Debre Markos University

College of Natural and Computational Science

Department of Mathematics

Linear algebra II work sheet I

1) determine whether the statements are true or false.

a. If $Av = \lambda v$ for some vector v, then λ is an eigenvalue of the matrix A.

b. If Av = AV for some vector v, then v is an eigenvector of the matrix A.

c. If v is an eigenvector of a matrix, then there is a unique eigenvalue of the matrix that corresponds to v.

d. If λ is an eigenvalue of a linear operator, then there are infinitely many eigenvectors of the operator that correspond to λ.

e. Every linear operator on $\mathbb{R}^n$ has real eigenvalues.

f. If v is an eigenvector of a matrix A, then cv is also an eigenvector for any scalar c.

g. If v is an eigenvector of a matrix A, then cv is also an eigenvector for any nonzero scalar c.

h. If A and B are n x n matrices and λ is an eigenvalue of both A and B , then λ is an e igenvalue of A + B.

i. If A and B are n x n matrices and v is an eigenvector of both A and B, then v is an eigenvector of A + B.

j. If A and B are n x n matrices and λ is an eigenvalue of both A and B, then λ is an e igenvalue of AB.

2) Find the Eigen values and the corresponding eigenvectors of the matrices over $\mathcal{R}$.

c. $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$ b. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ c. $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

d. $A = \begin{pmatrix} -1 & -3 & 9 \\ 0 & 5 & 18 \\ 0 & -2 & 7 \end{pmatrix}$ e. $A = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$

3) Prove that a square matrix A is invertible if and only if 0 is not an eigenvalue of A.

4) Prove that if λ is an eigenvalue of an invertible matrix A, then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}.

5) Suppose that A is a square matrix in which the sum of the entries of each row equals the same scalar r. Show that r is an eigenvalue of A by finding an eigenvector of A corresponding to r.

6) Prove that if λ is an eigenvalue of a matrix A , then λ^2 is an eigenvalue of A^2.

7) Let $A = (a_{ij})_{nxn}$ and $A^m = A$ for some positive integer m greater than 1. Show that if λ is the eigenvalue of A, then $\lambda = 0\ or |\lambda| = 1$.

8) Let A be a 3x3 matrix whose characteristic polynomial $P_A(\lambda) = \lambda(\lambda - 1)(\lambda + 2)$. What is the characteristic polynomial of A^2? Is A^2 invertible? Why?

9) If A is an nxn matrix, show that $\det(A - \lambda I) = (-1)^n \det(\lambda I - A)$.

10) Show that the Eigen values of a triangular matrix are the diagonal elements of the matrix.

11) Let A be a square matrix. Show that A is non-invertible if and only if $\lambda = 0$ is an eigenvalue of A.

12) a) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Find all eigenvalues and a basis of the eigenspace of A.

b)Find all Eigen values and a basis for each eigenspace over $\mathcal{R}$ of the linear operator $T: \Re^3 \longrightarrow \Re^3$ defined by T(x, y, z) = (2x + y, y-x, 2y + 4x)

13) For each of the following linear operators T on a vector space V and ordered bases β, compute

i. $[T]_\beta$

ii. The Eigen values and the corresponding eigenvectors of T.\

a) V = $\Re^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$.

b) V = $P_1(\Re)$, $T(a + bx) = (6a - 6b) + (12a - 11b)x$, and $\beta = \{3 + 4x, 2 + 3x\}$.

c) V = $\Re^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$, and $\beta = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$.

d) V = $M_{2x2}(\Re)$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$,

and $\beta = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\right\}$

14) If A is diagonalizable, show that A^t is diagonalizable.

15) If A is invertible and diagonalizable matrix, then show that A^{-1} is diagonalizable.

16) Suppose A and B are similar matrices, with A non-singular. Prove that B is non-singular, and that A^{-1} is similar to B^{-1}.

17) For each of the following matrices A, test A for diagonalizability, and if it is diagonalizable, find an invertible matrix P and a diagonal matrix D such that D = Q^{-1}AQ.

a) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

18) For each of the following linear operators T on a vector space V, test T for diagonalizability by using the standard ordered basis.

a) V = P_2(R) = and T is defined by T(f(x)) = f(0) + f(1)(x + x^2).

b) V = R^3, and T is defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ -x \\ 2z \end{pmatrix}$