Paper: Linear Programming and Theory of Games

Lesson: Introduction to Linear Programing Problems

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LINEAR PROGRAMMING PROBLEM

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Linear Programing Problem

1. **OBJECTIVES:**

   After studying this chapter, you should be able to :
   - Formulate a management problem in suitable cases
   - Identify the characteristics of a linear programing problem
   - Make a graphical analysis of the problem
   - Solve the problem graphically
   - Identify various types of solutions
   - Explain various applications of linear programing in business and industry

2. INTRODUCTION:

The idea of Linear Programing is conceived by George B. Bantzing in 1947 and the work named " Programing in Liner Structure" done by Kantorovich (1939) was published in 1959. Koopmans coined the term linear programing in 1948.

Linear Programing is a versatile technique which can be applied to a variety of problems of management such as production, refinery operation, advertising, transportation, distribution and investment analysis. Over the years linear programing has been found useful not only in business and industry but also in non-profit organizations such as government, hospitals, libraries and education.

2.1. Terminology:

- The problem variable X and Y are called decision variables and they represent the solution or output decision from the problem.
- The profit function that the manufacture wishes to increase, represents the objective of making the decisions on the production quantities and it is called ***objective function***.
- The conditions matching the resource requirements are called ***constraints.***
- The decision variables should take non negative values. This is called ***non-negativity restriction.***
- The problem written in algebraic form represents the mathematical model of the given system and is called ***Problem Formulation.***

2.2. Formulation:

The problem formulation has the following steps:

- Identifying the decision variables.
- Writing the objective function.
- Writing the constraints.
- Writing the non-negativity restrictions.

In the above formulations, the objective function and the constraints are linear therefore the formulated model is known as *Linear Programming Problem.*

The formulation of a linear programming problem can be illustrated through what is known as the product mix problem. Typically, it occurs in a manufacturing industry where it is possible to manufacture a variety of products. Each of the products has a certain margin of profit per unit. These products use a common pool of resources whose availability is limited. The linear programming techniques identify the combination of the products which will maximize the profit without violating the resource constraints. The formulation is illustrated with the help of the following examples:

EXAMPLE-1:

Suppose a small manufacture produces two products say A & B and two resources say R1 and R2 require to make these products. Each unit of product A requires 1 unit of R1 and 3 units of R2. Each unit of product B requires 1unit of R1 and 2 units of R2. The manufacturer has 5 units of R1 and 12 units of R2 available. Manufacturer makes a profit of Rs. 6 per unit of product and Rs. 5 per unit on selling of product A & B respectively. **(*Product mix problem)***

Table-1 showing the units manufactured, requirement of units and the profit of manufacturer.

Table-1

Resources / **Units**	**R1**	**R2**	**Profit**
A	1	3	**6**
B	1	2	**5**
Availability	**5**	**12**	

First of all manufacturer has to decide to produce the units A & B in such a way so that he can maximize the profit.

Suppose X & Y be the number of units of A & B produced by the manufacture respectively. The profit Z(say), the manufacturer makes will be Z= 6X+5Y.

Resource R1 and R2 required X+Y units that should be always less or equal to 5 and 12 respectively and produced units can't be negative. X ≥0,Y≥ 0.

Therefore the formulation of problem can be written as follow:

Maximize. $Z= 6X+5Y$

Conditions

$X+Y \leq 5$

$3X+2Y \le 12$

And $X, Y \ge 0$

2.3. STANDARD and CANONICAL form of the model: sometimes referred to as the *canonical* form:

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
STANDARD FORM	Minimize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1,\dots,m$; $x_j \ge 0$, $j = 1,\dots,n$	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1,\dots,m$; $x_j \ge 0$, $j = 1,\dots,n$
CANONICAL FORM	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \ge b_i$, $i = 1,\dots,m$; $x_j \ge 0$, $j = 1,\dots,n$	Maximize $z = \sum_{j=1}^{n} c_j x_j$ subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i$, $i = 1,\dots,m$; $x_j \ge 0$, $j = 1,\dots,n$

1. All RHS parameters $b_i \ge 0$ and $n > m$.
2. Use the n-dimensional vector x to represent the decision variables; i.e., $x = (x_1,\dots,x_n)$.
3. Might have simple upper bounds, say, $x_j \le u_j$.
4. Convert inequalities to equalities in (2).
5. Vector form of constraint: $a_{i1}x_1 + \bullet\ \bullet\ \bullet + a_{in}x_n = b_i$; $a_i x = b_i$; $Ax = b$

   Maximize $\{z = cx : Ax = b, x \ge 0\}$.

2.4. Feasible Solution and Feasible Region:

Any no-negative value of (X_1,X_2) i.e. $X_1 \ge 0$, $X_2 \ge 0$ is a feasible solution of the linear programing problem if it satisfies all the constraints. The collection of all **feasible solutions** is known as the **feasible region.**

The feasible region of the linear programming problem with the given problem is shown by the shaded area in the figure-4.

Although the method is quite simple the principle of solution is based on certain analytical concepts. Following are some basic terminologies that are used in graphical analysis:

2.5. Convex Set:

A **region** or set **R** is convex if and only if for any two points on the set R the line segment connecting those points lies entirely in R. The collections of feasible solutions in a linear programming problem form a convex set. In fact, it is a special type of convex set known as **convex polygon** as this is formed by the intersection of a finite number of **closed half planes.**

2.6. Extreme Points:

The extreme point E of a convex set R is a point such that it is not possible to locate two distinct points in or on **R** so that the line joining them will include E. Extreme points are also referred to as vertices or corner points.

3. GRAPHICAL ANALYSIS:

A linear programming problem involving two decision variables can be conveniently solved graphically. The graphical analysis of a linear programming problem is illustrated with the help of the following example

Example-1

Maximize $50X_1 + 60X_2$

Subject to :

$$2X_1+X_2 \le 300$$
$$3X_1+4X_2 \le 509$$
$$4X_1+7X_2 \le 812$$
$$X_1 \ge 0, \quad X_2 \ge 0$$

If we draw the line $2X_1+X_2 \le 300$,which passes through(0, 300) and(150, 0)

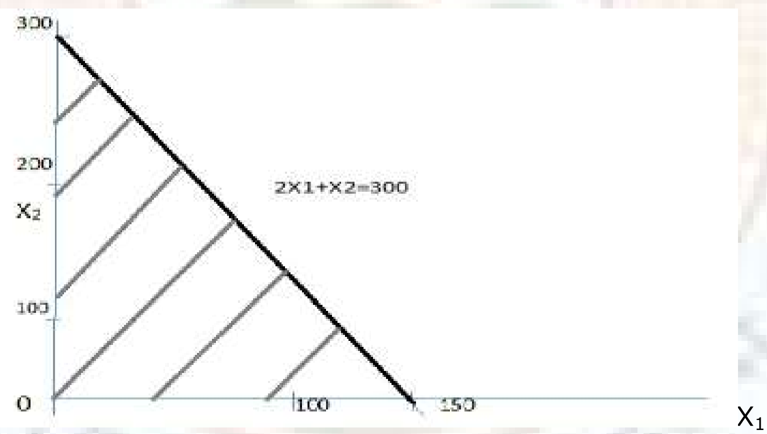


FIGURE:1- Closed half plane

A linear equality in two variables is known as a **Half Plane.** The corresponding equality or the line is known as the boundary of the half plane. The half plane along with its boundary is called a **closed half plane.** We must decide on which side of the line $2X_1+X_2 \le 300$ the half plane is located. It is easy to solve the inequality for X_2.

$X_1 \le 300 - 2X_2$

For fixed X_1, the ordinates satisfying this inequality are smaller than the corresponding ordinate on the line and thus the inequality is satisfied for all points below the line. This is shown in the shaded area in the figure-1.

Similarly, we can find the closed half for the other inequalities $3X_1+4X_2 \le 509$ and $4X_1+7X_2 \le 812$. These are shown in the fig-2 and fig-3 respectively.

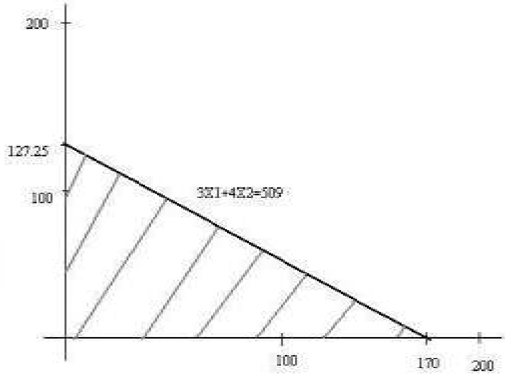


FIGURE-2: Closed half plane

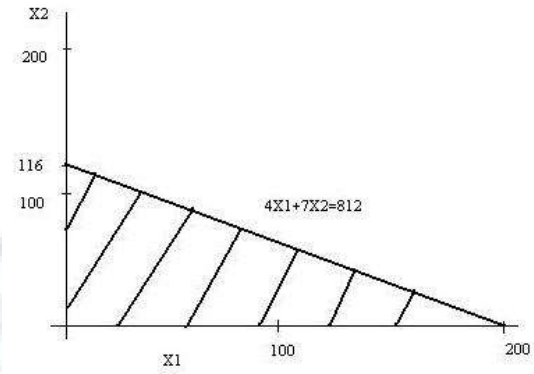


FIGURE-3: Closed half plane

Since all the three constraints must be satisfied simultaneously, we consider the intersection of these three closed half planes in figure-3.

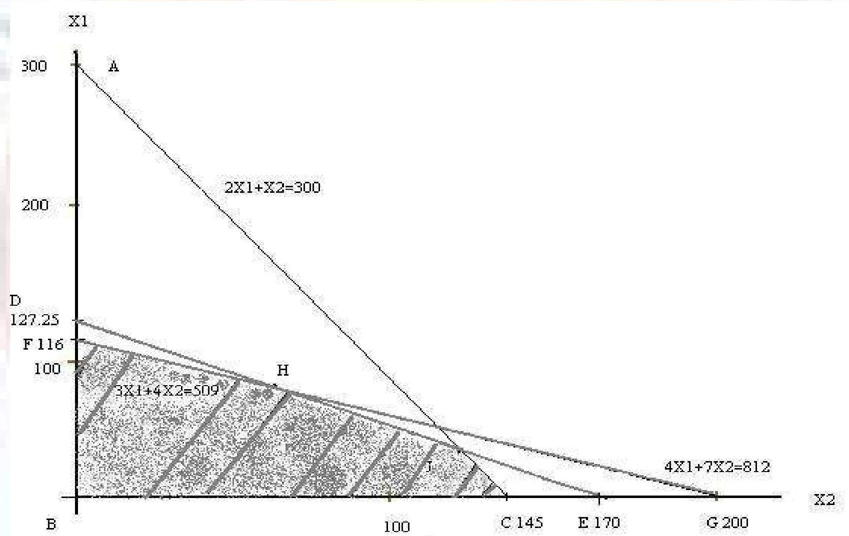


Figure-4- Feasible Region

If we can find the values of the decision variables x_1, x_2, x_3, ... x_n, which can optimize (maximize or minimize) the objective function Z, then we say that these values of x_i are the optimal solution of the Linear Program (LP).

The graphical method is applicable to solve the LPP involving two decision variables x_1, and x_2, we usually take these decision variables as x, y instead of x_1, x_2. To solve an LPP, the graphical method includes two major steps.

a) The determination of the solution space that defines the feasible solution. Note that the set of values of the variable x_1, x_2, x_3,....x_n which satisfy all the constraints and also the non-negative conditions is called the *feasible solution* of the LP.

b) The determination of the optimal solution from the *feasible region*.

3.1. Feasible solution of an LPP:

To determine the feasible solution, we have the following steps.

Step 1: Since the two decision variable x and y are non-negative, consider only the first quadrant of xy-coordinate plane

STEP 2: Each constraint is of the form $ax + by \leq c$ or $ax + by \geq c$

Draw the line $ax + by = c$

For each constraint, the line (1) divides the first quadrant in to two regions say R_1 and R_2, suppose $(x_1, 0)$ is a point in R_1. If this point satisfies the in equation $ax + by \leq c$ or $(\geq c)$, then shade the region R_1. If $(x_1, 0)$ does not satisfy the inequality, shade the region R_2.

Step 3: Corresponding to each constant, we obtain a shaded region. The intersection of all these shaded regions is the feasible region or feasible solution of the LP.

3.2. OPTIMAL SOLUTION:

There are two techniques to find the optimal solution of an LPP.

(i) Corner Point Method
(ii) ISO profit or ISO cost method

3.2.1 <u>Corner Point Method</u>:

The optimal solution to a LPP, if it exists, occurs at the corners of the feasible region.

The method includes the following steps

Step 1: Find the feasible region of the LLP.

Step 2: Find the co-ordinates of each vertex of the feasible region.

These co-ordinates can be obtained from the graph or by solving the equation of the lines.

Step 3: At each vertex (corner point) compute the value of the objective function.

Step 4: Identify the corner point at which the value of the objective function is maximum (or minimum depending on the LP)

The co-ordinates of this vertex is the optimal solution and the value of Z is the optimal value

Example-2 : Find the optimal solution in the above problem of decorative item dealer whose objective function is $Z = 50x + 18y$

In the graph, the corners of the feasible region are O (0, 0), A (0, 80), B(20, 60), C(50, 0)

At (0, 0) Z = 0

At (0, 80) Z = 50 (0) + 18(80) = 1440

At (20, 60), Z = 50 (20) +18 (60) = 1000 + 1080 = Rs.2080

At (50, 0) Z = 50 (50)+ 18 (0) = 2500.

Since our object is to maximize Z and Z has maximum at (50, 0) the optimal solution is x = 50 and y = 0

The optimal value is 2500

Example-3

Let us find the feasible solution for the problem of a decorative item dealer whose LPP is to maximize profit function.

$$Z = 50x + 18y \qquad (1)$$

Subject to the constraints

$2x + y \leq 100$

$x + y \leq 80$

$x \geq 0, y \geq 0$

Step 1: Since $x \geq 0$, $y \geq 0$, we consider only the first quadrant of the xy – plane

Step 2: We draw straight lines for the equation

$$2x + y = 100 \qquad (2)$$

$$x + y = 80$$

To determine two points on the straight line 2x + y = 100

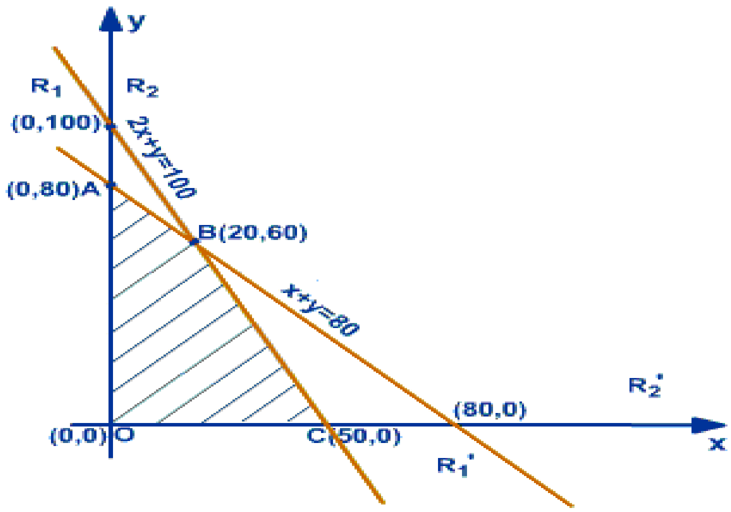
Put y = 0, 2x = 100 $\Rightarrow$ x = 50

$\Rightarrow$ (50, 0) is a point on the line (2)

put x = 0 in (2), y =100

$\Rightarrow$ (0, 100) is the other point on the line (2)

Plotting these two points on the graph paper draw the line which represent the line 2x + y =100.



This line divides the 1st quadrant into two regions, say R_1 and R_2. Choose a point say (1, 0) in R_1. (1, 0) satisfy the inequality $2x + y \leq 100$. Therefore R_1 is the required region for the constraint $2x + y \leq$ A100.

Similarly draw the straight line $x + y = 80$ by joining the point (0, 80) and (80, 0). Find the required region say R_1', for the constraint $x + y \leq 80$

The intersection of both the region R_1 and R_1' is the feasible solution of the LPP. Therefore every point in the shaded region OABC is a feasible solution of the LPP, since this point satisfies all the constraints including the non-negative constraints.

Step-3: In the graph, the corners of the feasible region are O (0, 0), A (0, 80), B(20, 60), C(50, 0)

At (0, 0) Z = 0

At (0, 80) Z = 50 (0) + 18(80) = 1440

At (20, 60), Z = 50 (20) +18 (60) = 1000 + 1080 = Rs.2080

At (50, 0) Z = 50 (50)+ 18 (0) = 2500.

Step-4: Since our object is to maximize Z and Z has maximum at (50, 0) the optimal solution is x = 50 and y = 0

The optimal value is 2500.

If an LPP has many constraints, then it may be long and tedious to find all the corners of the feasible region. There is another alternate and more general method to find the optimal solution of an LP, known as 'ISO profit or ISO cost method'.

3.2.2. ISO- PROFIT (OR ISO-COST) Method of Solving LPP:

Suppose the LPP is to
Optimize $Z = ax + by$ subject to the constraints

$a_1x+b_1y \leq (\text{or} \geq)\ c_1$
$A_2x+b_2y \geq (\text{or} \geq)\ c_2$
$x \geq 0,\ y \geq 0$

This method of optimization involves the following method:

Step 1: Draw the half planes of all the constraints

Step 2: Shade the intersection of all the half planes which is the feasible region

Step 3: Since the objective function is Z = ax + by, draw a dotted line for the equation ax + by = k, where k is any constant. Sometimes it is convenient to take k as the LCM of a and b.

Step 4: To maximize Z draw a line parallel to ax + by = k and farthest from the origin. This line should contain at least one point of the feasible region. Find the coordinates of this point by solving the equations of the lines on which it lies

To minimize Z draw a line parallel to ax + by = k and nearest to the origin. This line should contain at least one point of the feasible region. Find the co-ordinates of this point by solving the equation of the line on which it lies

Step 5: If (x_1, y_1) is the point found in step 4, then

$x = x_1$, $y = y_1$, is the optimal solution of the LPP and $Z = ax_1 + by_1$ is the optimal value

The above method of solving an LPP is more clear with the following example

Example-4 : Solve the following LPP graphically using ISO- profit method.

maximize Z =100 + 100y

Subject to the constraints

$10x+5y\leq 80$, $6x+6y\leq 66$

$4x+8y\leq 24$, $5x+6y\leq 90$

$x\geq 0$, $y\geq 0$

Solution:

since $x \geq 0$, $y \geq 0$, consider only the first quadrant of the plane graph the following straight lines on a graph paper

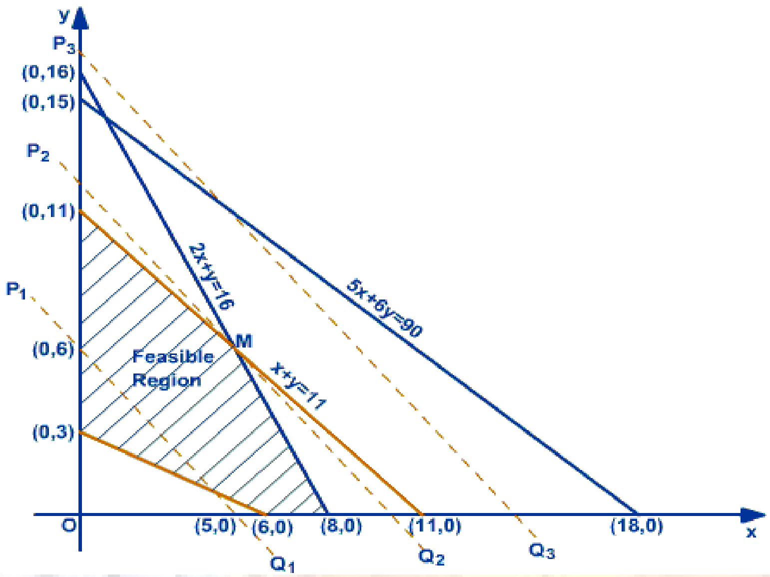
$10x + 5y = 80$ or $2x+y =16$

$6x + 6y = 66$ or $x +y =11$

$4x+ 8y = 24$ or $x+ 2y = 6$

$5x + 6y = 90$

Identify all the half planes of the constraints. The intersection of all these half planes is the feasible region as shown in the figure.



Give a constant value 600 to Z in the objective function, then we have an equation of the line

$$120x + 100y = 600 \qquad (1)$$

or $6x + 5y = 30$ (Dividing both sides by 20)

P_1Q_1 is the line corresponding to the equation $6x + 5y = 30$. We give a constant 1200 to Z then the $P_2\,Q_2$ represents the line.

$$120x + 100y = 1200$$

$$6x + 5y = 60$$

P_2Q_2 is a line parallel to P_1Q_1 and has one point 'M' which belongs to feasible region and farthest from the origin. If we take any line P_3Q_3 parallel to P_2Q_2 away from the origin, it does not touch any point of the feasible region.

The co-ordinates of the point M can be obtained by solving the equation $2x + y = 16$

$x + y = 11$ which give

$x = 5$ and $y = 6$

- $\Rightarrow$ The optimal solution for the objective function is $x = 5$ and $y = 6$
- $\Rightarrow$ The optimal value of Z
- $\Rightarrow$ $120\,(5) + 100\,(6) = 600 + 600 = 1200$

The following results (Hadley, 1969) , (Mittal,1976) provides the solution of a linear programming problem.

If the maximum or minimum value of a linear function defined over a convex polygon exists, then it must be on one of the extreme points.

We can understand the graphical analysis of a linear programing problem with the following example:

We can understand the graphical analysis of Linear Programming Problem with the help of some more examples of product mix problem:

3.3. Multiple Optimal Solution:

Some times in Linear Programming Problem ,the objective function coincides with one of the half planes generated by a constraints provides the MULTIPLE SOLUTION. We can understand this kind of situation through the example as follows:

EXAMPLE-5:

A company buying scrap metal has two types of scrap metal available to him. The first type of scrap metal has 30% of metal A,20% of metal B and 50% of metal C by weight. The second scrap has 40% of metal A, 10% of metal B and 30% of metal C. The company requires at least 240 kg. of metal A,100 kg. of metal B and 290 kg of metal C. The prices per kg of the two scraps are Rs. 120 and Rs. 160 respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the three metals are satisfied at a minimum cost.

Solution:

If the decision variable are X_1 and X_2 indicating the amount of scrap metal to be purchased respectively.

The problem can be formulated as:

Min. $Z = 120X_1 + 160X_2$

S.t.

$0.3\ X_1 + 0.4\ X_2 \geq 240$

$0.2\ X_1 + 0.1\ X_2 \geq 100$

$0.5X_1 + 0.3\ X_2 \geq 290$

$X_1, X_2 \geq 0$

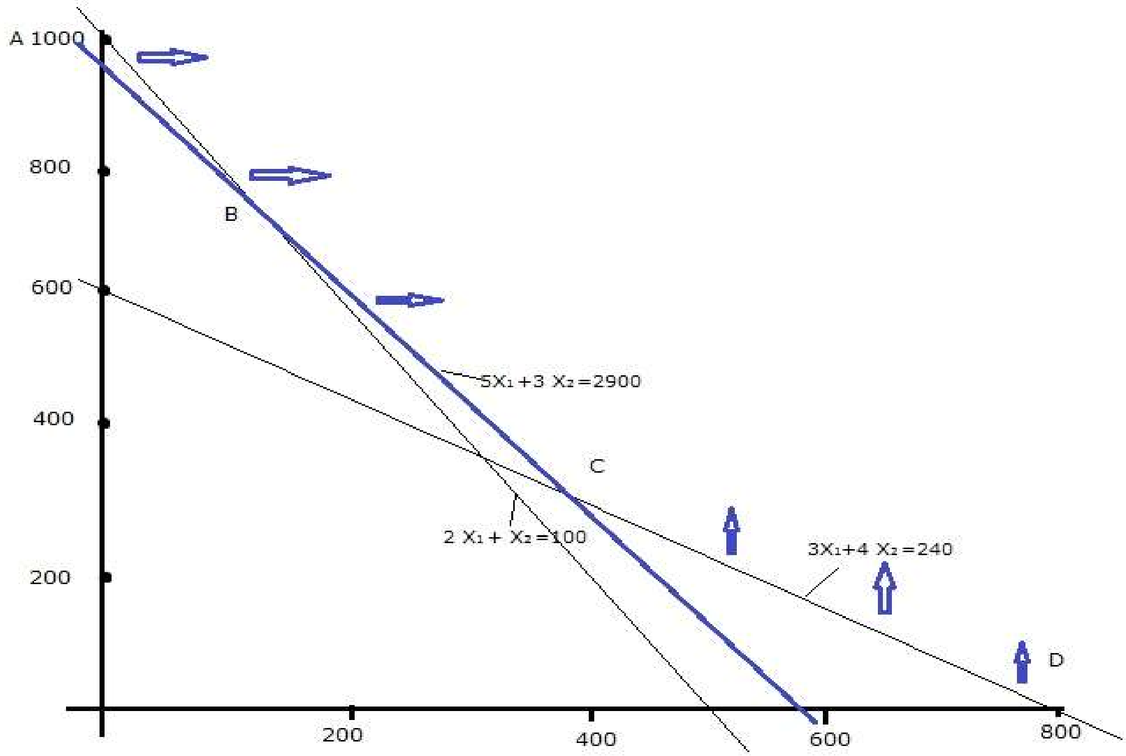
These inequalities can be written after multiplying by 10 as:

$3\ X_1 + 4\ X_2 \geq 240$

$2\ X_1 + 1\ X_2 \geq 100$

$5X_1 + 3\ X_2 \geq 290$

$X_1, X_2 \geq 0$



As shown in the figure, The points A, B, C , D are the extreme points of the lower boundary of the convex set of feasible solutions. One of the members of the family of objective functions 120X1+ 160X2=Z coincides with the line CD with Z=96000. It shows the fact that both the point C with co-ordinates x1=400, x2= 300 and D with coordinates x1=800, x2=0 are on the line 120X1+ 160X2= 96000. Therefore, It can be infer that every point on the line CD minimizes the value of the objective function and the problem has multiple solutions.

So far we discussed linear programming problem which can be easily solved and provide the solutions .Some time we may encounter with the situations where no solution exists or for which the only solution obtained is an unbounded one. Therefore we have two conditions in this situation. i.e.

(a)Unbounded Solution (b) No Solution

3.3.1.Unbounded solution:

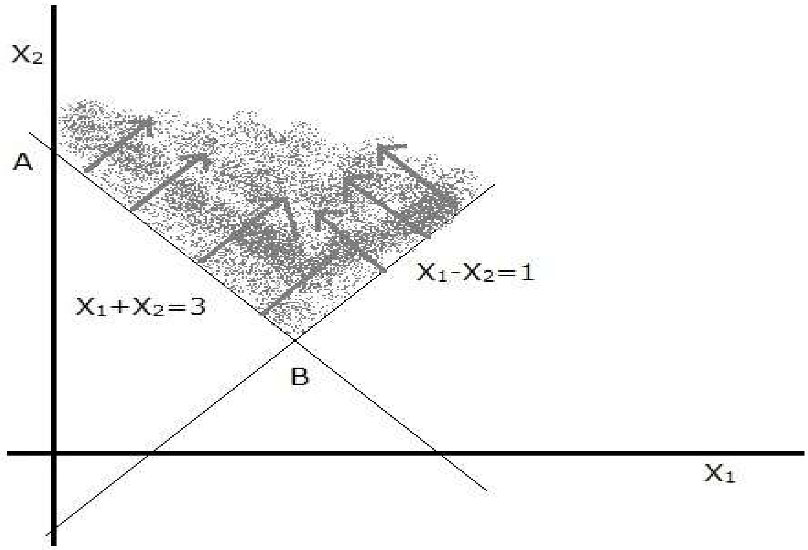
When the value of decision variables in linear programming is permitted to increase infinitely without violating the feasible condition, then the solution is known as UNBOUNDED SOLUTION. In such type of situation, the objective function value can also be increased infinitely. We can understand it with the following example:

Example-6:

Max. $Z= 3X_1 +2 X_2$

S.t.

$X_1 - X_2 \leq 1$, $X_1 + X_2 \geq 3$
$X_1 \geq 0$ and $X_2 \geq 0$



If we draw if graphically then it can be seen easily that the shaded area shown in the figure is unbounded. There are two vertices of the region in the finite plane

A(0,3) and B= (2,1)

The value of objective function at these vertices are given as

Z(A) = 6 and Z(B) = 8

But there exist points in the convex region for which the value of the objective function is more than 8. We can check here that the point (10,10) lies in the region and the function value at this point is 50 which is more than 8. Therefore the maximum value of Z lies at infility only and we can say that this problem has an unbounded solution.

3.3.2. No Solution (No Feasible Solution):

Sometime it happens that no feasible region is formed by the constraints in conjunction with the non-negativity conditions then no solution of linear programming exists in this situation.

Example-7:

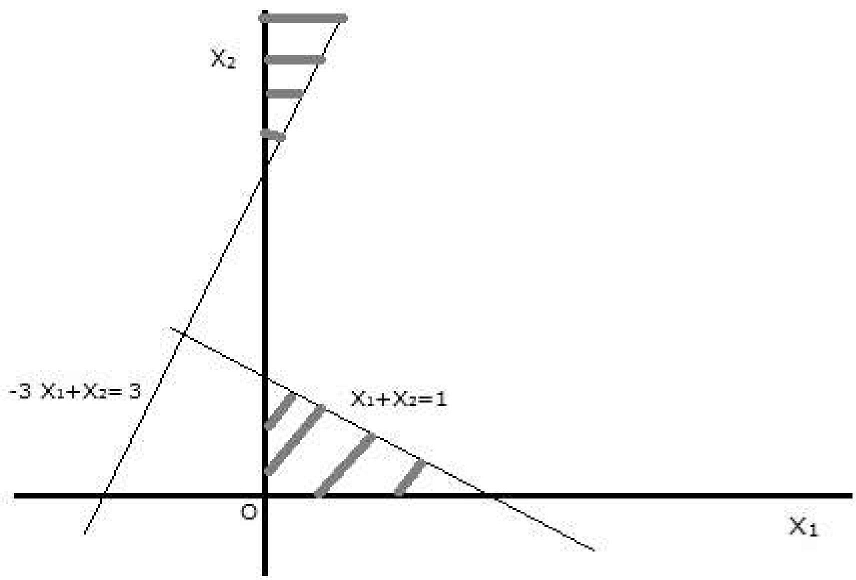
Max. $Z = X_1 + X_2$

S.t

$X_1 + X_2 \leq 1$

$-3 X_1 + X_2 \geq 3$

$X_1 \geq 0$, $X_2 \geq 0$

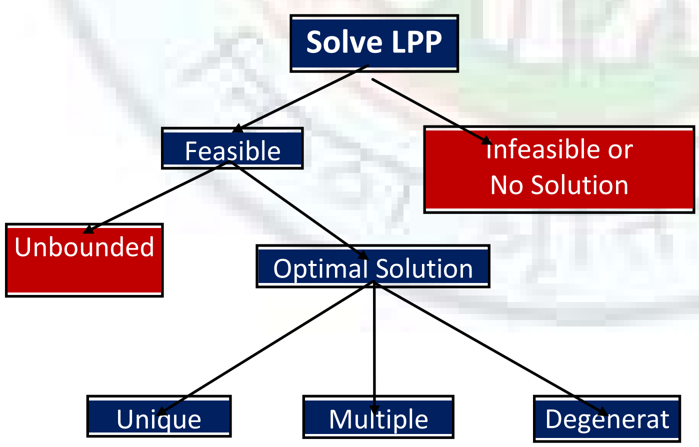


In the given problem, It can be seen graphically in the figure that the desired number pair (x1,x2) lies in the first quadrant only. In this case the two half planes $X_1+X_2 \leq 1$ and $-3X_1+X_2 \geq 3$ don't intersect and no point is common. The region common to the two half planes and the first quadrant are shown in shaded area in the given figure.

We can see here that no point (x1,x2) lie in both the region ,this kind of situation leads to no solution to the given linear programing problem.

Possible outcomes from solving an LPP

We can understand about the possible outcomes during the solving Linear Programming Problem through the following algorithm:



SUMMARY:

Linear programming is a fascinating topic in operational research with wide applications in various problems of managements. Regardless of the functional area a linear programming problem has a number of characteristics. We first identify the decision variables which are some economic or physical quantities whose values are of interest to the management. The problem must have a well-defined objective function expressed in terms of the decision variables. The objective function may have to be maximized when it expresses the profit or contribution. In case the objective function indicates a cost, it has to be minimized. The decision variables interact with each other through some constraints. These constraints occur due to limited resources, stipulation on quality, technical, legal or variety of other reasons. The objective function and the constraints are linear functions of the decision variables.
When a problem of management is expressed in terms of the decision variables with appropriate objective function and constraints we say that the problem has been formulated. A linear programming problem with two decision variables can be solved graphically. Any non-negative solution which satisfies all the constraints is known as feasible solution of the problem. The collection of all the feasible solutions is known as a feasible region. The feasible region of a linear programming problem is a convex set. The value of the decision variables which maximize or minimize the objective function is located on the extreme point of the convex set formed by the feasible solutions. This point and hence the solution of linear programming problem with two decision variables can be identified graphically. In some problem has no finite solution. Sometimes the problem may be infeasible indicating that no feasible solution of the problem exists.

EXCERSISES:

(1) Maximize (Z) = $9x_1 + 12x_2$

subject to:

$4x_1 + 8x_2 \le 64$

$5x_1 + 5x_2 \le 50$

$15x_1 + 8x_2 \le 120$

$x_1 \le 7 \quad x_2 \le 7$

$x_1, x_2 \ge 0$

(2) Minimize (Z) = $8x_1 + 6x_2$

subject to:

$4x_1 + 2x_2 \ge 20$

$-6x_1 + 4x_2 \le 12$

$x_1 + x_2 \ge 6$

$x_1, x_2 \ge 0$

(3) Maximize (Z) = $3x_1 + 4x_2$
subject to:
$3x_1 + 2x_2 \leq 18$
$2x_1 + 4x_2 \leq 20$
$x_2 \leq 4$, $x_1 + x_2 \geq 2$
$x_1, x_2 \geq 0$

(4) Maximize (Z) = $10X_1 + 12X_2$ (objective)
subject to
$5X_1 + 6X_2 \leq 60$ (resource A)
$8X_1 + 4X_2 \leq 72$ (resource B)
$3X_1 + 5X_2 \leq 45$ (resource C)
where
$X_1, X_2 \geq 0$

(5) Maximize (Z) = $6X_1 + 4X_2$
subject to
$X_1 + X_2 \leq 5$
$X_2 \geq 8$
where
$X_1, X_2 \geq 0$

(6) Maximize (Z) = $3X_1 + 6X_2$
subject to
$3X_1 + 4X_2 \geq 12$
$-2X_1 + X_2 \leq 4$
Where $X_1, X_2 \geq 0$

(7) A diet for a sick person must contain at least 1400 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B are available at a cost of Rs. 4 and Rs. 3 per unit respectively. If one unit of A contains 200 units of vitamins, one unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, two units of minerals and 40 calories. Find what combination of food be used to have least cost?

(8) Maximize (Z) = $3X_1 + 6X_2$
s.t.
$3X_1 + 4X_2 \geq 12$
$-2X_1 + X_2 \leq 4$
where
$X_1, X_2 \geq 0$

(9) Max. Z= $3X_1-2X_2$
S.t
$X_1- X_2\geq 0$
$3 X_1-X_2\leq 3$
$X_1\geq0$, $X_2\geq0$.

(10) Max. Z= $6X_1+X_2$

S.t

$2X_1+X_2 \geq 3$

$X_1- X_2 \geq 0$

$X_1 \geq 0$, $X_2 \geq 0$.

(11) Max. $Z= -5X_2$

S.t

$X_1+ X_2 \leq 1$

$0.5X_1+5X_2 \geq 10$

$X_1 \geq 0$, $X_2 \geq 0$.

(12) Max. $Z= X_1+X_2$

S.t

$X_1- X_2 \geq 0$

$3X_1- X_2 \leq -3$

$X_1 \geq 0$, $X_2 \geq 0$.

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