of the matrix. The resultant matrix is

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix} \end{array}$$

LPP

Max $1/V = Y_1 + Y_2 + Y_3$

Subject to

$2Y_1 + 0Y_2 + 0Y_3 \leq 1$

$0Y_1 + 0Y_2 + 4Y_3 \leq 1$

$0Y_1 + 3Y_2 + 0Y_3 \leq 1$

$Y_1, Y_2, Y_3 \geq 0$

SLPP

Max $1/V = Y_1 + Y_2 + Y_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$2Y_1 + 0Y_2 + 0Y_3 + s_1 = 1$

$0Y_1 + 0Y_2 + 4Y_3 + s_2 = 1$

$0Y_1 + 3Y_2 + 0Y_3 + s_3 = 1$

$Y_1, Y_2, Y_3, s_1, s_2, s_3 \geq 0$

C_j→ 1 1 1 0 0 0

Basic Variables	C_B	Y_B	Y_1	Y_2	Y_3	S_1	S_2	S_3	Min Ratio Y_B / Y_K
S_1	0	1	$\boxed{2}$	0	0	1	0	0	1/2→
S_2	0	1	0	0	4	0	1	0	-
S_3	0	1	0	3	0	0	0	1	-
	1/V =0		↑ -1	-1	-1	0	0	0	
Y_1	1	1/2	1	0	0	1/2	0	0	-
S_2	0	1	0	0	4	0	1	0	-
S_3	0	1	0	$\boxed{3}$	0	0	0	1	1/3→
	1/V =1/2		0	↑ -1	-1	1/2	0	0	
Y_1	1	1/2	1	0	0	1/2	0	0	-
S_2	0	1	0	0	$\boxed{4}$	0	1	0	1/4→
Y_2	1	1/3	0	1	0	0	0	1/3	-
	1/V = 5/6		0	0	↑ -1	1/2	0	1/3	
Y_1	1	1/2	1	0	0	1/2	0	0	
Y_3	1	1/4	0	0	1	0	1/4	0	
Y_2	1	1/3	0	1	0	0	0	1/3	
	1/V =13/12		0	0	0	1/2	1/4	1/3	

1/V =13/12

V = 12/13

y_1 = 1/2 * 12/13 = 6/13

y_2 = 1/3 * 12/13 = 4/13

y_3 = 1/4 * 12/13 = 3/13

x_1 = 1/2*12/13 = 6/13

x_2 = 1/4 * 12/13 = 3/13

x_3 = 1/3 * 12/13 = 4/13

S_A = (6/13, 3/13, 4/13)

S_B = (6/13, 4/13, 3/13)

Value = 12/13 – C =12/13 -1 = -1/13

Exercise

1. Explain the method of solving a problem with mixed strategy using algebraic method.

2. Solve the following game graphically

1.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{bmatrix} \end{array}$$

2.

$$\begin{array}{c|cc} & B1 & B2 \\ \hline A1 & 1 & -5 \\ A2 & 3 & 5 \\ A3 & -1 & 6 \\ A4 & 4 & 1 \\ A5 & 2 & 2 \\ A6 & -5 & 0 \end{array}$$

3. Use simplex to solve the following

1.

$$\begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$$

2.

$$\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

4. Two companies A and B are competing for the same product. Their different strategies are given in the following pay off matrix

$$\begin{bmatrix} 2 & -2 & 3 \\ -3 & 5 & -1 \end{bmatrix}$$

Module 6

Unit 1

1.1 Shortest Route Problem

1.2 Minimal Spanning Tree Problem

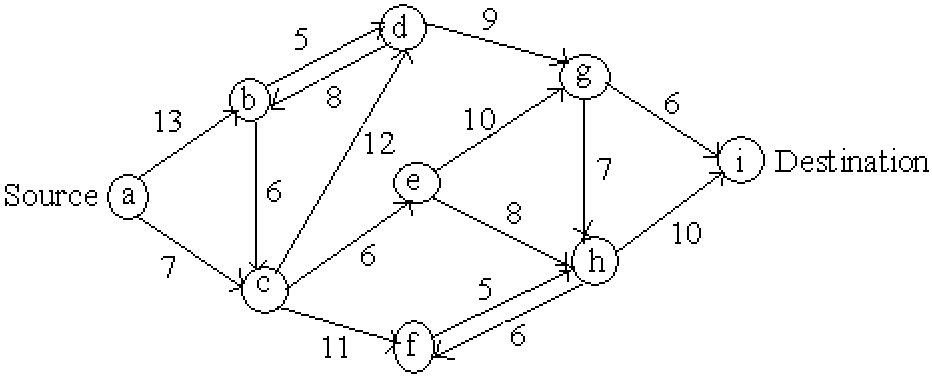
1.3 Maximal Flow Problem

1.1 Shortest Route Problem

The criterion of this method is to find the shortest distance between two nodes with minimal cost.

Example 1

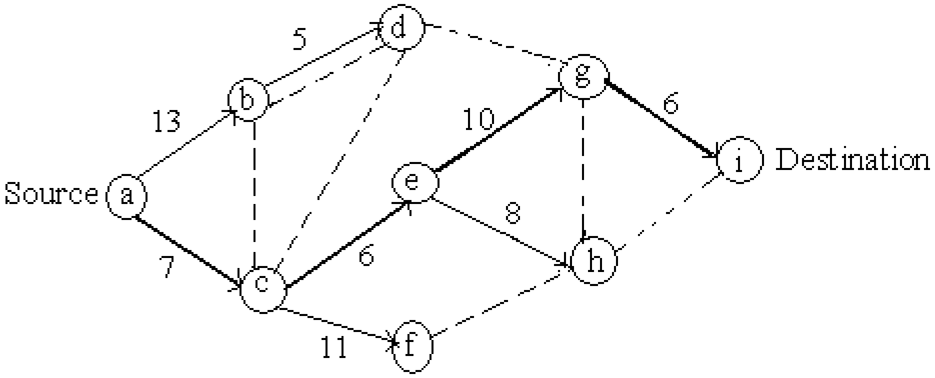
Find the shortest path



Solution

n	Solved nodes directly connected to unsolved nodes	Closest connected unsolved node	Total distance involved	n^{th} nearest node	Minimum distance	Last connection
1	a	c	7	c	7	a-c
2	a c	b e	13 7+6 =13	b e	13 13	a-b c-e
3	b c e	d f h	13+5 =18 7+11 =18 13+8 =21	d f -	18 18 -	b-d c-f -
4	e	h	13+8 =21	h	21	e-h

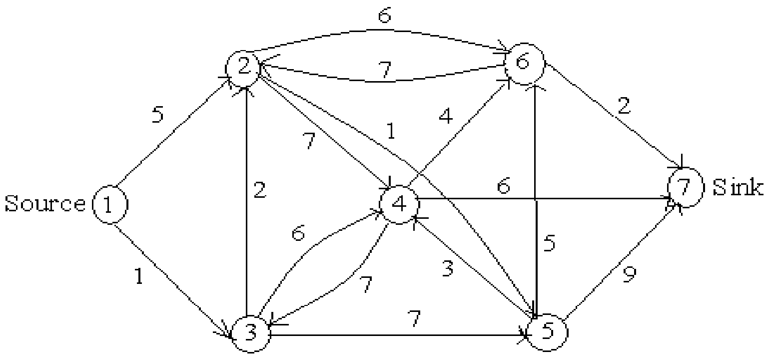
	d	g	18+9 =27	-	-	-
	f	h	18+5 =23	-	-	-
5	e	g	13+10 =23	g	23	e-g
	h	i	21+10 =31	-	-	-
	d	g	18+9 =27	-	-	-
6	g	i	23+6 =29	i	29	g-i
	h	i	21+10 =31	-	-	-



The shortest path from a to i is a → c →e →g → i

Distance = 7 + 6 + 10 + 6 = 29 units

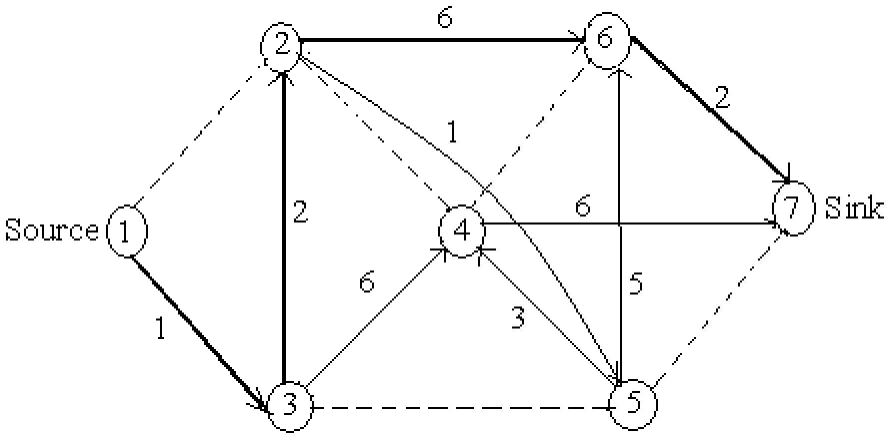
Example 2



Solution

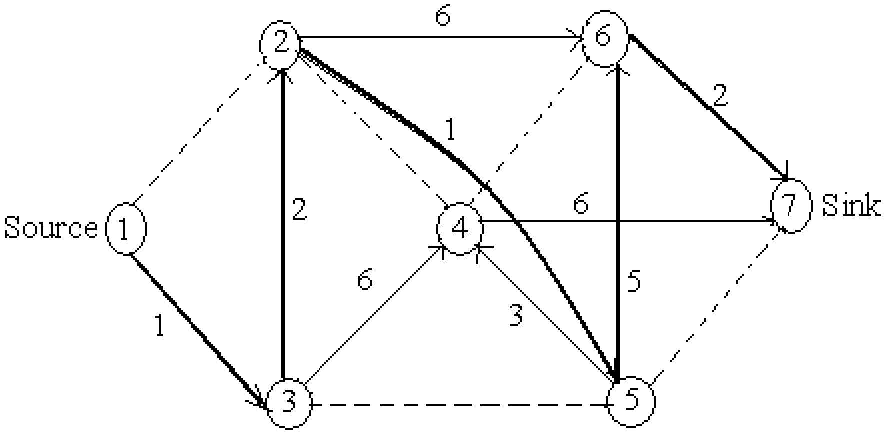
n	Solved nodes directly connected to unsolved nodes	Closest connected unsolved node	Total distance involved	n^{th} nearest node	Minimum distance	Last connection
1	1	3	1	3	1	1-3
2	1 3	2 2	5 1+2 =3	- 2	- 3	- 3-2
3	2 3	5 4	3+1 =4 1+6 =7	5 -	4 -	2-5 -
4	2 3 5	6 4 4	3+6 =9 1+6 =7 4+3 =7	- 4 4	- 7 7	- 3-4 5-4
5	2 4 5	6 6 6	3+6 =9 7+4 =11 4+5 =9	6 - 6	9 - 9	2-6 - 5-6
6	4 5 6	7 7 7	7+6 =13 4+9 =13 9+2 =11	- - 7	- - 11	- - 6-7

The shortest path from 1 to 7 can be



1 →3 → 2 → 6 →7

Total distance is 11 units



1 → 3 → 2 →5 → 6 →7

Total distance = 11 units

1.2 Minimal Spanning Tree Problem

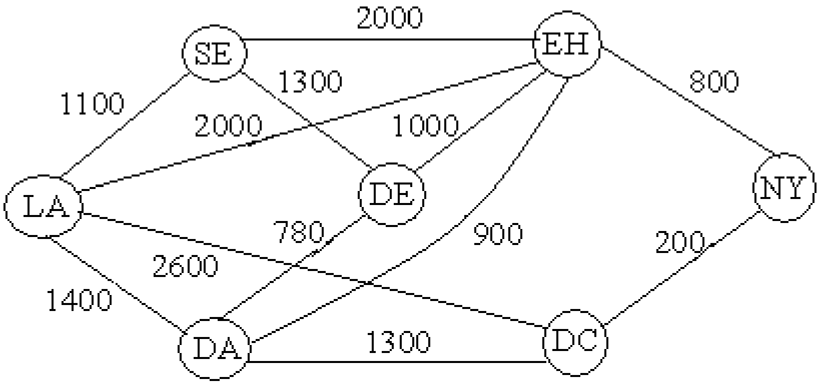
A tree is defined to be an undirected, acyclic and connected graph. A spanning tree is a subgraph of G (undirected, connected graph), is a tree and contains all the vertices of G. A minimum spanning tree is a spanning tree but has weights or lengths associated with edges and the total weight is at the minimum.

Prim's Algorithm

- It starts at any vertex (say A) in a graph and finds the least cost vertex (say B) connected to the start vertex.
- Now either from A or B, it will find the next least costly vertex connection, without creating cycle (say C)
- Now either from A, B or C find the next least costly vertex connection, without creating a cycle and so on.
- Eventually all the vertices will be connected without any cycles and a minimum spanning tree will be the result.

Example 1

Suppose it is desired to establish a cable communication network that links major cities, which is shown in the figure. Determine how the cities are connected such that the total cable mileage is minimized.



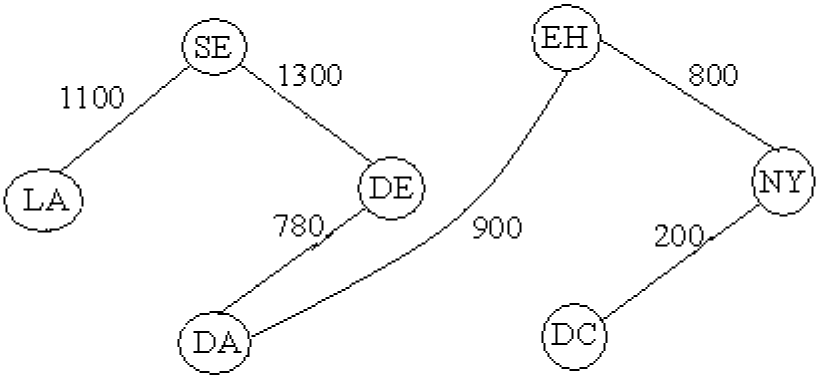
Solution

C = {LA} C' = {SE, DE, DA, EH, NY, DC}

C = {LA, SE} C' = {DE, DA, EH, NY, DC}

C = {LA, SE, DE}	C' = {DA, EH, NY, DC}
C = {LA, SE, DE, DA}	C' = {EH, NY, DC}
C = {LA, SE, DE, DA, EH}	C' = {NY, DC}
C = {LA, SE, DE, DA, EH, NY}	C' = {DC}
C = {LA, SE, DE, DA, EH, NY, DC}	C' = { }

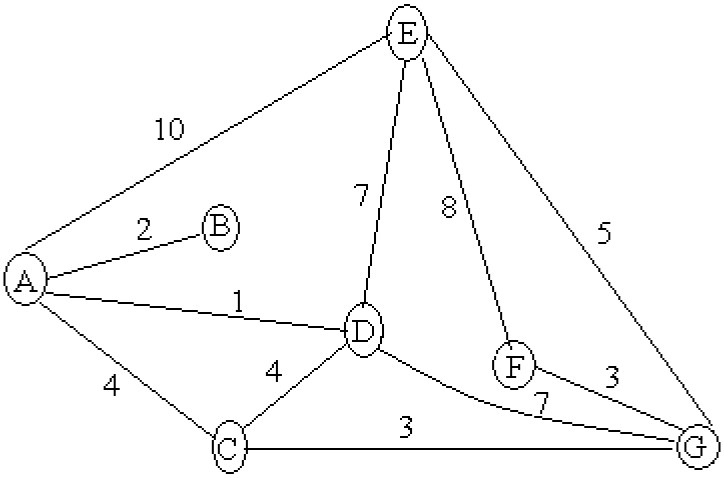
The resultant network is



Thus the total cable mileage is 1100 + 1300 + 780 + 900 + 800 + 200 = 5080

Example 2

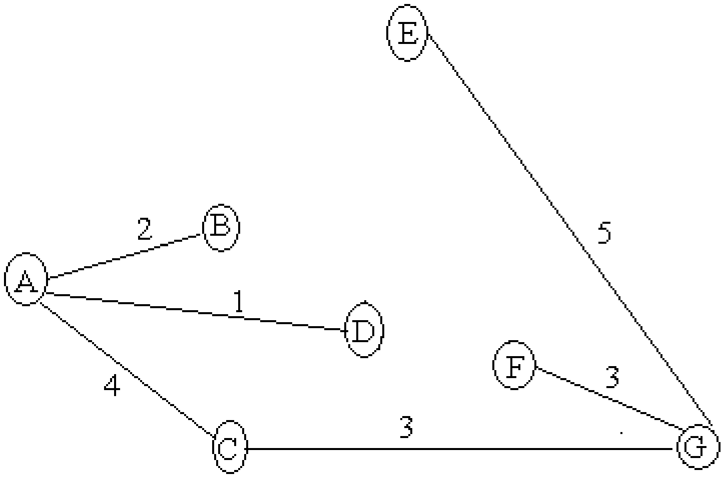
For the following graph obtain the minimum spanning tree. The numbers on the branches represent the cost.



Solution

C = {A}   C' = {B, C, D, E, F, G}
C = {A, D}   C' = {B, C, E, F, G}
C = {A, D, B}   C' = {C, E, F, G}
C = {A, D, B, C}   C' = {E, F, G}
C = {A, D, B, C, G}   C' = {E, F}
C = {A, D, B, C, G, F}   C' = {E}
C = {A, D, B, C, G, F, E}   C' = { }

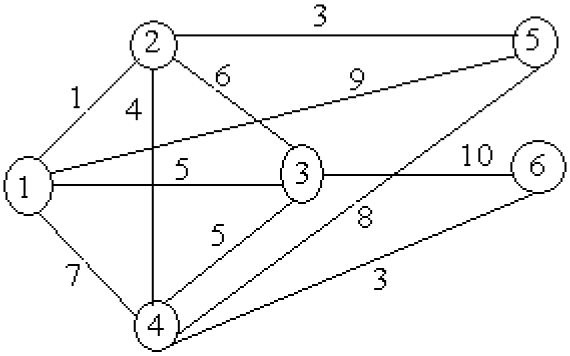
The resultant network is



Cost = 2 + 1 + 4 + 3 + 3 + 5 = 18 units

Example 3

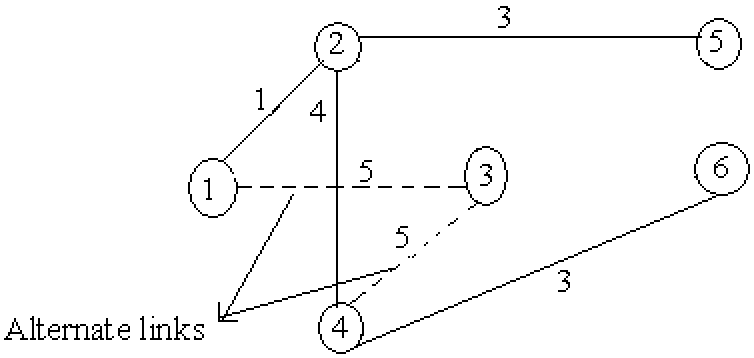
Solve the minimum spanning problem for the given network. The numbers on the branches represent in terms of miles.



Solution

C = {1}	C' = {2, 3, 4, 5, 6}
C = {1, 2}	C' = {3, 4, 5, 6}
C = {1, 2, 5}	C' = {3, 4, 6}
C = {1, 2, 5, 4}	C' = {3, 6}
C = {1, 2, 5, 4, 6}	C' = {3}
C = {1, 2, 5, 4, 6, 3}	C' = {}

The resultant network is



1 + 4 + 5+ 3 + 3 = 16 miles

1.3 Maximal Flow Problem

Algorithm

Step1

Find a path from source to sink that can accommodate a positive flow of material. If no path exists go to step 5

Step2

Determine the maximum flow that can be shipped from this path and denote by ‘k’ units.

Step3

Decrease the direct capacity (the capacity in the direction of flow of k units) of each branch of this path ‘k’ and increase the reverse capacity k_1. Add ‘k’ units to the amount delivered to sink.

Step4

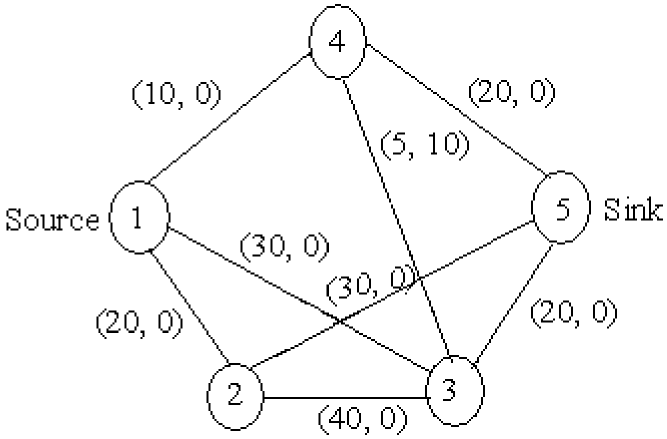
Goto step1

Step5

The maximal flow is the amount of material delivered to the sink. The optimal shipping schedule is determined by comparing the original network with the final network. Any reduction in capacity signifies shipment.

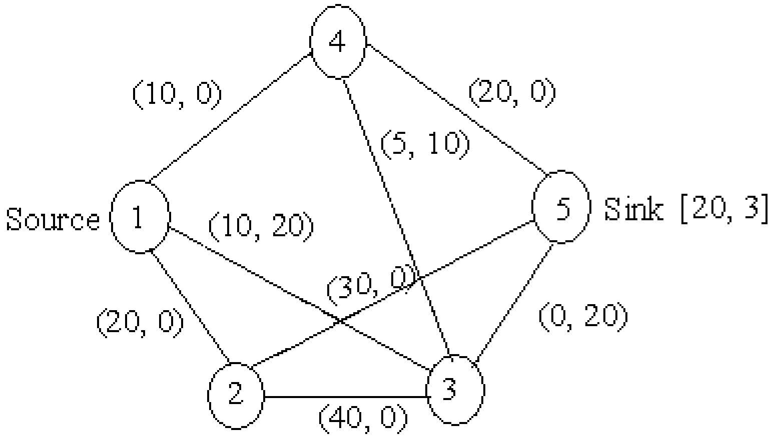
Example 1

Consider the following network and determine the amount of flow between the networks.

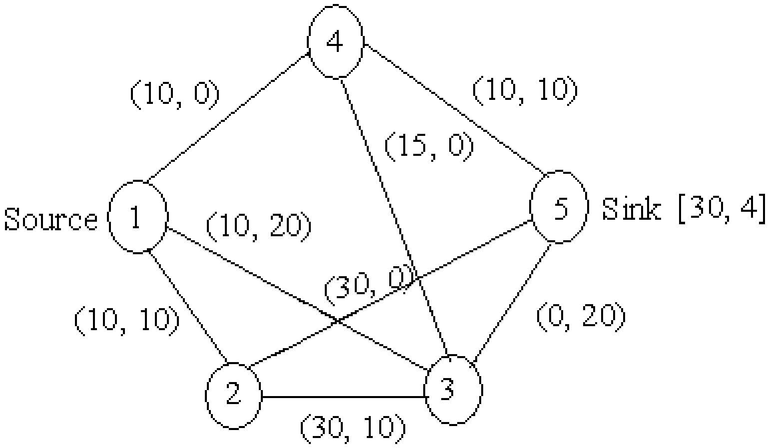


Solution

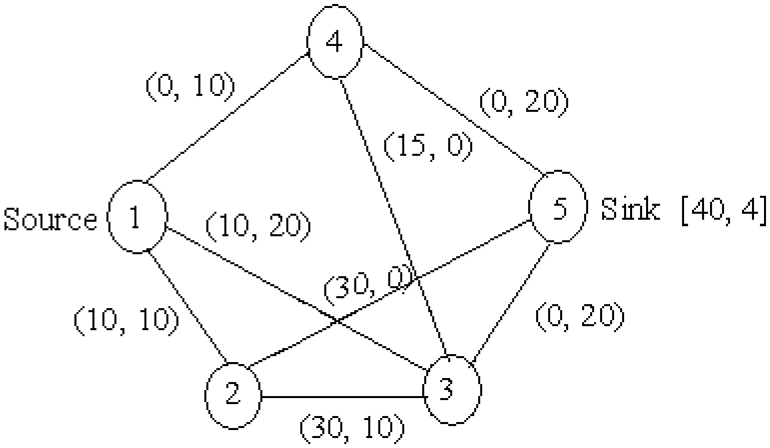
Iteration 1: 1 – 3 – 5



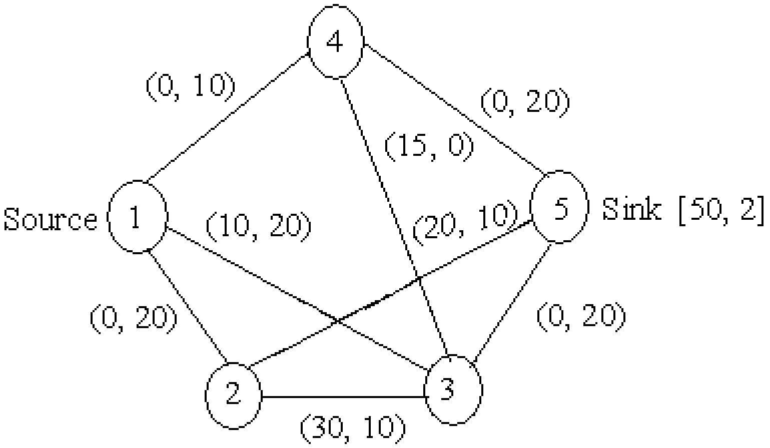
Iteration 2: 1 – 2 – 3 – 4 – 5



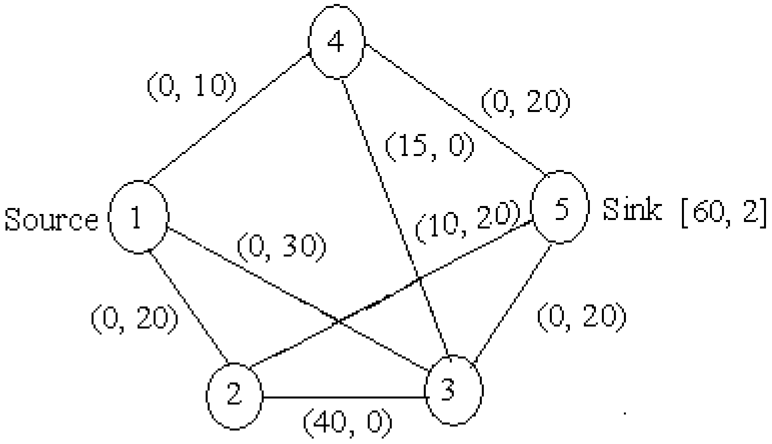
Iteration 3: 1 – 4 – 5



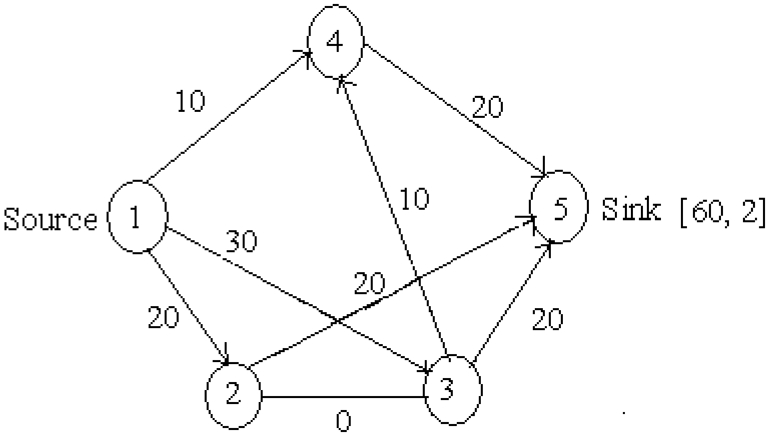
Iteration 4: 1 – 2 – 5



Iteration 5: 1 – 3 – 2 – 5

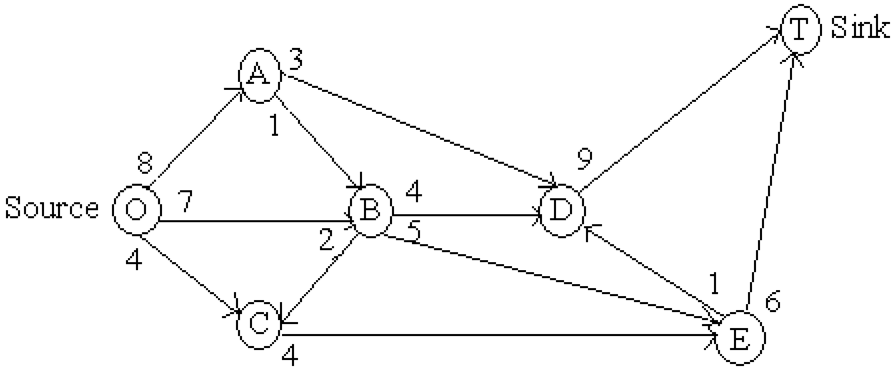


Maximum flow is 60 units. Therefore the network can be written as



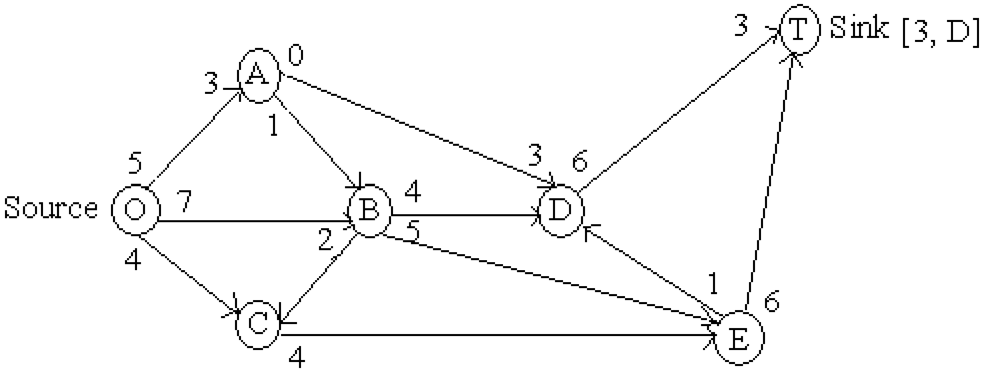
Example 2

Solve the maximal flow problem

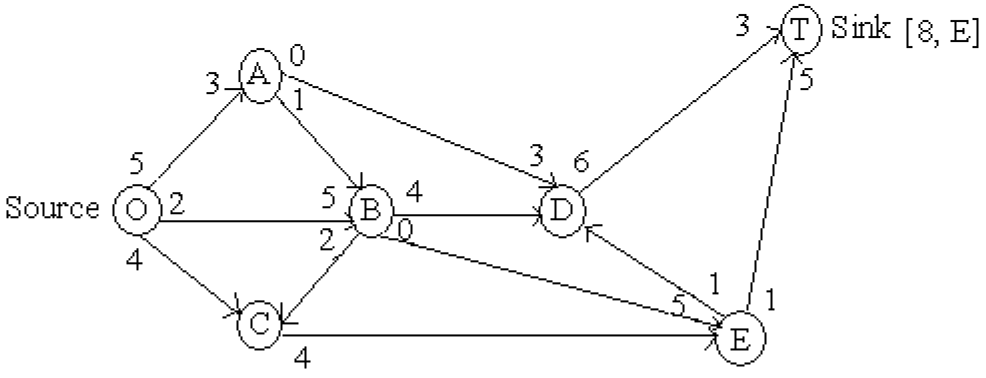


Solution

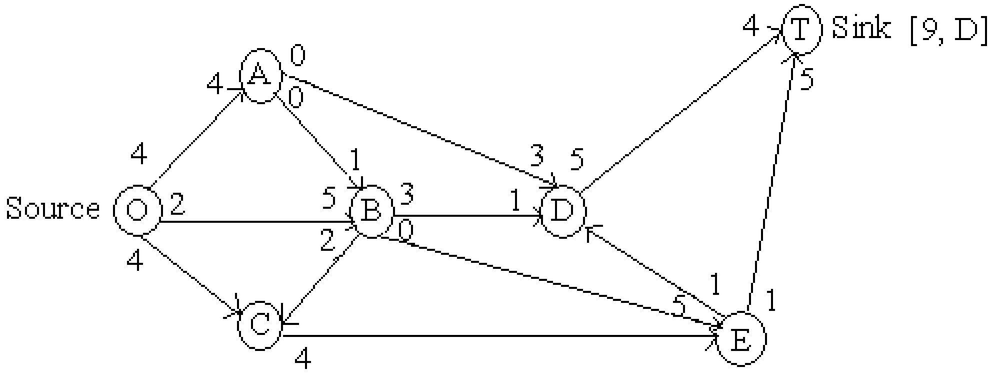
Iteration 1: O – A – D – T



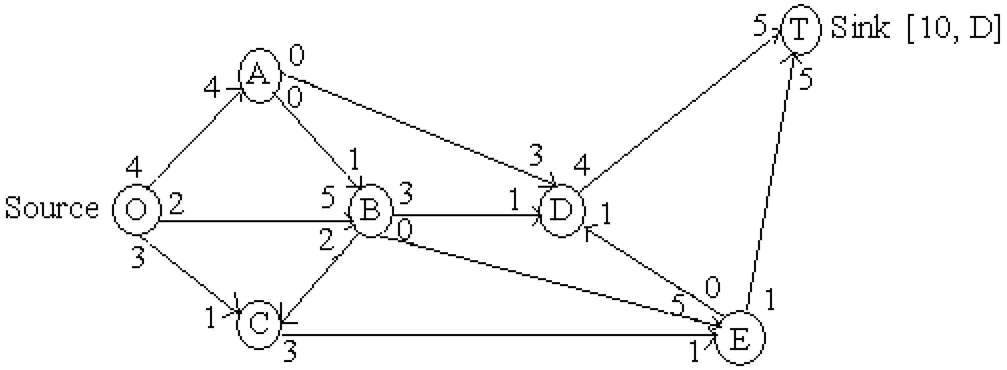
Iteration 2: O – B – E – T



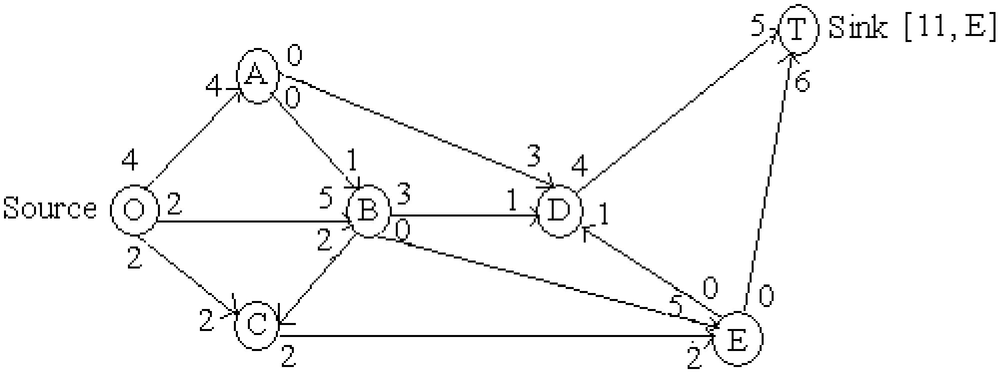
Iteration 3: O – A – B – D – T



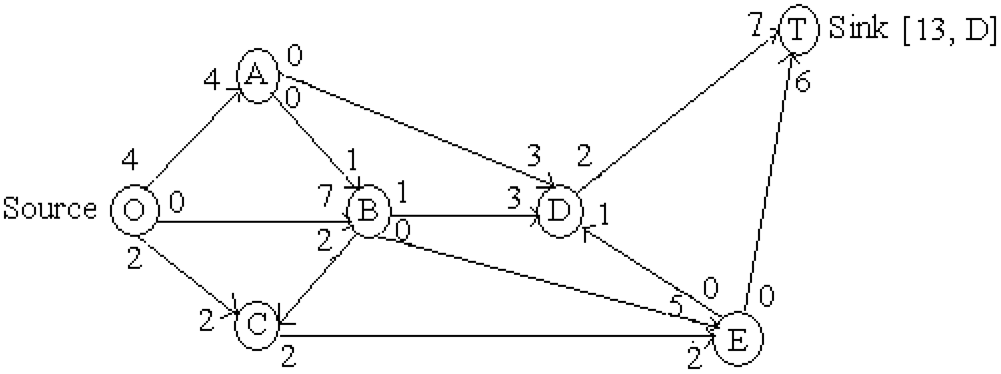
Iteration 4: O – C – E – D – T



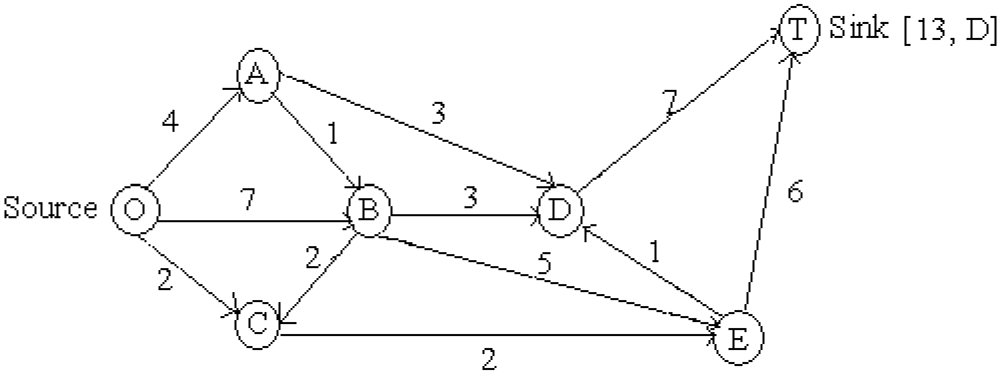
Iteration 5: O – C – E – T



Iteration 6: O – B – D – T

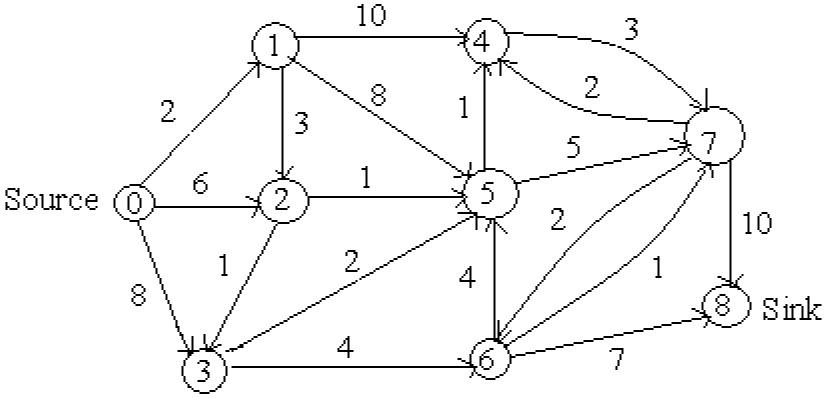


Therefore there are no more augmenting paths. So the current flow pattern is optimal. The maximum flow is 13 units.

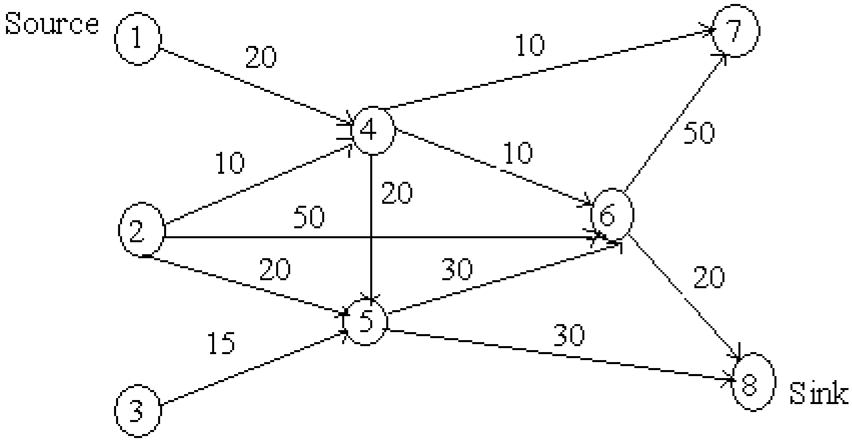


Exercise

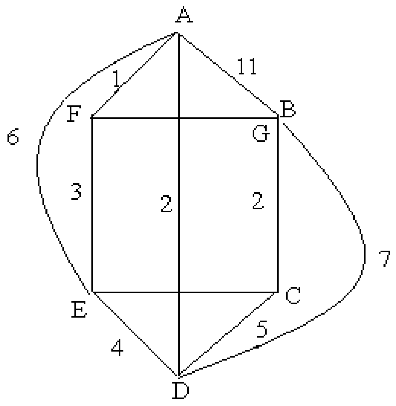
1. Find the shortest path



2. Solve the maximal flow problem



3. Explain prim’s algorithm
4. Solve the minimal spanning tree



Unit 2

2.1 Introduction to CPM / PERT Techniques

CPM/PERT or Network Analysis as the technique is sometimes called, developed along two parallel streams, one industrial and the other military.

CPM (Critical Path Method) was the discovery of M.R.Walker of E.I.Du Pont de Nemours & Co. and J.E.Kelly of Remington Rand, circa 1957. The computation was designed for the UNIVAC-I computer. The first test was made in 1958, when CPM was applied to the construction of a new chemical plant. In March 1959, the method was applied to maintenance shut-down at the Du Pont works in Louisville, Kentucky. Unproductive time was reduced from 125 to 93 hours.

PERT (Project Evaluation and Review Technique) was devised in 1958 for the POLARIS missile program by the Program Evaluation Branch of the Special Projects office of the U.S.Navy, helped by the Lockheed Missile Systems division and the Consultant firm of Booz-Allen & Hamilton. The calculations were so arranged so that they could be carried out on the IBM Naval Ordinance Research Computer (NORC) at Dahlgren, Virginia.

The methods are essentially **network-oriented techniques** using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be **deterministic** while in PERT these are described **probabilistically**. These techniques are referred as **project scheduling** techniques.

In **CPM** activities are shown as a network of precedence relationships using activity-on-node network construction

- Single estimate of activity time
- Deterministic activity times

USED IN: Production management - for the jobs of repetitive in nature where the activity time estimates can be predicted with considerable certainty due to the existence of past experience.

In **PERT** activities are shown as a network of precedence relationships using activity-on-arrow network construction

- Multiple time estimates
- Probabilistic activity times

USED IN: Project management - for non-repetitive jobs (research and development work), where the time and cost estimates tend to be quite uncertain. This technique uses probabilistic time estimates.

Benefits of PERT/CPM

- Useful at many stages of project management
- Mathematically simple
- Give critical path and slack time
- Provide project documentation
- Useful in monitoring costs

Limitations of PERT/CPM

- Clearly defined, independent and stable activities
- Specified precedence relationships
- Over emphasis on critical paths

2.2 Applications of CPM / PERT

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations. These include

- Construction of a dam or a canal system in a region
- Construction of a building or highway
- Maintenance or overhaul of airplanes or oil refinery
- Space flight
- Cost control of a project using PERT / COST
- Designing a prototype of a machine
- Development of supersonic planes

2.3 Basic Steps in PERT / CPM

Project scheduling by PERT / CPM consists of four main steps

1. **Planning**

- The planning phase is started by splitting the total project in to small projects. These smaller projects in turn are divided into activities and are analyzed by the department or section.
- The relationship of each activity with respect to other activities are defined and established and the corresponding responsibilities and the authority are also stated.
- Thus the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

2. **Scheduling**

- The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project.
- Moreover the schedule must pinpoint the critical path activities which require special attention if the project is to be completed in time.
- For non-critical activities, the schedule must show the amount of slack or float times which can be used advantageously when such activities are delayed or when limited resources are to be utilized effectively.

3. **Allocation of resources**

- Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labour, finance, equipment and space which will impose a limitation on time for the project.

- When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential.
- Resource allocation usually incurs a compromise and the choice of this compromise depends on the judgment of managers.

4. Controlling

- The final phase in project management is controlling. Critical path methods facilitate the application of the principle of management by expectation to identify areas that are critical to the completion of the project.
- By having progress reports from time to time and updating the network continuously, a better financial as well as technical control over the project is exercised.
- Arrow diagrams and time charts are used for making periodic progress reports. If required, a new course of action is determined for the remaining portion of the project.

2.4 The Framework for PERT and CPM

Essentially, there are six steps which are common to both the techniques. The procedure is listed below:

I. Define the Project and all of its significant activities or tasks. The Project (made up of several tasks) should have only a single start activity and a single finish activity.

II. Develop the relationships among the activities. Decide which activities must precede and which must follow others.

III. Draw the "Network" connecting all the activities. Each Activity should have unique event numbers. Dummy arrows are used where required to avoid giving the same numbering to two activities.

IV. Assign time and/or cost estimates to each activity

V. Compute the longest time path through the network. This is called the critical path.

VI. Use the Network to help plan, schedule, and monitor and control the project.

The Key Concept used by CPM/PERT is that a small set of activities, which make up the longest path through the activity network control the entire project. If these "critical" activities could be identified and assigned to responsible persons, management resources could be optimally used by concentrating on the few activities which determine the fate of the entire project.

Non-critical activities can be replanned, rescheduled and resources for them can be reallocated flexibly, without affecting the whole project.

Five useful questions to ask when preparing an activity network are:

- Is this a Start Activity?
- Is this a Finish Activity?
- What Activity Precedes this?
- What Activity Follows this?
- What Activity is Concurrent with this?

2.5 Network Diagram Representation

In a network representation of a project certain definitions are used

1. Activity

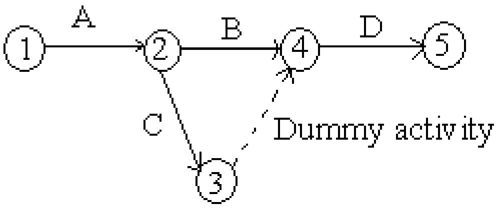
Any individual operation which utilizes resources and has an end and a beginning is called activity. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. These are classified into four categories

1. **Predecessor activity** – Activities that must be completed immediately prior to the start of another activity are called predecessor activities.
2. **Successor activity** – Activities that cannot be started until one or more of other activities are completed but immediately succeed them are called successor activities.
3. **Concurrent activity** – Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of other activities.
4. **Dummy activity** – An activity which does not consume any kind of resource but merely depicts the technological dependence is called a dummy activity.

The dummy activity is inserted in the network to clarify the activity pattern in the following two situations

- To make activities with common starting and finishing points distinguishable
- To identify and maintain the proper precedence relationship between activities that is not connected by events.

For example, consider a situation where A and B are concurrent activities. C is dependent on A and D is dependent on A and B both. Such a situation can be handled by using a dummy activity as shown in the figure.

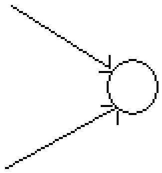


2. Event

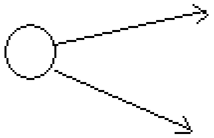
An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle in a network which is also called a node or connector.

The events are classified in to three categories

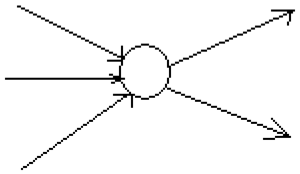
1. **Merge event** – When more than one activity comes and joins an event such an event is known as merge event.
2. **Burst event** – When more than one activity leaves an event such an event is known as burst event.
3. **Merge and Burst event** – An activity may be merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.



Merge event



Burst event



Merge and Burst event

3. Sequencing

The first prerequisite in the development of network is to maintain the precedence relationships. In order to make a network, the following points should be taken into considerations

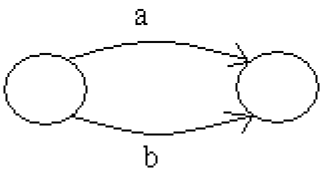
- What job or jobs precede it?
- What job or jobs could run concurrently?
- What job or jobs follow it?
- What controls the start and finish of a job?

Since all further calculations are based on the network, it is necessary that a network be drawn with full care.

2.6 Rules for Drawing Network Diagram

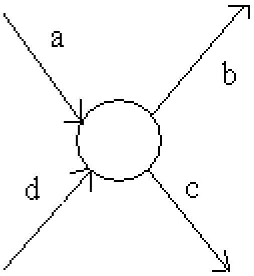
Rule 1

Each activity is represented by one and only one arrow in the network



Rule 2

No two activities can be identified by the same end events



Rule 3

In order to ensure the correct precedence relationship in the arrow diagram, following questions must be checked whenever any activity is added to the network

- What activity must be completed immediately before this activity can start?
- What activities must follow this activity?
- What activities must occur simultaneously with this activity?

In case of large network, it is essential that certain good habits be practiced to draw an easy to follow network

- Try to avoid arrows which cross each other
- Use straight arrows
- Do not attempt to represent duration of activity by its arrow length
- Use arrows from left to right. Avoid mixing two directions, vertical and standing arrows may be used if necessary.
- Use dummies freely in rough draft but final network should not have any redundant dummies.

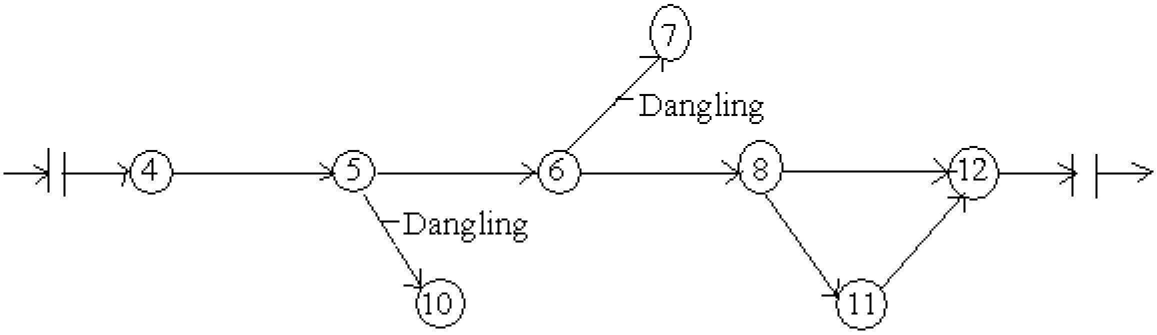
- The network has only one entry point called start event and one point of emergence called the end event.

2.7 Common Errors in Drawing Networks

The three types of errors are most commonly observed in drawing network diagrams

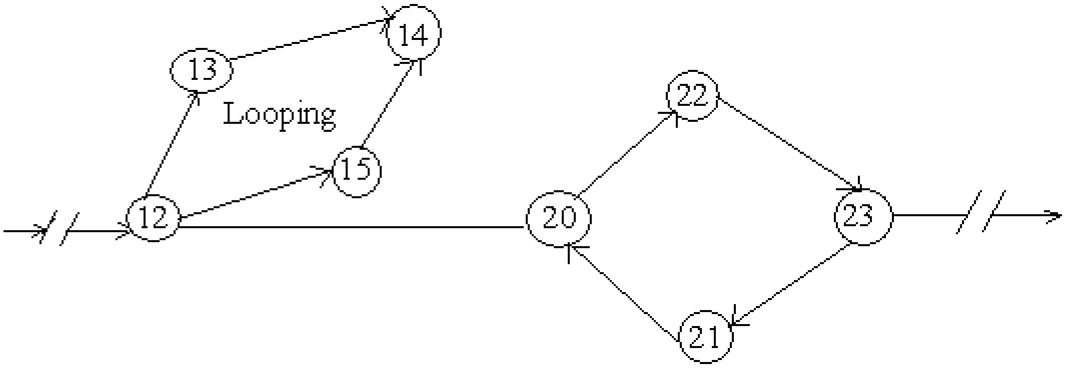
1. Dangling

To disconnect an activity before the completion of all activities in a network diagram is known as dangling. As shown in the figure activities (5 – 10) and (6 – 7) are not the last activities in the network. So the diagram is wrong and indicates the error of dangling



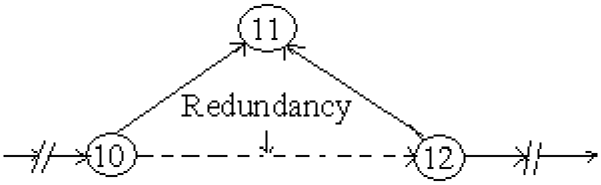
2. Looping or Cycling

Looping error is also known as cycling error in a network diagram. Drawing an endless loop in a network is known as error of looping as shown in the following figure.



3. Redundancy

Unnecessarily inserting the dummy activity in network logic is known as the error of redundancy as shown in the following diagram



2.8 Advantages and Disadvantages

PERT/CPM has the following advantages

- A PERT/CPM chart explicitly defines and makes visible dependencies (precedence relationships) between the elements,
- PERT/CPM facilitates identification of the critical path and makes this visible,
- PERT/CPM facilitates identification of early start, late start, and slack for each activity,
- PERT/CPM provides for potentially reduced project duration due to better understanding of dependencies leading to improved overlapping of activities and tasks where feasible.

PERT/CPM has the following disadvantages:

- There can be potentially hundreds or thousands of activities and individual dependency relationships,
- The network charts tend to be large and unwieldy requiring several pages to print and requiring special size paper,
- The lack of a timeframe on most PERT/CPM charts makes it harder to show status although colours can help (e.g., specific colour for completed nodes),

- When the PERT/CPM charts become unwieldy, they are no longer used to manage the project.

2.9 Critical Path in Network Analysis

Basic Scheduling Computations

The notations used are

(i, j) = Activity with tail event i and head event j

E_i = Earliest occurrence time of event i

L_j = Latest allowable occurrence time of event j

D_{ij} = Estimated completion time of activity (i, j)

$(Es)_{ij}$ = Earliest starting time of activity (i, j)

$(Ef)_{ij}$ = Earliest finishing time of activity (i, j)

$(Ls)_{ij}$ = Latest starting time of activity (i, j)

$(Lf)_{ij}$ = Latest finishing time of activity (i, j)

The procedure is as follows

1. **Determination of Earliest time (E_j): Forward Pass computation**

- **Step 1**

  The computation begins from the start node and move towards the end node. For easiness, the forward pass computation starts by assuming the earliest occurrence time of zero for the initial project event.

- **Step 2**
  i. Earliest starting time of activity (i, j) is the earliest event time of the tail end event i.e. $(Es)_{ij} = E_i$

ii. Earliest finish time of activity (i, j) is the earliest starting time + the activity time i.e. $(Ef)_{ij} = (Es)_{ij} + D_{ij}$ or $(Ef)_{ij} = E_i + D_{ij}$

iii. Earliest event time for event j is the maximum of the earliest finish times of all activities ending in to that event i.e. $E_j = \max [(Ef)_{ij}$ for all immediate predecessor of (i, j)] or $E_j = \max [E_i + D_{ij}]$

2. Backward Pass computation (for latest allowable time)

- **Step 1**

  For ending event assume E = L. Remember that all E's have been computed by forward pass computations.

- **Step 2**

  Latest finish time for activity (i, j) is equal to the latest event time of event j i.e. $(Lf)_{ij} = L_j$

- **Step 3**

  Latest starting time of activity (i, j) = the latest completion time of (i, j) – the activity time or $(Ls)_{ij} = (Lf)_{ij} - D_{ij}$ or $(Ls)_{ij} = L_j - D_{ij}$

- **Step 4**

  Latest event time for event 'i' is the minimum of the latest start time of all activities originating from that event i.e. $L_i = \min [(Ls)_{ij}$ for all immediate successor of (i, j)] $= \min [(Lf)_{ij} - D_{ij}] = \min [L_j - D_{ij}]$

3. Determination of floats and slack times

There are three kinds of floats

- **Total float** – The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

  Mathematically

  $(Tf)_{ij}$ = (Latest start – Earliest start) for activity (i – j)

  $(Tf)_{ij} = (Ls)_{ij} - (Es)_{ij}$ or $(Tf)_{ij} = (L_j - D_{ij}) - E_i$

- **Free float –** The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent activity.

  Mathematically

  $(Ff)_{ij}$ = (Earliest time for event j – Earliest time for event i) – Activity time for (i, j)

  $(Ff)_{ij} = (E_j - E_i) - D_{ij}$

- **Independent float** – The amount of time by which the start of an activity can be delayed without effecting the earliest start time of any immediately following activities, assuming that the preceding activity has finished at its latest finish time.

  Mathematically

  $(If)_{ij} = (E_j - L_i) - D_{ij}$

  The negative independent float is always taken as zero.

- **Event slack -** It is defined as the difference between the latest event and earliest event times.

  Mathematically

  Head event slack = $L_j – E_j$, Tail event slack = $L_i - E_i$

4. Determination of critical path

- **Critical event** – The events with zero slack times are called critical events. In other words the event i is said to be critical if $E_i = L_i$

- **Critical activity** – The activities with zero total float are known as critical activities. In other words an activity is said to be critical if a delay in its start will cause a further delay in the completion date of the entire project.

- **Critical path** – The sequence of critical activities in a network is called critical path. The critical path is the longest path in the network from the starting event to ending event and defines the minimum time required to complete the project.

Exercise

1. What is PERT and CPM?
2. What are the advantages of using PERT/CPM?
3. Mention the applications of PERT/CPM
4. Explain the following terms
    a. Earliest time
    b. Latest time
    c. Total activity slack
    d. Event slack
    e. Critical path
5. Explain the CPM in network analysis.
6. What are the rules for drawing network diagram? Also mention the common errors that occur in drawing networks.
7. What is the difference between PERT and CPM/
8. What are the uses of PERT and CPM?
9. Explain the basic steps in PERT/CPM techniques.
10. Write the framework of PERT/CPM.

Unit 3

3.1 Worked Examples on CPM

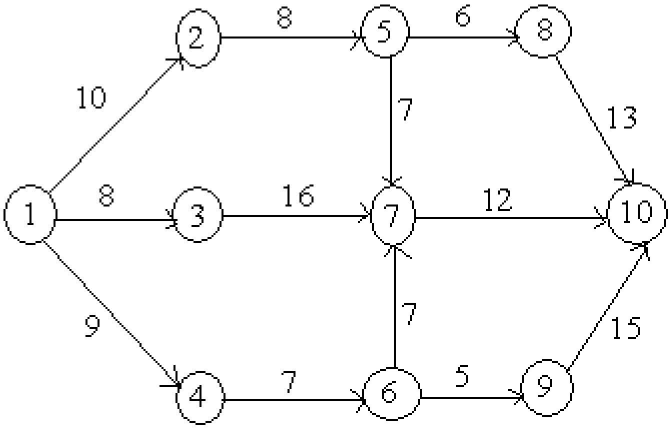
3.2 PERT

3.3 Worked Examples

3.1 Worked Examples on CPM

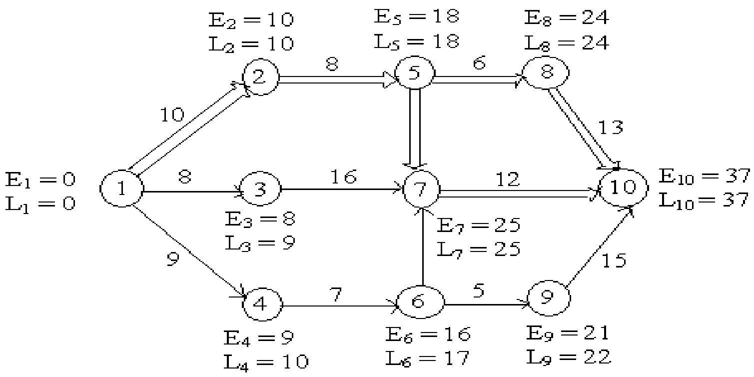
Example 1

Determine the early start and late start in respect of all node points and identify critical path for the following network.



Solution

Calculation of E and L for each node is shown in the network



Activity(i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time $(L_i - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_i - D_{ij})$	Finish (L_i)	
(1, 2)	10	0	10	0	10	0
(1, 3)	8	0	8	1	9	1
(1, 4)	9	0	9	1	10	1
(2, 5)	8	10	18	10	18	0
(4, 6)	7	9	16	10	17	1
(3, 7)	16	8	24	9	25	1
(5, 7)	7	18	25	18	25	0
(6, 7)	7	16	23	18	25	2
(5, 8)	6	18	24	18	24	0
(6, 9)	5	16	21	17	22	1
(7, 10)	12	25	37	25	37	0
(8, 10)	13	24	37	24	37	0
(9, 10)	15	21	36	22	37	1

Network Analysis Table

From the table, the critical nodes are (1, 2), (2, 5), (5, 7), (5, 8), (7, 10) and (8, 10)

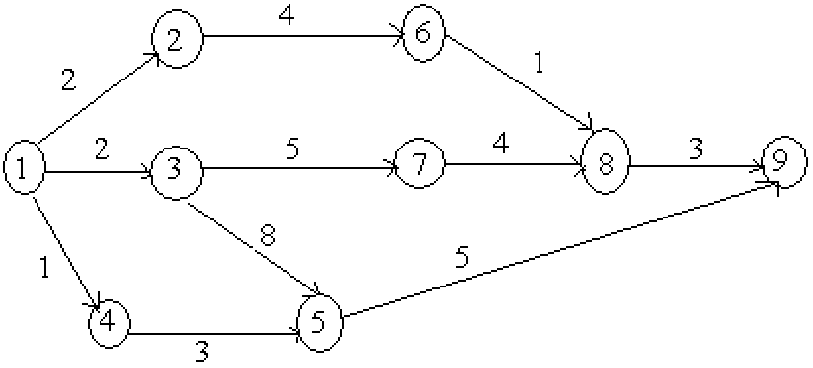
From the table, there are two possible critical paths

i. 1 → 2 → 5 → 8 → 10

ii. 1 → 2 → 5 → 7 → 10

Example 2

Find the critical path and calculate the slack time for the following network



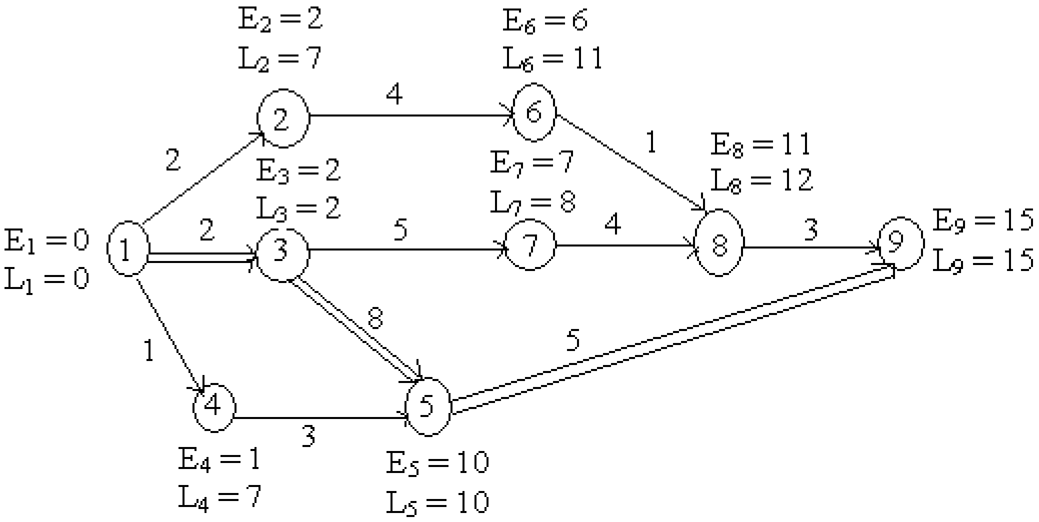
Solution

The earliest time and the latest time are obtained below

Activity(i, j)	Normal Time (D_{ij})	Earliest Time		Latest Time		Float Time $(L_i - D_{ij}) - E_i$
		Start (E_i)	Finish $(E_i + D_{ij})$	Start $(L_i - D_{ij})$	Finish (L_i)	
(1, 2)	2	0	2	5	7	5
(1, 3)	2	0	2	0	2	0
(1, 4)	1	0	1	6	7	6
(2, 6)	4	2	6	7	11	5
(3, 7)	5	2	7	3	8	1
(3, 5)	8	2	10	2	10	0
(4, 5)	3	1	4	7	10	6
(5, 9)	5	10	15	10	15	0
(6, 8)	1	6	7	11	12	5
(7, 8)	4	7	11	8	12	1

(8, 9)	3	11	14	12	15	1

From the above table, the critical nodes are the activities (1, 3), (3, 5) and (5, 9)



The critical path is 1 → 3 → 5 → 9

Example 3

A project has the following times schedule

Activity	Times in weeks	Activity	Times in weeks
(1 – 2)	4	(5 – 7)	8
(1 – 3)	1	(6 – 8)	1
(2 – 4)	1	(7 – 8)	2
(3 – 4)	1	(8 – 9)	1
(3 – 5)	6	(8 – 10)	8
(4 – 9)	5	(9 – 10)	7
(5 – 6)	4		

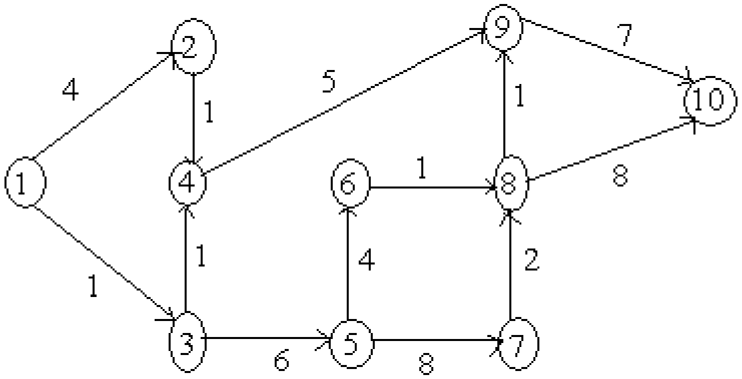
Construct the network and compute

1. T_E and T_L for each event

2. Float for each activity
3. Critical path and its duration

Solution

The network is



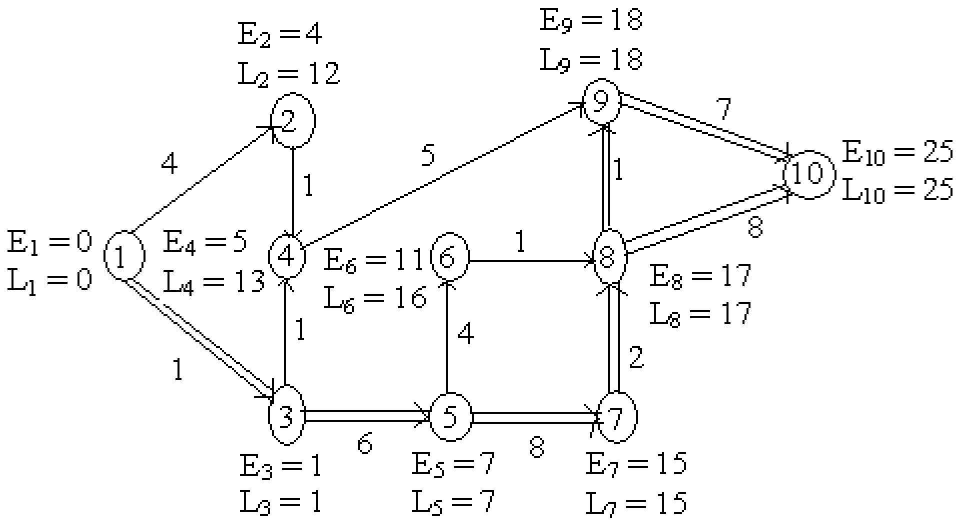
Event No.:	1	2	3	4	5	6	7	8	9	10
T_E:	0	4	1	5	7	11	15	17	18	25
T_L:	0	12	1	13	7	16	15	17	18	25

Float = T_L (Head event) – T_E (Tail event) – Duration

Activity	Duration	T_E (Tail event)	T_L (Head event)	Float
(1 – 2)	4	0	12	8
(1 – 3)	1	0	1	0
(2 – 4)	1	4	13	8
(3 – 4)	1	1	13	11
(3 – 5)	6	1	7	0
(4 – 9)	5	5	18	8
(5 – 6)	4	7	16	5
(5 – 7)	8	7	15	0

(6 – 8)	1	11	17	5
(7 – 8)	2	15	17	0
(8 – 9)	1	17	18	0
(8 – 10)	8	17	25	0
(9 – 10)	7	18	25	0

The resultant network shows the critical path



The two critical paths are

i. 1 → 3 → 5 →7 → 8 → 9 →10

ii. 1 → 3 → 5 → 7 → 8 →10

3.2 Project Evaluation and Review Technique (PERT)

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. The PERT approach takes into account the uncertainties. The three time values are associated with each activity

1. **Optimistic time** – It is the shortest possible time in which the activity can be finished. It assumes that every thing goes very well. This is denoted by t_0.

2. **Most likely time** – It is the estimate of the normal time the activity would take. This assumes normal delays. If a graph is plotted in the time of completion and the frequency of completion in that time period, then most likely time will represent the highest frequency of occurrence. This is denoted by t_m.
3. **Pessimistic time** – It represents the longest time the activity could take if everything goes wrong. As in optimistic estimate, this value may be such that only one in hundred or one in twenty will take time longer than this value. This is denoted by t_p.

In PERT calculation, all values are used to obtain the percent expected value.

1. **Expected time** – It is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution, this is given by

$$t_e = (t_0 + 4 t_m + t_p) / 6$$

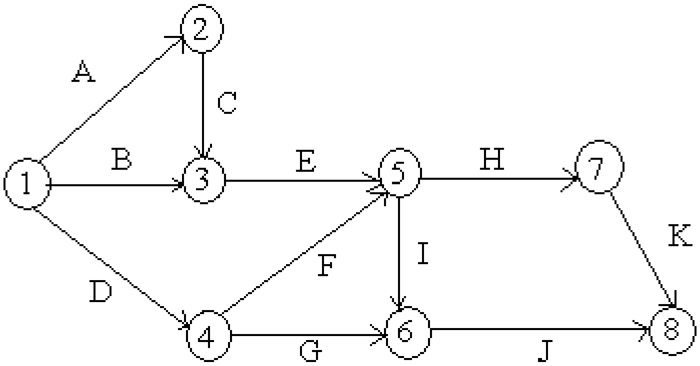
2. The **variance** for the activity is given by

$$\sigma^2 = [(t_p – t_o) / 6]^2$$

3.3 Worked Examples

Example 1

For the project



Task:	A	B	C	D	E	F	G	H	I	J	K
Least time:	4	5	8	2	4	6	8	5	3	5	6
Greatest time:	8	10	12	7	10	15	16	9	7	11	13
Most likely time:	5	7	11	3	7	9	12	6	5	8	9

Find the earliest and latest expected time to each event and also critical path in the network.

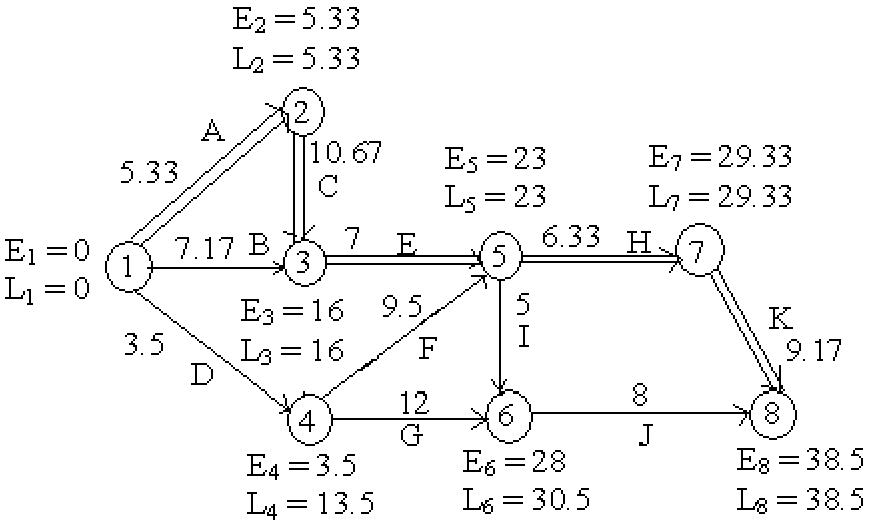
Solution

Task	Least time(t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time (to + t_p + 4t_m)/6
A	4	8	5	5.33
B	5	10	7	7.17
C	8	12	11	10.67
D	2	7	3	3.5
E	4	10	7	7
F	6	15	9	9.5
G	8	16	12	12
H	5	9	6	6.33
I	3	7	5	5
J	5	11	8	8
K	6	13	9	9.17

Task	Expected time (t_e)	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
A	5.33	0	0	5.33	5.33	0
B	7.17	0	8.83	7.17	16	8.83

C	10.67	5.33	5.33	16	16	0
D	3.5	0	10	3.5	13.5	10
E	7	16	16	23	23	0
F	9.5	3.5	13.5	13	23	10
G	12	3.5	18.5	15.5	30.5	15
H	6.33	23	23	29.33	29.33	0
I	5	23	25.5	28	30.5	2.5
J	8	28	30.5	36	38.5	2.5
K	9.17	29.33	29.33	31.5	38.5	0

The network is



The critical path is A →C →E → H → K

Example 2

A project has the following characteristics

Activity	Most optimistic time (a)	Most pessimistic time (b)	Most likely time (m)
(1 – 2)	1	5	1.5

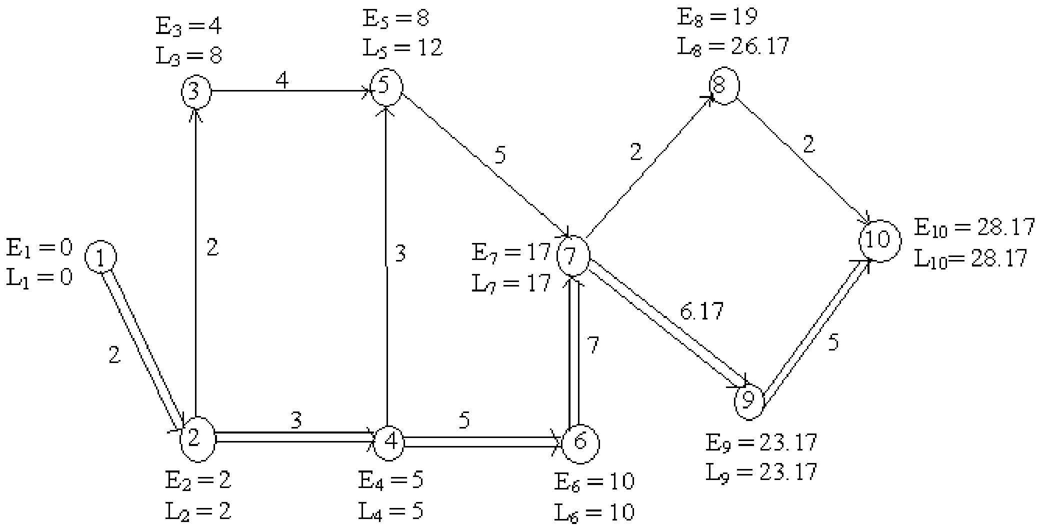
(2 – 3)	1	3	2
(2 – 4)	1	5	3
(3 – 5)	3	5	4
(4 – 5)	2	4	3
(4 – 6)	3	7	5
(5 – 7)	4	6	5
(6 – 7)	6	8	7
(7 – 8)	2	6	4
(7 – 9)	5	8	6
(8 – 10)	1	3	2
(9 – 10)	3	7	5

Construct a PERT network. Find the critical path and variance for each event.

Solution

Activity	(a)	(b)	(m)	(4m)	t_e (a + b + 4m)/6	v [(b – a) / 6]^2
(1 – 2)	1	5	1.5	6	2	4/9
(2 – 3)	1	3	2	8	2	1/9
(2 – 4)	1	5	3	12	3	4/9
(3 – 5)	3	5	4	16	4	1/9
(4 – 5)	2	4	3	12	3	1/9
(4 – 6)	3	7	5	20	5	4/9
(5 – 7)	4	6	5	20	5	1/9
(6 – 7)	6	8	7	28	7	1/9
(7 – 8)	2	6	4	16	4	4/9
(7 – 9)	5	8	6	24	6.17	1/4
(8 – 10)	1	3	2	8	2	1/9
(9 – 10)	3	7	5	20	5	4/9

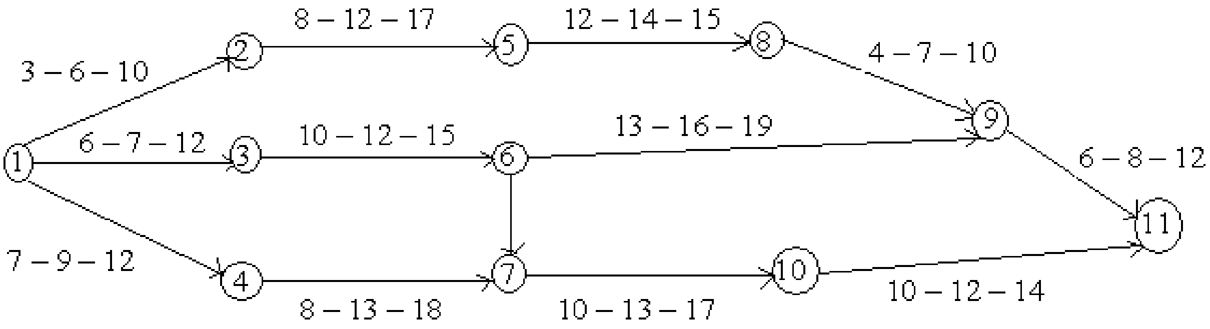
The network is constructed as shown below



The critical path = 1 → 2 → 4 → 6 → 7 →9 →10

Example 3

Calculate the variance and the expected time for each activity



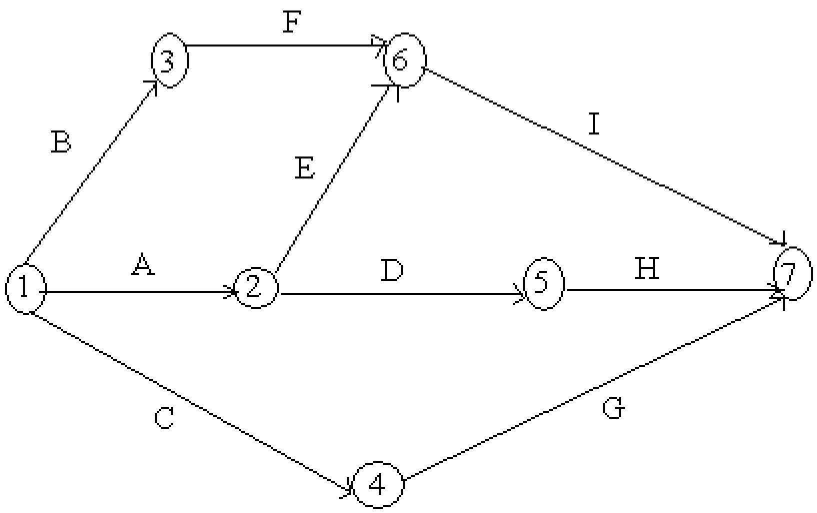
Solution

Activity	(t_o)	(t_m)	(t_p)	t_e $(t_o + t_p + 4t_m)/6$	v $[(t_p – t_o) / 6]^2$
(1 – 2)	3	6	10	6.2	1.36
(1 – 3)	6	7	12	7.7	1.00
(1 – 4)	7	9	12	9.2	0.69
(2 – 3)	0	0	0	0.0	0.00

(2 – 5)	8	12	17	12.2	2.25
(3 – 6)	10	12	15	12.2	0.69
(4 – 7)	8	13	19	13.2	3.36
(5 – 8)	12	14	15	13.9	0.25
(6 – 7)	8	9	10	9.0	0.11
(6 – 9)	13	16	19	16.0	1.00
(8 – 9)	4	7	10	7.0	1.00
(7 – 10)	10	13	17	13.2	1.36
(9 – 11)	6	8	12	8.4	1.00
(10 – 11)	10	12	14	12.0	0.66

Example 4

A project is represented by the network as shown below and has the following data



Task:	A	B	C	D	E	F	G	H	I
Least time:	5	18	26	16	15	6	7	7	3
Greatest time:	10	22	40	20	25	12	12	9	5

Most likely time: 15 20 33 18 20 9 10 8 4

Determine the following

1. Expected task time and their variance
2. Earliest and latest time

Solution

1.

Activity	Least time (t_0)	Greatest time (t_p)	Most likely time (t_m)	Expected time $(to + t_p + 4t_m)/6$	Variance (σ^2)
(1-2)	5	10	8	7.8	0.69
(1-3)	18	22	20	20.0	0.44
(1-4)	26	40	33	33.0	5.43
(2-5)	16	20	18	18.0	0.44
(2-6)	15	25	20	20.0	2.78
(3-6)	6	12	9	9.0	1.00
(4-7)	7	12	10	9.8	0.69
(5-7)	7	9	8	8.0	0.11
(6-7)	3	5	4	4.0	0.11

2.

Earliest time

$E_1 = 0$

$E_2 = 0 + 7.8 = 7.8$

$E_3 = 0 + 20 = 20$

$E_4 = 0 + 33 = 33$

$E_5 = 7.8 + 18 = 25.8$

$E_6 = \max [7.8 + 20, 20 + 9] = 29$

$E_7 = \max [33 + 9.8, 25.8 + 8, 29 + 4] = 42.8$

Latest time

$L_7 = 42.8$

$L_6 = 42.8 – 4 = 38.8$

$L_5 = 42.8 – 8 = 34.3$

$L_4 = 42.8 – 9.8 = 33$

$L_3 = 38.8 – 9 = 29.8$

$L_2 = \min [34.8 – 18, 38.8 – 20] = 16.8$

$L_1 = \min [16.8 – 7.8, 29.8 – 20, 33 - 33] = 0$

Exercise

1. What is PERT?
2. For the following data, draw network. Find the critical path, slack time after calculating the earliest expected time and the latest allowable time

Activity	Duration	Activity	Duration
(1 – 2)	5	(5 – 9)	3
(1 – 3)	8	(6 – 10)	5
(2 – 4)	6	(7 – 10)	4
(2 – 5)	4	(8 – 11)	9
(2 – 6)	4	(9 – 12)	2
(3– 7)	5	(10 – 12)	4
(3 – 8)	3	(11 – 13)	1
(4 – 9)	1	(12 – 13)	7

[Ans. Critical path: 1 → 3 → 7 → 10 → 12 →13]

3. A project schedule has the following characteristics

Activity	Most optimistic time	Most likely time	Most pessimistic time
(1 – 2)	1	2	3
(2 – 3)	1	2	3
(2 – 4)	1	3	5
(3 – 5)	3	4	5
(4 – 5)	2	5	4
(4 – 6)	3	5	7
(5 – 7)	4	5	6
(6 – 7)	6	7	8
(7 – 8)	2	4	6
(7 – 9)	4	6	8
(8 – 10)	1	2	3
(9 – 10)	3	5	7

Construct a PERT network and find out

   a. The earliest possible time
   b. Latest allowable time
   c. Slack values
   d. Critical path

4. Explain the following terms
   a. optimistic time
   b. Most likely time
   c. Pessimistic time
   d. Expected time
   e. Variance
5. Calculate the variance and the expected time for each activity