Mechanical Engineering
Department

Fluid Mechanics

(MEng 2113)

Chapter 5

Dimensional Analysis And Similitude

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Introduction. DIMENSIONS AND UNITS

- A dimension is a measure of a physical quantity (without numerical values), while a unit is a way to assign a number to that dimension. For example, length is a dimension that is measured in units such as microns (μm), feet (ft), centimeters (cm), meters (m), kilometers (km), etc.
- There are seven primary dimensions (also called fundamental or basic dimensions)—mass, length, time, temperature, electric current, amount of light, and amount of matter.
- All nonprimary dimensions can be formed by some combination of the seven primary dimensions.
- For example, force has the same dimensions as mass times acceleration (by Newton's second law). Thus, in terms of primary dimensions,

$$\text{Dimensions of force:} \quad \{\text{Force}\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\right\} = \{mL/t^2\}$$

Introduction. DIMENSIONS AND UNITS

Primary dimensions and their associated primary SI and English units

Dimension	Symbol*	SI Unit	English Unit
Mass	m	kg (kilogram)	lbm (pound-mass)
Length	L	m (meter)	ft (foot)
Time†	t	s (second)	s (second)
Temperature	T	K (kelvin)	R (rankine)
Electric current	I	A (ampere)	A (ampere)
Amount of light	C	cd (candela)	cd (candela)
Amount of matter	N	mol (mole)	mol (mole)

- Surface tension (σ_s), has dimensions of force per unit length. The dimensions of surface tension in terms of primary dimensions is

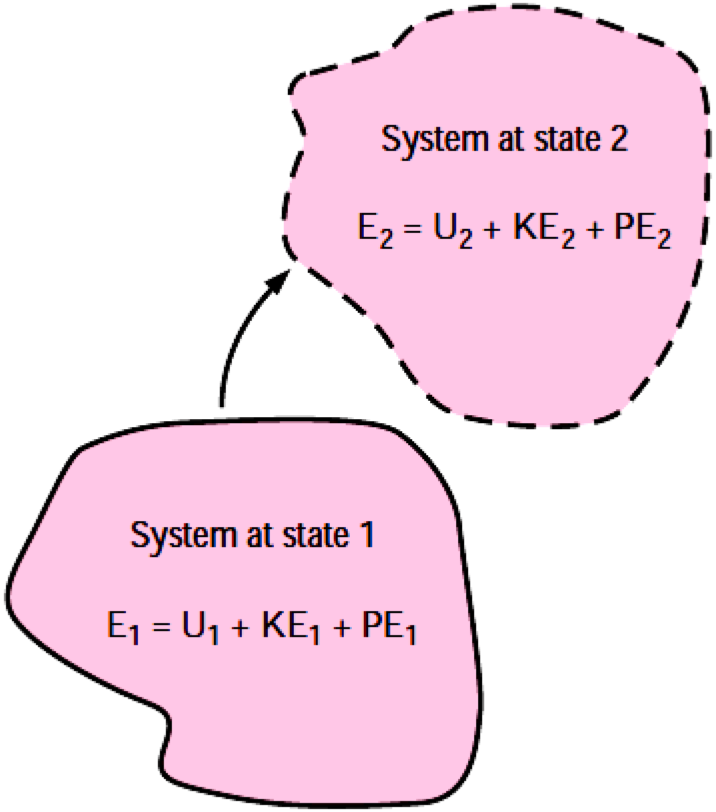
$$\text{Dimensions of surface tension:} \quad \{\sigma_s\} = \left\{\frac{\text{Force}}{\text{Length}}\right\} = \left\{\frac{m \cdot L/t^2}{L}\right\} = \{m/t^2\}$$

DIMENSIONAL HOMOGENEITY

- **Law of dimensional homogeneity:** Every additive term in an equation must have the same dimensions.
- Consider, for example, the change in total energy of a simple compressible closed system from one state and/or time (1) to another (2), as shown in the figure
- The change in total energy of the system (ΔE) is given by

$$\Delta E = \Delta U + \Delta KE + \Delta PE$$

- where E has three components: internal energy (U), kinetic energy (KE), and potential energy (PE).



DIMENSIONAL HOMOGENEITY

- These components can be written in terms of the system mass (m); measurable quantities and thermodynamic properties at each of the two states, such as speed (V), elevation (z), and specific internal energy (u); and the known gravitational acceleration constant (g),

$$\Delta U = m(u_2 - u_1) \qquad \Delta KE = \frac{1}{2}m(V_2^2 - V_1^2) \qquad \Delta PE = mg(z_2 - z_1)$$

- It is straightforward to verify that the left side of the change in Energy equation and all three additive terms on the right side have the same dimensions—energy.

$$\{\Delta E\} = \{\text{Energy}\} = \{\text{Force} \cdot \text{Length}\} \quad \rightarrow \quad \{\Delta E\} = \{mL^2/t^2\}$$

$$\{\Delta U\} = \left\{\text{Mass}\,\frac{\text{Energy}}{\text{Mass}}\right\} = \{\text{Energy}\} \quad \rightarrow \quad \{\Delta U\} = \{mL^2/t^2\}$$

DIMENSIONAL HOMOGENEITY

$$\{\Delta KE\} = \left\{\text{Mass}\,\frac{\text{Length}^2}{\text{Time}^2}\right\} \quad \rightarrow \quad \{\Delta KE\} = \{mL^2/t^2\}$$

$$\{\Delta PE\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} \quad \rightarrow \quad \{\Delta PE\} = \{mL^2/t^2\}$$

- In addition to dimensional homogeneity, calculations are valid only when the units are also homogeneous in each additive term.
- For example, units of energy in the above terms may be J, N·m , or kg·m^2/s^2, all of which are equivalent.
- Suppose, however, that kJ were used in place of J for one of the terms. This term would be off by a factor of 1000 compared to the other terms.
- It is wise to write out all units when performing mathematical calculations in order to avoid such errors.

Example 1. Dimensional Homogeneity of the Bernoulli Equation

- Probably the most well-known equation in fluid mechanics is the Bernoulli equation . One standard form of the Bernoulli equation for incompressible irrotational fluid flow is

$$P + \frac{1}{2}\rho V^2 + \rho g z = C$$

- (a) Verify that each additive term in the Bernoulli equation has the same dimensions. (b) What are the dimensions of the constant *C*?

SOLUTION We are to verify that the primary dimensions of each additive term in Eq. 1 are the same, and we are to determine the dimensions of constant *C*.

Analysis (*a*) Each term is written in terms of primary dimensions,

$$\{P\} = \{\text{Pressure}\} = \left\{\frac{\text{Force}}{\text{Area}}\right\} = \left\{\text{Mass}\,\frac{\text{Length}}{\text{Time}^2}\,\frac{1}{\text{Length}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\left\{\frac{1}{2}\rho V^2\right\} = \left\{\frac{\text{Mass}}{\text{Volume}}\left(\frac{\text{Length}}{\text{Time}}\right)^2\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

$$\{\rho g z\} = \left\{\frac{\text{Mass}}{\text{Volume}}\,\frac{\text{Length}}{\text{Time}^2}\,\text{Length}\right\} = \left\{\frac{\text{Mass}\times\text{Length}^2}{\text{Length}^3\times\text{Time}^2}\right\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Indeed, **all three additive terms have the same dimensions.**

(*b*) From the law of dimensional homogeneity, the constant must have the same dimensions as the other additive terms in the equation. Thus,

Primary dimensions of the Bernoulli constant: $$\{C\} = \left\{\frac{\text{m}}{\text{t}^2\text{L}}\right\}$$

Example 2 . Dimensional Homogeneity

- In Chap. 4 we discussed the differential equation for conservation of mass, the *continuity equation. In cylindrical* coordinates, and for steady flow,

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

- Write the primary dimensions of each additive term in the equation, and verify that the equation is dimensionally homogeneous.

- **Solution.** We are to determine the primary dimensions of each additive term, and we are to verify that the equation is dimensionally homogeneous.
- **Analysis .**The primary dimensions of the velocity components are length/time. The primary dimensions of coordinates *r* and *z* are length, and the primary dimensions of coordinate θ are unity (it is a dimensionless angle). Thus each term in the equation can be written in terms of primary dimensions,

Example 2 . Dimensional Homogeneity

$$\left\{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right\} = \left\{\frac{1}{\text{length}} \times \frac{\text{length}\,\dfrac{\text{length}}{\text{time}}}{\text{length}}\right\} = \left\{\frac{1}{\text{time}}\right\} \qquad \left\{\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right\} = \left\{\frac{1}{\text{t}}\right\}$$

$$\left\{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right\} = \left\{\frac{1}{\text{length}} \times \frac{\dfrac{\text{length}}{\text{time}}}{1}\right\} = \left\{\frac{1}{\text{time}}\right\} \qquad \left\{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right\} = \left\{\frac{1}{\text{t}}\right\}$$

$$\left\{\frac{\partial u_z}{\partial z}\right\} = \left\{\frac{\dfrac{\text{length}}{\text{time}}}{\text{length}}\right\} = \left\{\frac{1}{\text{time}}\right\} \qquad \left\{\frac{\partial u_z}{\partial z}\right\} = \left\{\frac{1}{\text{t}}\right\}$$

- Indeed, all three additive terms have the same dimensions, namely $\{t^{-1}\}$.

Nondimensionalization of Equations

- The law of dimensional homogeneity guarantees that every additive term in an equation has the same dimensions.
- It follows that if we divide each term in the equation by a collection of variables and constants whose product has those same dimensions, the equation is rendered **nondimensional**.

The nondimensionalized Bernoulli equation

$$\underbrace{\frac{P}{P_\infty}}_{\{1\}} + \underbrace{\frac{\rho V^2}{2P_\infty}}_{\{1\}} + \underbrace{\frac{\rho g z}{P_\infty}}_{\{1\}} = \underbrace{\frac{C}{P_\infty}}_{\{1\}}$$

A *nondimensionalized form of the* Bernoulli equation is formed by dividing each additive term by a pressure (here we use P_∞). *Each* resulting term is *dimensionless* (dimensions of {1}).

- If, in addition, the nondimensional terms in the equation are of order unity, the equation is called **normalized.**
- Each term in a nondimensional equation is dimensionless.

Nondimensionalization of Equations

- In the process of nondimensionalizing an equation of motion, **nondimensional parameters** often appear—most of which are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number).
- This process is sometimes called **inspectional analysis.**
- As a simple example, consider the equation of motion describing the elevation *z* of an object falling by gravity through a vacuum (no air drag).
- The initial location of the object is z_0 and its initial velocity is w_0 in the z-direction. From high school physics,

$$\frac{d^2z}{dt^2} = -g \qquad \text{..................} \ (1)$$

- **Dimensional variables** are defined as dimensional quantities that change or vary in the problem.

Nondimensionalization of Equations

- For the simple differential equation given in Eq. 1, there are two dimensional variables: z (dimension of length) and t (dimension of time).
- **Nondimensional (or dimensionless)** variables are defined as quantities that change or vary in the problem, but have no dimensions; an example is angle of rotation, measured in degrees or radians which are dimensionless units. Gravitational constant g, while dimensional, remains constant and is called a **dimensional constant.**
- Other dimensional constants are relevant to this particular problem are initial location z_0 and initial vertical speed w_0.
- While dimensional constants may change from problem to problem, they are fixed for a particular problem and are thus distinguished from dimensional variables.

Nondimensionalization of Equations

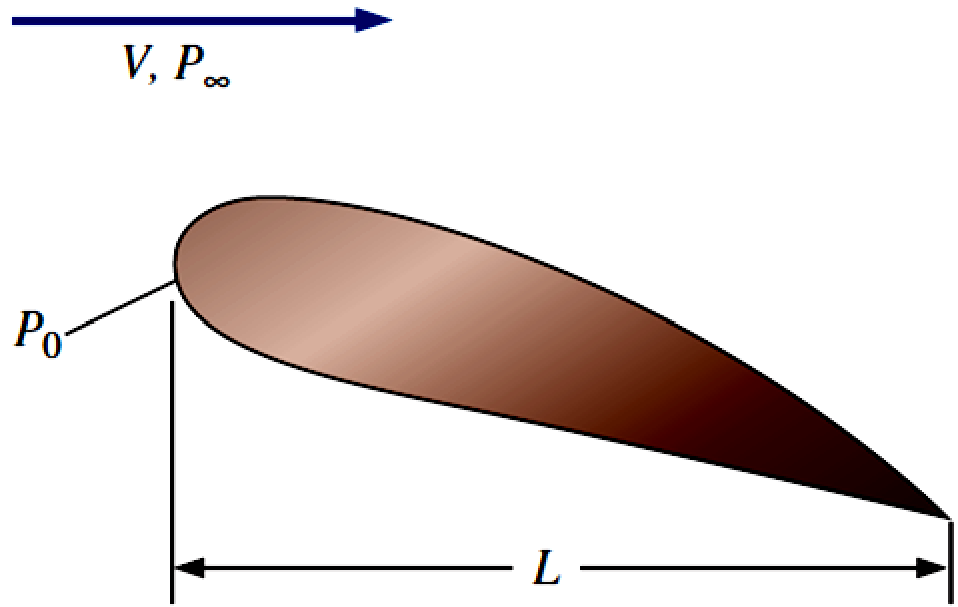
- We use the term **parameters** for the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.
- Equation 1 is easily solved by integrating twice and applying the initial conditions. The result is an expression for elevation *z* at any time *t*:

$$z = z_0 + w_0 t - \frac{1}{2} g t^2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

- The constant ½ and the exponent 2 in Eq. 2 are dimensionless results of the integration. Such constants are called **pure constants.** Other common examples of pure constants are Π and *e*.

Nondimensionalization of Equations

- To nondimensionalize Eq. 1, we need to select **scaling parameters,** based on the primary dimensions contained in the original equation.
- In fluid flow problems there are typically at least three scaling parameters, e.g., L, V, and $P_0 - P_\infty$, since there are at least three primary dimensions in the general problem (e.g., mass, length, and time).



In a typical fluid flow problem, the scaling parameters usually include a characteristic length L, a characteristic velocity V, and a reference pressure difference $P_0 - P_\infty$. Other parameters and fluid properties such as density, viscosity, and gravitational acceleration enter the problem as well.

Nondimensionalization of Equations

- In the case of the falling object being discussed here, there are only two primary dimensions, length and time, and thus we are limited to selecting only *two scaling parameters.*
- We have some options in the selection of the scaling parameters since we have three available dimensional constants g, z_0, *and* w_0. We choose z_0 and w_0. we can also do the analysis using g and z_0 and/or with g and w_0
- With these two chosen scaling parameters we nondimensionalize the dimensional variables z *and* t.
- The first step is to list the primary dimensions of all dimensional variables and dimensional constants in the problem,

Primary dimensions of all parameters:

$$\{z\} = \{\mathrm{L}\} \qquad \{t\} = \{\mathrm{t}\} \qquad \{z_0\} = \{\mathrm{L}\} \qquad \{w_0\} = \{\mathrm{L/t}\} \qquad \{g\} = \{\mathrm{L/t^2}\}$$

Nondimensionalization of Equations

- The second step is to use our two scaling parameters to nondimensionalize *z* and *t* (by inspection) into nondimensional variables z^* and t^*,

Nondimensionalized variables:

$$z^* = \frac{z}{z_0} \qquad t^* = \frac{w_0 t}{z_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

- Substitution of Eq. 3 into Eq. 1 gives

$$\frac{d^2 z}{dt^2} = \frac{d^2(z_0 z^*)}{d(z_0 t^*/w_0)^2} = \frac{w_0^2}{z_0}\frac{d^2 z^*}{dt^{*2}} = -g \quad \rightarrow \quad \frac{w_0^2}{g z_0}\frac{d^2 z^*}{dt^{*2}} = -1 \ldots(4)$$

- which is the desired nondimensional equation. The grouping of dimensional constants in Eq. 4 is the square of a well-known **nondimensional parameter** or **dimensionless group** called the **Froude number,**

Nondimensionalization of Equations

- Froude number: $\mathrm{Fr} = \dfrac{w_0}{\sqrt{g z_0}}$ ………… (5)
- Substitution of Eq. 5 into Eq. 4 yields
- *Nondimensionalized equation of motion:* $\dfrac{d^2 z^*}{dt^{*2}} = -\dfrac{1}{\mathrm{Fr}^2}$ …..(6)
- In dimensionless form, only one parameter remains, namely the Froude number.
- Equation 6 is easily solved by integrating twice and applying the initial conditions. The result is an expression for dimensionless elevation *z** as a function of dimensionless time *t*:*
- *Nondimensional result:*
- $$z^* = 1 + t^* - \frac{1}{2\mathrm{Fr}^2} t^{*2}$$ ………….(7)

Nondimensionalization of Equations

- There are two key advantages of nondimensionalization
- First, it increases our insight about the relationships between key parameters. Equation 5 reveals, for example, that doubling w_0 has the same effect as decreasing z_0 by a factor of 4.
- Second, it reduces the number of parameters in the problem. For example, the original problem contains one dependent variable, z; one independent variable, t; and three additional dimensional constants, g, w_0, and z_0. The nondimensionalized problem contains one dependent parameter, z*; one independent parameter, t*; and only one additional parameter, namely the dimensionless Froude number, Fr. The number of additional parameters has been reduced from three to one!

Dimensional Analysis and Similarity

- Nondimensionalization of an equation by inspection is useful only when we know the equation to begin with.
- However, in many cases in real-life engineering, the equations are either not known or too difficult to solve; often times experimentation is the only method of obtaining reliable information.
- In most experiments, to save time and money, tests are performed on a geometrically scaled **model,** rather than on the full-scale **prototype**. In such cases, care must be taken to properly scale the results. We introduce here a powerful technique called **dimensional analysis.**

Dimensional Analysis and Similarity

- The three primary purposes of dimensional analysis are
  - ✓To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
  - ✓To obtain scaling laws so that prototype performance can be predicted from model performance
  - ✓To (sometimes) predict trends in the relationship between parameters
- There are three necessary conditions for complete similarity between a model and a prototype.
- The first condition is **geometric similarity**—the model must be the same shape as the prototype, but may be scaled by some constant scale factor.

Dimensional Analysis and Similarity

- The second condition is **kinematic similarity**, which means that the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow.
- Specifically, for kinematic similarity the velocity at corresponding points must scale in magnitude and must point in the same relative direction.

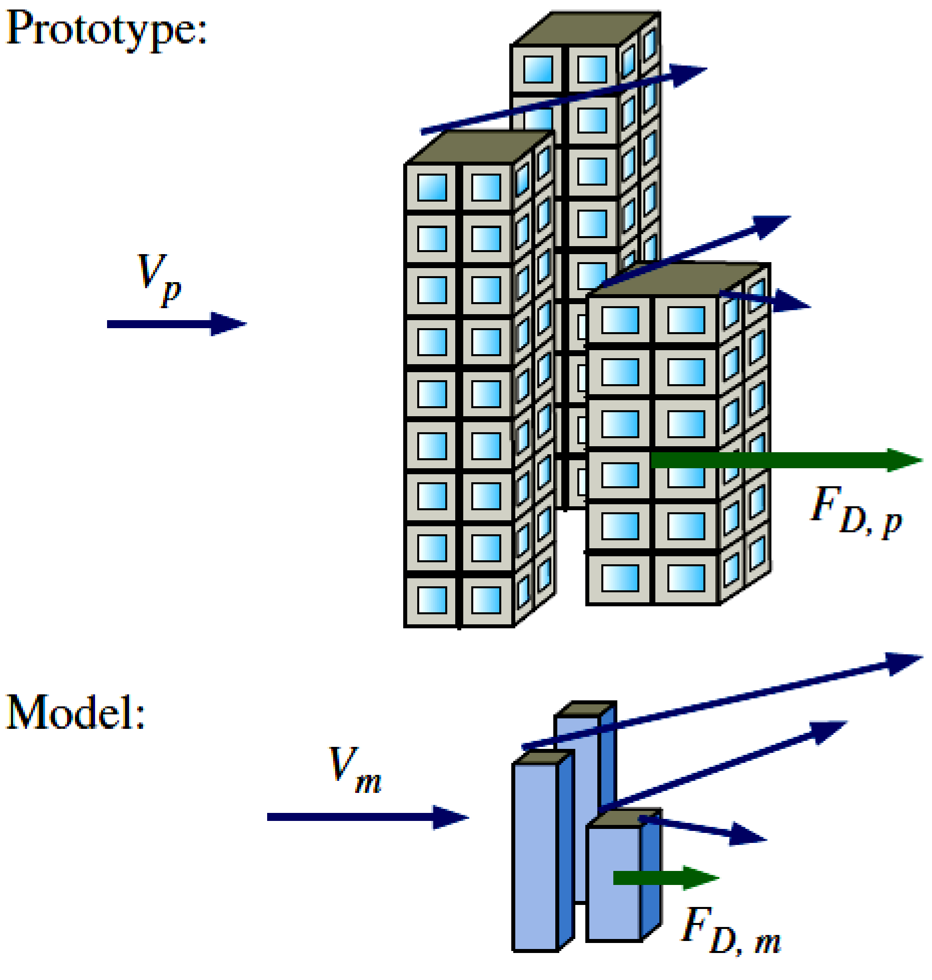


Fig. Kinematic similarity

Dimensional Analysis and Similarity

- *Kinematic similarity* is achieved when, at all locations, the speed in the model flow is proportional to that at corresponding locations in the prototype flow, and points in the same direction.
- *Geometric similarity is a prerequisite for kinematic similarity*
- Just as the geometric scale factor can be less than, equal to, or greater than one, so can the velocity scale factor.
- In Fig. above, for example, the geometric scale factor is less than one (model smaller than prototype), but the velocity scale is greater than one (velocities around the model are greater than those around the prototype).
- The third and most restrictive similarity condition is that of **dynamic similarity**. Dynamic similarity is achieved when all *forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow (*force-scale equivalence)*.

Dimensional Analysis and Similarity

- As with geometric and kinematic similarity, the scale factor for forces can be less than, equal to, or greater than one.
- In Fig. shown in slide 20 above for example, the force-scale factor is less than one since the force on the model building is less than that on the prototype.
- *Kinematic similarity is a necessary but insufficient condition for dynamic similarity.*
- It is thus possible for a model flow and a prototype flow to achieve both geometric and kinematic similarity, yet not dynamic similarity. All three similarity conditions must exist for complete similarity to be ensured.
- In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

Dimensional Analysis and Similarity

- We let uppercase Greek letter Pi (Π) denote a nondimensional parameter. We have already discussed one Π, namely the Froude number, Fr.
- In a general dimensional analysis problem, there is one Π that we call the **dependent Π**, giving it the notation **Π_1**. The parameter Π_1 is in general a function of several other **Π**'s, which we call **independent** Π's. The functional relationship is
- *Functional relationship between Π's:*

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots, \Pi_k)$$

- where k is the total number of Π's.
- Consider an experiment in which a scale model is tested to simulate a prototype flow.

Dimensional Analysis and Similarity

- To ensure complete similarity between the model and the prototype, each independent P of the model (subscript *m)* must be identical to the corresponding independent Π of the prototype (subscript p),

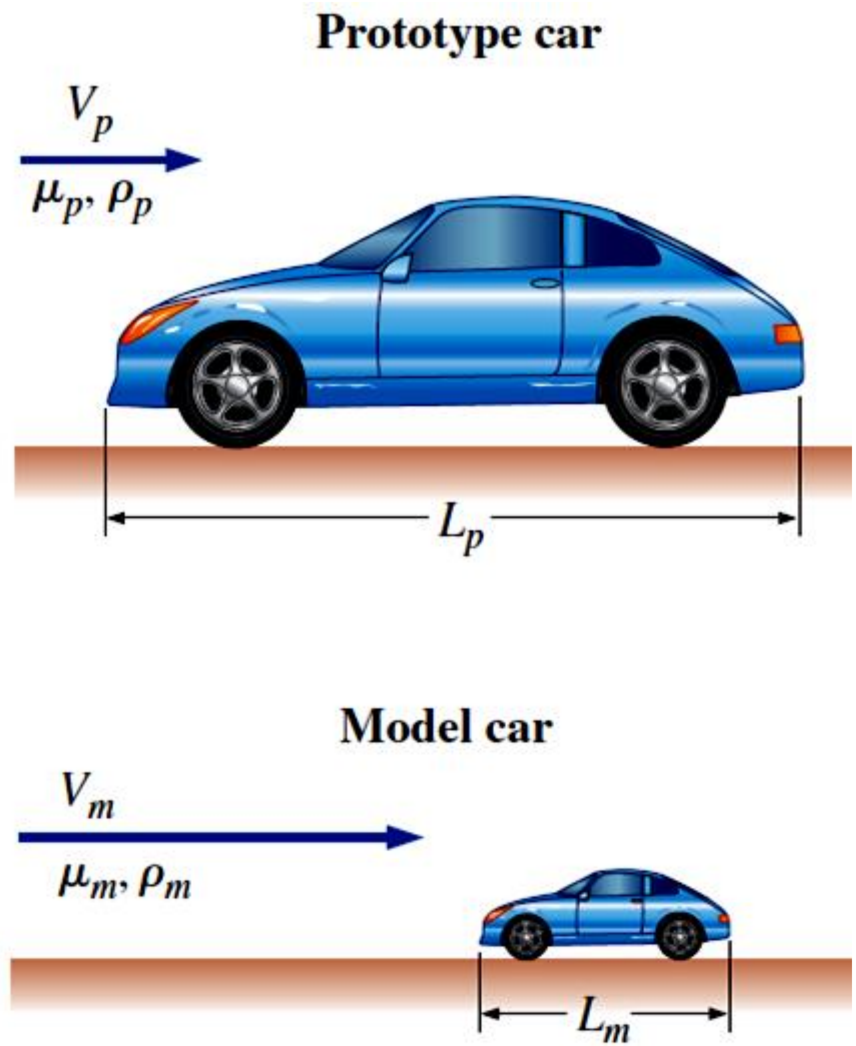
  i.e., $\Pi_{2,\,m} = \Pi_{2,\,p}$, $\Pi_{3,\,m} = \Pi_{3,\,p}$, ..., $\Pi_{k,\,m} = \Pi_{k,\,p}$.
- To ensure complete similarity, the model and prototype must be geometrically similar, and all independent Π groups must match between model and prototype.
- Under these conditions the *dependent* Π of the model ($\Pi_{1,\,m}$) is guaranteed to also equal the dependent Π of the prototype ($\Pi_{1,\,p}$).
- Mathematically, we write a conditional statement for achieving similarity,

$$\text{If} \quad \Pi_{2,m} = \Pi_{2,p} \quad \text{and} \quad \Pi_{3,m} = \Pi_{3,p} \ldots \quad \text{and} \quad \Pi_{k,m} = \Pi_{k,p}:$$

$$\text{then} \quad \Pi_{1,m} = \Pi_{1,p}$$

Dimensional Analysis and Similarity

- Consider, for example, the design of a new sports car, the aerodynamics of which is to be tested in a wind tunnel. To save money, it is desirable to test a small, geometrically scaled model of the car rather than a full-scale prototype of the car.
- In the case of aerodynamic drag on an automobile, it turns out that if the flow is approximated as incompressible, there are only two Π’s in the problem,



Dimensional Analysis and Similarity

$$\Pi_1 = f(\Pi_2)$$

- Where

$$\Pi_1 = \frac{F_D}{\rho V^2 L^2} \quad \text{and} \quad \Pi_2 = \frac{\rho V L}{\mu}$$

- The procedure used to generate these Π's will be discussed later in this chapter.
- In the above equation F_D is the magnitude of the aerodynamic drag on the car, ρ is the air density, V is the car's speed (or the speed of the air in the wind tunnel), L is the length of the car, and μ is the viscosity of the air. Π_1 is a nonstandard form of the drag coefficient, and Π_2 is the **Reynolds number, Re.**
- The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics

Dimensional Analysis and Similarity

- In the problem at hand there is only one independent Π, and the above Eq. ensures that if the independent Π's match (the Reynolds numbers match: $\Pi_{2,\,m} = \Pi_{2,\,p}$), then the dependent Π's also match ($\Pi_{1,\,m} = \Pi_{1,\,p}$).
- This enables engineers to measure the aerodynamic drag on the model car and then use this value to predict the aerodynamic drag on the prototype car.

Example 3: Similarity between Model and Prototype Cars

- The aerodynamic drag of a new sports car is to be predicted at a speed of 50.0 mi/h at an air temperature of 25°C. Automotive engineers build a one fifth scale model of the car to test in a wind tunnel. It is winter and the wind tunnel is located in an unheated building; the temperature of the wind tunnel air is only about 5°C. Determine how fast the engineers should run the wind tunnel in order to achieve similarity between the model and the prototype.

Solution:

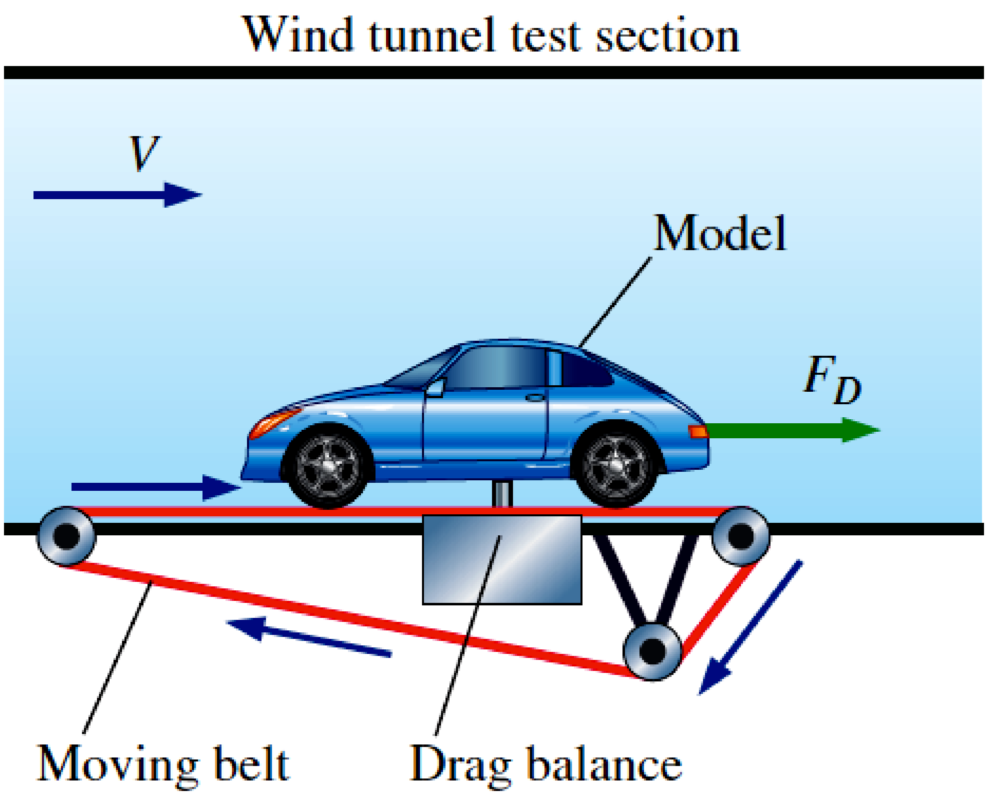
- We are to utilize the concept of similarity to determine the speed of the wind tunnel.

Assumptions:

- The model is geometrically similar to the prototype
- The wind tunnel walls are far enough away so as to not interfere with the aerodynamic drag on the model car.

Example 3: Similarity between Model and Prototype Cars

- The wind tunnel has a moving belt to simulate the ground under the car. (The moving belt is necessary in order to achieve kinematic similarity everywhere in the flow, in particular underneath the car.)



A *drag balance is a device used* in a wind tunnel to measure the aerodynamic drag of a body. When testing automobile models, a *moving belt is* often added to the floor of the wind tunnel to simulate the moving ground (from the car's frame of reference).

Example 3: Similarity between Model and Prototype Cars

- ***Properties:*** For air at atmospheric pressure and at T = 25°C, ρ = 1.184 kg/m^3 and μ = 1.849 x 10^{-5} kg/m·s. Similarly, at T = 5°C, ρ = 1.269 kg/m^3 and μ = 1.754 x 10^{-5} kg/m·s.
- **Analysis:** Since there is only one independent Π in this problem, the similarity equation holds if $\Pi_{2,\,m} = \Pi_{2,\,p}$, where Π_2 is the Reynolds number. Thus, we write

$$\Pi_{2,\,m} = \mathrm{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \Pi_{2,\,p} = \mathrm{Re}_p = \frac{\rho_p V_p L_p}{\mu_p}$$

- Thus

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0\ \mathrm{mi/h})\left(\frac{1.754\times10^{-5}\ \mathrm{kg/m\cdot s}}{1.849\times10^{-5}\ \mathrm{kg/m\cdot s}}\right)\left(\frac{1.184\ \mathrm{kg/m^3}}{1.269\ \mathrm{kg/m^3}}\right)(5) = \mathbf{221\ mi/h}$$

Example 3: Similarity between Model and Prototype Cars

- The power of using dimensional analysis and similarity to supplement experimental analysis is further illustrated by the fact that the actual values of the dimensional parameters (density, velocity, etc.) are irrelevant. As long as the corresponding independent Π's are set equal to each other, similarity is achieved *even if different fluids are used.*
- *This explains why automobile* or aircraft performance can be simulated in a water tunnel, and the performance of a submarine can be simulated in a wind tunnel.
- Suppose, for example, that the engineers in Example above use a water tunnel instead of a wind tunnel to test their one-fifth scale model. Using the properties of water at room temperature (20°C is assumed), the water tunnel speed required to achieve similarity is easily calculated as

Example 3: Similarity between Model and Prototype Cars

$$V_m = V_p\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{L_p}{L_m}\right)$$

$$= (50.0\ \text{mi/h})\left(\frac{1.002 \times 10^{-3}\ \text{kg/m·s})}{1.849 \times 10^{-5}\ \text{kg/m·s}}\right)\left(\frac{1.184\ \text{kg/m}^3}{998.0\ \text{kg/m}^3}\right)(5) = 16.1\ \text{mi/h}$$

- As can be seen, one advantage of a water tunnel is that the required water tunnel speed is much lower than that required for a wind tunnel using the same size model

The Method of Repeating Variables and the Buckingham Pi Theorem

- In this section we will learn how to generate the nondimensional parameters, i.e., the Π's.
- There are several methods that have been developed for this purpose, but the most popular (and simplest) method is the **method of repeating variables,** popularized by Edgar Buckingham (1867–1940).
- We can think of this method as a step-by-step procedure or "recipe" for obtaining nondimensional parameters. There are six steps in this method as described below in detail

The Method of Repeating Variables and the Buckingham Pi Theorem

Step 1 List the parameters (dimensional variables, nondimensional variables, and dimensional constants) and count them. Let *n* be the total number of parameters in the problem, including the dependent variable. Make sure that any listed independent parameter is indeed independent of the others, i.e., it cannot be expressed in terms of them. (E.g., don't include radius *r* and area $A = \pi r^2$, since *r* and *A* are *not* independent.)

Step 2 List the primary dimensions for each of the *n* parameters.

Step 3 Guess the **reduction** *j*. As a first guess, set *j* equal to the number of primary dimensions represented in the problem. The expected number of Π's (*k*) is equal to *n* minus *j*, according to the **Buckingham Pi theorem,**

The Buckingham Pi theorem: $$k = n - j$$

If at this step or during any subsequent step, the analysis does not work out, verify that you have included enough parameters in step 1. Otherwise, go back and *reduce j by one* and try again.

The Method of Repeating Variables and the Buckingham Pi Theorem

Step 4 Choose j **repeating parameters** that will be used to construct each Π. Since the repeating parameters have the potential to appear in each Π, be sure to choose them *wisely*

Step 5 Generate the Π's one at a time by grouping the j repeating parameters with one of the remaining parameters, forcing the product to be dimensionless. In this way, construct all k Π's. By convention the first Π, designated as Π_1, is the *dependent* Π (the one on the left side of the list). Manipulate the Π's as necessary to achieve established dimensionless groups

Step 6 Check that all the Π's are indeed dimensionless.

The Method of Repeating Variables and the Buckingham Pi Theorem

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

- *Fig*. A concise summary of the six steps that comprise the *method of repeating variables*

The Method of Repeating Variables and the Buckingham Pi Theorem

- As a simple first example, consider a ball falling in a vacuum. Let us pretend that we do not know that Eq. 1 is appropriate for this problem, nor do we know much physics concerning falling objects.
- In fact, suppose that all we know is that the instantaneous elevation z of the ball must be a function of time t, initial vertical speed w_0, initial elevation z_0, and gravitational constant g.
- The beauty of dimensional analysis is that the only other thing we need to know is the primary dimensions of each of these quantities.
- As we go through each step of the method of repeating variables, we explain some of the subtleties of the technique in more detail using the falling ball as an example.

The Method of Repeating Variables

Step 1

- There are five parameters (dimensional variables, nondimensional variables, and dimensional constants) in this problem; *n = 5*. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:
- *List of relevant parameters:*

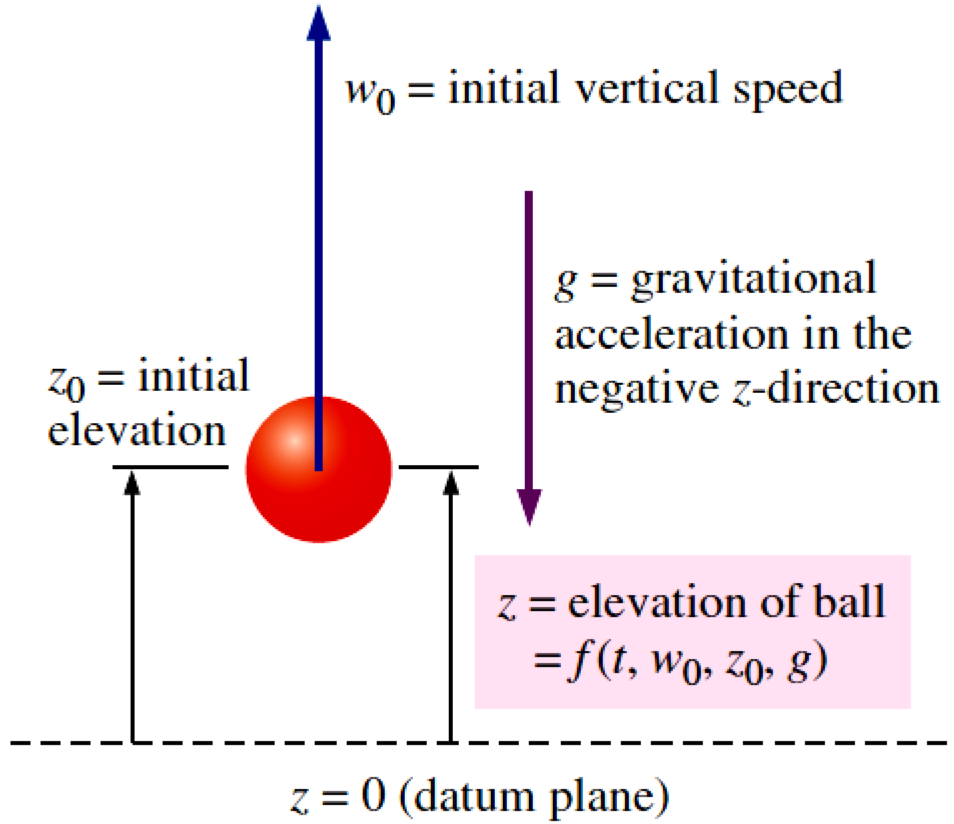


Fig. Setup for dimensional analysis of a ball falling in a vacuum. Elevation *z* is a function of time *t*, *initial vertical* speed w_0, *initial elevation* z_0*, and* gravitational constant *g*.

$$z = f(t, w_0, z_0, g) \qquad n = 5$$

The Method of Repeating Variables

Step 2

The primary dimensions of each parameter are listed here. We recommend writing each dimension with exponents since this helps with later algebra.

z	t	w_0	z_0	g
$\{L^1\}$	$\{t^1\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{L^1t^{-2}\}$

Step 3

As a first guess, j is set equal to 2, the number of primary dimensions represented in the problem (L and t).

Reduction: $j = 2$

If this value of j is correct, the number of Π's predicted by the Buckingham Pi theorem is

Number of expected Π*'s*: $k = n - j = 5 - 2 = 3$

The Method of Repeating Variables

Step 4

- We need to choose two repeating parameters since $j = 2$. Since this is often the hardest (or at least the most mysterious) part of the method of repeating variables, several guidelines about choosing repeating parameters are listed in Table 1.
- Following the guidelines of Table 1 on the next page, the wisest choice of two repeating parameters is w_0 *and* z_0.

Repeating parameters: w_0 *and* z_0

Step 5

- Now we combine these repeating parameters into products with each of the remaining parameters, one at a time, to create the Π's. The first Π is always the *dependent* Π and is formed with the dependent variable *z*.

Dependent Π : $\Pi_1 = z w_0^{a_1} z_0^{b_1}$ ....(1)

- where a_1 and b_1 are constant exponents that need to be determined.

The Method of Repeating Variables

- We apply the primary dimensions of step 2 into Eq. 1 and *force the Π to be* dimensionless by setting the exponent of each primary dimension to zero:
- *Dimensions of Π_1:*

$$\{\Pi_1\} = \{L^0 t^0\} = \{z w_0^{a_1} z_0^{b_1}\} = \{L^1 (L^1 t^{-1})^{a_1} L^{b_1}\}$$

- Since primary dimensions are by definition independent of each other, we equate the exponents of each primary dimension independently to solve for exponents a_1 and b_1

Time: $\{t^0\} = \{t^{-a_1}\}$ $\quad 0 = -a_1 \quad a_1 = 0$

Length: $\{L^0\} = \{L^1 L^{a_1} L^{b_1}\}$ $\quad 0 = 1 + a_1 + b_1 \quad b_1 = -1 - a_1 \quad b_1 = -1$

- Thus

$$\Pi_1 = \frac{z}{z_0}$$

The Method of Repeating Variables

- In similar fashion we create the first independent Π (Π_2) by combining the repeating parameters with independent variable *t*.

First independent Π: $$\Pi_2 = t w_0^{a_2} z_0^{b_2}$$

Dimensions of Π_2: $$\{\Pi_2\} = \{\mathrm{L}^0 \mathrm{t}^0\} = \{t w_0^{a_2} z_0^{b_2}\} = \{\mathrm{t}(\mathrm{L}^1\mathrm{t}^{-1})^{a_2} \mathrm{L}^{b_2}\}$$

Equating exponents,

Time: $$\{\mathrm{t}^0\} = \{\mathrm{t}^1\mathrm{t}^{-a_2}\} \qquad 0 = 1 - a_2 \qquad a_2 = 1$$

Length: $$\{\mathrm{L}^0\} = \{\mathrm{L}^{a_2}\mathrm{L}^{b_2}\} \qquad 0 = a_2 + b_2 \qquad b_2 = -a_2 \qquad b_2 = -1$$

Π_2 is thus

$$\Pi_2 = \frac{w_0 t}{z_0}$$

The Method of Repeating Variables

- Finally we create the second independent Π (Π_3) by combining the repeating parameters with *g and forcing the P to be dimensionless*

Second independent Π: $\quad \Pi_3 = g w_0^{a_3} z_0^{b_3}$

Dimensions of Π_3: $\quad \{\Pi_3\} = \{L^0 t^0\} = \{g w_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2} (L^1 t^{-1})^{a_3} L^{b_3}\}$

Equating exponents,

Time: $\quad \{t^0\} = \{t^{-2} t^{-a_3}\} \quad 0 = -2 - a_3 \quad a_3 = -2$

Length: $\quad \{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \quad 0 = 1 + a_3 + b_3 \quad b_3 = -1 - a_3 \quad b_3 = 1$

Π_3 is thus

$$\Pi_3 = \frac{g z_0}{w_0^2}$$

The Method of Repeating Variables

- We can see that Π_1 and Π_2 are the same as the nondimensionalized variables *z* and t* defined by* Eq. 3 (See slide number 15)—no manipulation is necessary for these.
- However, we recognize that the third P must be raised to the power of -1/2 to be of the same form as an established dimensionless parameter, namely the Froude number of

$$\Pi_{3,\,\text{modified}} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = \text{Fr}$$

- Such manipulation is often necessary to put the Π's into proper established form ("socially acceptable form" since it is a named, established nondimensional parameter that is commonly used in the literature.

The Method of Repeating Variables

Step 6

- We should double-check that the Π's are indeed dimensionless
- We are finally ready to write the functional relationship between the nondimensional parameters

Relationship between Π's:

$$\Pi_1 = f(\Pi_2, \Pi_3) \quad \rightarrow \quad \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{g z_0}}\right)$$

- The method of repeating variables properly predicts the functional relationship between dimensionless groups.
- However, the method of repeating variables cannot predict the exact mathematical form of the equation. This is a fundamental limitation of dimensional analysis and the method of repeating variables.

The Method of Repeating Variables

Table 1

Guidelines for choosing *repeating parameters* in step 4 of the method of repeating variables*

Guideline	Comments and Application to Present Problem
1. Never pick the *dependent* variable. Otherwise, it may appear in all the Π's, which is undesirable.	In the present problem we cannot choose z, but we must choose from among the remaining four parameters. Therefore, we must choose two of the following parameters: t, w_0, z_0, and g.
2. The chosen repeating parameters must not *by themselves* be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.	In the present problem, any two of the independent parameters would be valid according to this guideline. For illustrative purposes, however, suppose we have to pick three instead of two repeating parameters. We could not, for example, choose t, w_0, and z_0, because these can form a Π all by themselves (tw_0/z_0).
3. The chosen repeating parameters must represent *all* the primary dimensions in the problem.	Suppose for example that there were *three* primary dimensions (m, L, and t) and *two* repeating parameters were to be chosen. You could not choose, say, a length and a time, since primary dimension mass would not be represented in the dimensions of the repeating parameters. An appropriate choice would be a density and a time, which together represent all three primary dimensions in the problem.
4. Never pick parameters that are already dimensionless. These are Π's already, all by themselves.	Suppose an angle θ were one of the independent parameters. We could not choose θ as a repeating parameter since angles have no dimensions (radian and degree are dimensionless units). In such a case, one of the Π's is already known, namely θ.
5. Never pick two parameters with the *same* dimensions or with dimensions that differ by only an exponent.	In the present problem, two of the parameters, z and z_0, have the same dimensions (length). We cannot choose both of these parameters. (Note that dependent variable z has already been eliminated by guideline 1.) Suppose one parameter has dimensions of length and another parameter has dimensions of volume. In dimensional analysis, volume contains only one primary dimension (length) and *is not dimensionally distinct from length*—we cannot choose both of these parameters.

The Method of Repeating Variables

6.	Whenever possible, choose dimensional constants over dimensional variables so that only *one* Π contains the dimensional variable.	If we choose time t as a repeating parameter in the present problem, it would appear in all three Π's. While this would not be *wrong*, it would not be *wise* since we know that ultimately we want some nondimensional height as a function of some nondimensional time and other nondimensional parameter(s). From the original four independent parameters, this restricts us to w_0, z_0, and g.
7.	Pick common parameters since they may appear in each of the Π's.	In fluid flow problems we generally pick a length, a velocity, and a mass or density (Fig. 7–25). It is unwise to pick less common parameters like viscosity μ or surface tension σ_s, since we would in general not want μ or σ_s to appear in each of the Π's. In the present problem, w_0 and z_0 are wiser choices than g.
8.	Pick simple parameters over complex parameters whenever possible.	It is better to pick parameters with only one or two basic dimensions (e.g., a length, a time, a mass, or a velocity) instead of parameters that are composed of several basic dimensions (e.g., an energy or a pressure).

The Method of Repeating Variables

Table 2

Guidelines for manipulation of the Π's resulting from the method of repeating variables*

Guideline	Comments and Application to Present Problem
1. We may impose a constant (dimensionless) exponent on a Π or perform a functional operation on a Π.	We can raise a Π to any exponent n (changing it to Π^n) without changing the dimensionless stature of the Π. For example, in the present problem, we imposed an exponent of $-1/2$ on Π_3. Similarly we can perform the functional operation $\sin(\Pi)$, $\exp(\Pi)$, etc., without influencing the dimensions of the Π.
2. We may multiply a Π by a pure (dimensionless) constant.	Sometimes dimensionless factors of π, 1/2, 2, 4, etc., are included in a Π for convenience. This is perfectly okay since such factors do not influence the dimensions of the Π.
3. We may form a product (or quotient) of any Π with any other Π in the problem to replace one of the Π's.	We could replace Π_3 by $\Pi_3\Pi_1$, Π_3/Π_2, etc. Sometimes such manipulation is necessary to convert our Π into an established Π. In many cases, the established Π would have been produced if we would have chosen different repeating parameters.
4. We may use any of guidelines 1 to 3 in combination.	In general, we can replace any Π with some new Π such as $A\Pi_3{}^B \sin(\Pi_1{}^C)$, where A, B, and C are pure constants.
5. We may substitute a dimensional parameter in the Π with other parameter(s) of the same dimensions.	For example, the Π may contain the square of a length or the cube of a length, for which we may substitute a known area or volume, respectively, in order to make the Π agree with established conventions.

Table 3. Some common established nondimensional parameters

Some common established nondimensional parameters or Π's encountered in fluid mechanics and heat transfer*

Name	Definition	Ratio of Significance
Archimedes number	$\mathrm{Ar} = \dfrac{\rho_s g L^3}{\mu^2}(\rho_s - \rho)$	$\dfrac{\text{Gravitational force}}{\text{Viscous force}}$
Aspect ratio	$\mathrm{AR} = \dfrac{L}{W}$ or $\dfrac{L}{D}$	$\dfrac{\text{Length}}{\text{Width}}$ or $\dfrac{\text{Length}}{\text{Diameter}}$
Biot number	$\mathrm{Bi} = \dfrac{hL}{k}$	$\dfrac{\text{Surface thermal resistance}}{\text{Internal thermal resistance}}$
Bond number	$\mathrm{Bo} = \dfrac{g(\rho_f - \rho_v)L^2}{\sigma_s}$	$\dfrac{\text{Gravitational force}}{\text{Surface tension force}}$
Cavitation number	$\mathrm{Ca}\ (\text{sometimes } \sigma_c) = \dfrac{P - P_v}{\rho V^2}$ $\left(\text{sometimes } \dfrac{2(P - P_v)}{\rho V^2}\right)$	$\dfrac{\text{Pressure} - \text{Vapor pressure}}{\text{Inertial pressure}}$
Darcy friction factor	$f = \dfrac{8\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$

Drag coefficient	$C_D = \dfrac{F_D}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Drag force}}{\text{Dynamic force}}$
Eckert number	$\mathrm{Ec} = \dfrac{V^2}{c_p T}$	$\dfrac{\text{Kinetic energy}}{\text{Enthalpy}}$
Euler number	$\mathrm{Eu} = \dfrac{\Delta P}{\rho V^2}\left(\text{sometimes } \dfrac{\Delta P}{\frac{1}{2}\rho V^2}\right)$	$\dfrac{\text{Pressure difference}}{\text{Dynamic pressure}}$
Fanning friction factor	$C_f = \dfrac{2\tau_w}{\rho V^2}$	$\dfrac{\text{Wall friction force}}{\text{Inertial force}}$
Fourier number	$\mathrm{Fo}\ (\text{sometimes } \tau) = \dfrac{\alpha t}{L^2}$	$\dfrac{\text{Physical time}}{\text{Thermal diffusion time}}$
Froude number	$\mathrm{Fr} = \dfrac{V}{\sqrt{gL}}\left(\text{sometimes } \dfrac{V^2}{gL}\right)$	$\dfrac{\text{Inertial force}}{\text{Gravitational force}}$
Grashof number	$\mathrm{Gr} = \dfrac{g\beta \lvert \Delta T \rvert L^3 \rho^2}{\mu^2}$	$\dfrac{\text{Buoyancy force}}{\text{Viscous force}}$
Jakob number	$\mathrm{Ja} = \dfrac{c_p(T - T_{sat})}{h_{fg}}$	$\dfrac{\text{Sensible energy}}{\text{Latent energy}}$
Knudsen number	$\mathrm{Kn} = \dfrac{\lambda}{L}$	$\dfrac{\text{Mean free path length}}{\text{Characteristic length}}$

Lewis number	$\mathrm{Le} = \dfrac{k}{\rho c_p D_{AB}} = \dfrac{\alpha}{D_{AB}}$	$\dfrac{\text{Thermal diffusion}}{\text{Species diffusion}}$
Lift coefficient	$C_L = \dfrac{F_L}{\frac{1}{2}\rho V^2 A}$	$\dfrac{\text{Lift force}}{\text{Dynamic force}}$
Mach number	$\mathrm{Ma}\ (\text{sometimes } M) = \dfrac{V}{c}$	$\dfrac{\text{Flow speed}}{\text{Speed of sound}}$
Nusselt number	$\mathrm{Nu} = \dfrac{Lh}{k}$	$\dfrac{\text{Convection heat transfer}}{\text{Conduction heat transfer}}$
Peclet number	$\mathrm{Pe} = \dfrac{\rho L V c_p}{k} = \dfrac{LV}{\alpha}$	$\dfrac{\text{Bulk heat transfer}}{\text{Conduction heat transfer}}$
Power number	$N_p = \dfrac{\dot{W}}{\rho D^5 \omega^3}$	$\dfrac{\text{Power}}{\text{Rotational inertia}}$
Prandtl number	$\mathrm{Pr} = \dfrac{\nu}{\alpha} = \dfrac{\mu c_p}{k}$	$\dfrac{\text{Viscous diffusion}}{\text{Thermal diffusion}}$
Pressure coefficient	$C_p = \dfrac{P - P_\infty}{\frac{1}{2}\rho V^2}$	$\dfrac{\text{Static pressure difference}}{\text{Dynamic pressure}}$
Rayleigh number	$\mathrm{Ra} = \dfrac{g\beta \lvert\Delta T\rvert L^3 \rho^2 c_p}{k\mu}$	$\dfrac{\text{Buoyancy force}}{\text{Viscous force}}$

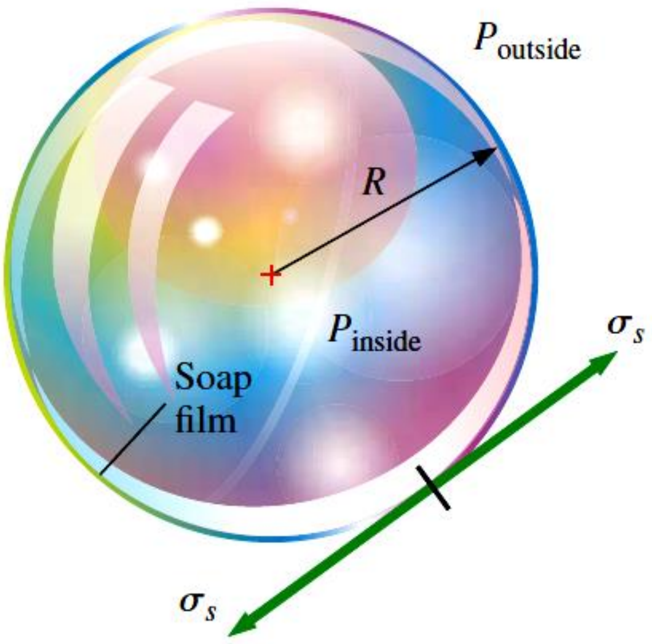
Reynolds number	$\mathrm{Re} = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$	$\frac{\text{Inertial force}}{\text{Viscous force}}$
Richardson number	$\mathrm{Ri} = \frac{L^5 g \Delta\rho}{\rho \dot{V}^2}$	$\frac{\text{Buoyancy force}}{\text{Inertial force}}$
Schmidt number	$\mathrm{Sc} = \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$	$\frac{\text{Viscous diffusion}}{\text{Species diffusion}}$
Sherwood number	$\mathrm{Sh} = \frac{V L}{D_{AB}}$	$\frac{\text{Overall mass diffusion}}{\text{Species diffusion}}$
Specific heat ratio	$k \text{ (sometimes } \gamma) = \frac{c_p}{c_v}$	$\frac{\text{Enthalpy}}{\text{Internal energy}}$
Stanton number	$\mathrm{St} = \frac{h}{\rho c_p V}$	$\frac{\text{Heat transfer}}{\text{Thermal capacity}}$
Stokes number	$\mathrm{Stk} \text{ (sometimes St)} = \frac{\rho_p D_p^2 V}{18 \mu L}$	$\frac{\text{Particle relaxation time}}{\text{Characteristic flow time}}$
Strouhal number	$\mathrm{St} \text{ (sometimes S or Sr)} = \frac{f L}{V}$	$\frac{\text{Characteristic flow time}}{\text{Period of oscillation}}$
Weber number	$\mathrm{We} = \frac{\rho V^2 L}{\sigma_s}$	$\frac{\text{Inertial force}}{\text{Surface tension force}}$

EXAMPLE 4. Pressure in a Soap Bubble

- Some children are playing with soap bubbles, and you become curious as to the relationship between soap bubble radius and the pressure inside the soap bubble. You reason that the pressure inside the soap bubble must be greater than atmospheric pressure, and that the shell of the soap bubble is under tension, much like the skin of a balloon. You also know that the property surface tension must be important in this problem. Not knowing any other physics, you decide to approach the problem using dimensional analysis. Establish a relationship between pressure difference

$$\Delta P = P_{\text{inside}} - P_{\text{outside}}$$

- soap bubble radius R, *and the surface tension* σ_s *of the soap film.*



The pressure inside a soap bubble is greater than that surrounding the soap bubble due to surface tension in the soap film.

EXAMPLE 4. Pressure in a Soap Bubble

- **SOLUTION.** The pressure difference between the inside of a soap bubble and the outside air is to be analyzed by the method of repeating variables.
- ***Assumptions*** *1*. The soap bubble is neutrally buoyant in the air, and gravity is not relevant. 2 No other variables or constants are important in this problem.
- ***Analysis*** The step-by-step method of repeating variables is employed.

Step 1 There are three variables and constants in this problem; $n = 3$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $\Delta P = f(R, \sigma_s) \qquad n = 3$

EXAMPLE 4. Pressure in a Soap Bubble

- **Step 2** The primary dimensions of each parameter are listed.

ΔP	R	σ_s
$\{m^1L^{-1}t^{-2}\}$	$\{L^1\}$	$\{m^1t^{-2}\}$

- **Step 3** As a first guess, *j is set equal to 3, the number of primary dimensions* represented in the problem (m, L, and t).

Reduction (first guess): $j = 3$

- If this value of *j is correct, the expected number of Π's is*

$$k = n - j = 3 - 3 = 0.$$

But how can we have zero P's? Something is obviously not right

- At times like this, we need to first go back and make sure that we are not neglecting some important variable or constant in the problem.
- Since we are confident that the pressure difference should depend only on soap bubble radius and surface tension, we reduce the value of *j by one*,

EXAMPLE 4. Pressure in a Soap Bubble

Reduction (second guess): $j = 2$

- If this value of j is correct, $k = n - j = 3 - 2 = 1$. Thus we expect one Π, which is more physically realistic than zero Π's.
- **Step 4** We need to choose two repeating parameters since $j = 2$. Following the guidelines of Table 1, our only choices are R and σ_s, since ΔP *is the* dependent variable.
- **Step 5** We combine these repeating parameters into a product with the dependent variable ΔP to create the dependent Π,

Dependent Π: $\Pi_1 = \Delta P R^{a_1} \sigma_s^{b_1}$ ....(1)

- We apply the primary dimensions of step 2 into Eq. 1 and force the Π to be dimensionless.

Dimensions of Π_1:

$$\{\Pi_1\} = \{m^0 L^0 t^0\} = \{\Delta P R^{a_1} \sigma_s^{b_1}\} = \{(m^1 L^{-1} t^{-2}) L^{a_1} (m^1 t^{-2})^{b_1}\}$$

EXAMPLE 4. Pressure in a Soap Bubble

We equate the exponents of each primary dimension to solve for a_1 and b_1:

$$\begin{array}{llll} \textit{Time:} & \{t^0\} = \{t^{-2} t^{-2b_1}\} & 0 = -2 - 2b_1 & b_1 = -1 \\ \textit{Mass:} & \{m^0\} = \{m^1 m^{b_1}\} & 0 = 1 + b_1 & b_1 = -1 \\ \textit{Length:} & \{L^0\} = \{L^{-1} L^{a_1}\} & 0 = -1 + a_1 & a_1 = 1 \end{array}$$

Fortunately, the first two results agree with each other, and Eq. 1 thus becomes

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} \qquad \textbf{(2)}$$

- From Table 3, the established nondimensional parameter most similar to Eq. 2 is the **Weber number,** defined as a pressure (ρV^2) *times a length* divided by surface tension. There is no need to further manipulate this Π.

EXAMPLE 4. Pressure in a Soap Bubble

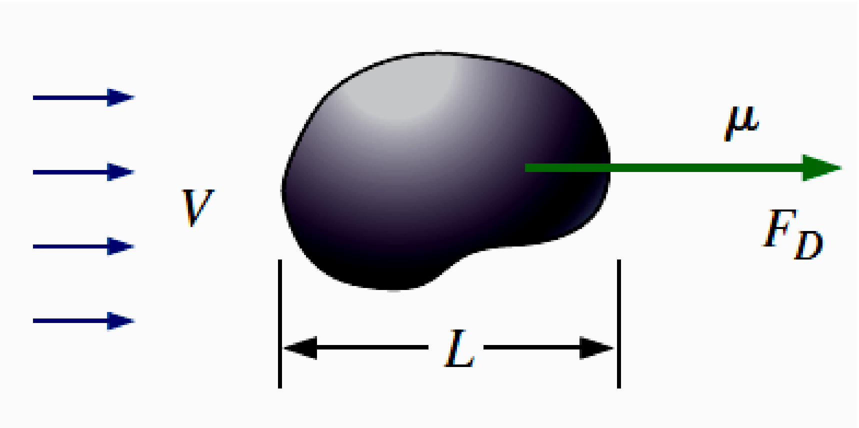
- **Step 6** We write the final functional relationship. In the case at hand, there is only one Π, which is a function of *nothing*. *This is possible only if* the Π is constant.
- *Relationship between Π's:*

$$\Pi_1 = \frac{\Delta P R}{\sigma_s} = f(\text{nothing}) = \text{constant} \quad \rightarrow \quad \boldsymbol{\Delta P = \text{constant}\,\frac{\sigma_s}{R}} \qquad \textbf{(3)}$$

- This is an example of how we can sometimes predict *trends with* dimensional analysis, even without knowing much of the physics of the problem. For example, we know from our result that if the radius of the soap bubble doubles, the pressure difference decreases by a factor of 2. Similarly, if the value of surface tension doubles, ΔP increases by a factor of 2.
- Dimensional analysis cannot predict the value of the constant in Eq. 3; further analysis (or one experiment) reveals that the constant is equal to 4 (Chap. 1).

Example 5

- When small aerosol particles or microorganisms move through air or water, the Reynolds number is very small (Re << 1). Such flows are called **creeping flows**. **The aerodynamic** drag on an object in creeping flow is a function only of its speed *V*, some characteristic length scale *L* of the object, and fluid viscosity μ. Use dimensional analysis to generate a relationship for F_D as a function of the independent variables.



Solution We are to use dimensional analysis to find a functional relationship between F_D and variables V, L, and μ.

Assumptions 1 We assume Re << 1 so that the creeping flow approximation applies. 2 Gravitational effects are irrelevant. 3 No parameters other than those listed in the problem statement are relevant to the problem.

Analysis We follow the step-by-step method of repeating variables.

Step 1 There are four variables and constants in this problem; $n = 4$. They are listed in functional form, with the dependent variable listed as a function of the independent variables and constants:

List of relevant parameters: $$F_D = f(V, L, \mu) \qquad n = 4$$

Step 2 The primary dimensions of each parameter are listed.

F_D	V	L	μ
$\{m^1L^1t^{-2}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-1}t^{-1}\}$

Step 3 As a first guess, we set j equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction: $$j = 3$$

If this value of j is correct, the number of Πs expected is

Number of expected Πs: $$k = n - j = 4 - 3 = 1$$

Step 4 Now we need to choose three repeating parameters since $j = 3$. Since we cannot choose the dependent variable, our only choices are V, L, and μ.

Step 5 Now we combine these repeating parameters into a product with the dependent variable F_D to create the dependent Π,

Dependent Π: $$\Pi_1 = F_D V^{a_1} L^{b_1} \mu^{c_1} \qquad (1)$$

We apply the primary dimensions of Step 2 into Eq. 1 and force the Π to be dimensionless,

Dimensions of Π_1:

$$\{\Pi_1\} = \{m^0L^0t^0\} = \{F_D V^{a_1} L^{b_1} \mu^{c_1}\} = \{(m^1L^1t^{-2})(L^1t^{-1})^{a_1}(L^1)^{b_1}(m^1L^{-1}t^{-1})^{c_1}\}$$

Now we equate the exponents of each primary dimension to solve for exponents a_1 through c_1.

mass:	$\{m^0\} = \{m^1 m^{c_1}\}$	$0 = 1 + c_1$	$c_1 = -1$
time:	$\{t^0\} = \{t^{-2} t^{-a_1} t^{-c_1}\}$	$0 = -2 - a_1 - c_1$	$a_1 = -1$
length:	$\{L^0\} = \{L^1 L^{a_1} L^{b_1} L^{-c_1}\}$	$0 = 1 + a_1 + b_1 - c_1$	$b_1 = -1$

Equation 1 thus becomes

Π_1:

$$\Pi_1 = \frac{F_D}{\mu V L} \qquad (2)$$

Step 6 We now write the functional relationship between the nondimensional parameters. In the case at hand, there is only one Π, which is a function of *nothing*. This is possible only if the Π is constant. Putting Eq. 2 into standard functional form,

Relationship between Πs:
$$\Pi_1 = \frac{F_D}{\mu V L} = f(\text{nothing}) = \text{constant} \qquad (3)$$

or

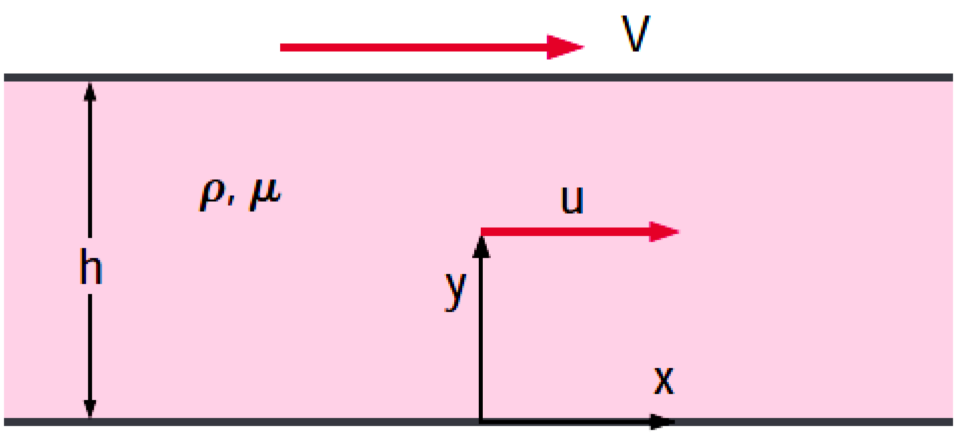
Result of dimensional analysis:
$$\boxed{F_D = \text{constant} \cdot \mu V L} \qquad (4)$$

Thus we have shown that for creeping flow around an object, the aerodynamic drag force is simply a constant multiplied by $\mu V L$, regardless of the shape of the object.

Discussion This result is very significant because all that is left to do is find the constant, which will be a function of the shape of the object (and its orientation with respect to the flow).

Example 6

- Consider fully developed **Couette flow**—flow between two infinite parallel plates separated by distance *h*, with the top plate moving and the bottom plate stationary as illustrated in the Fig. shown. The flow is steady, incompressible, and two-dimensional in the *xy*-plane. Use the method of repeating variables to generate a dimensionless relationship for the *x* component of fluid velocity *u* as a function of fluid viscosity μ, top plate speed *V*, distance h, fluid density *ρ*, *and* distance *y*.



Solution We are to use dimensional analysis to find the functional relationship between the given parameters.

Assumptions 1 The given parameters are the only relevant ones in the problem.

Analysis The step-by-step method of repeating variables is employed to obtain the nondimensional parameters (the Πs).

Step 1 There are six parameters in this problem; $n = 6$,

List of relevant parameters: $$u = f(\mu, V, h, \rho, y) \qquad n = 6 \qquad (1)$$

Step 2 The primary dimensions of each parameter are listed,

u	μ	V	h	ρ	y
$\{L^1t^{-1}\}$	$\{m^1L^{-1}t^{-1}\}$	$\{L^1t^{-1}\}$	$\{L^1\}$	$\{m^1L^{-3}\}$	$\{L^1\}$

Step 3 As a first guess, j is set equal to 3, the number of primary dimensions represented in the problem (m, L, and t).

Reduction: $$j = 3$$

If this value of j is correct, the expected number of Πs is

Number of expected Πs: $$k = n - j = 6 - 3 = 3$$

Step 4 We need to choose three repeating parameters since $j = 3$. Following the guidelines outlined in this chapter, we elect not to pick the viscosity. It is better to pick a fixed length (h) rather than a variable length (y); otherwise y would appear in each Pi, which would not be desirable. We choose

Repeating parameters: V, ρ, and h

Step 5 The dependent Π is generated:

$$\Pi_1 = uV^{a_1}\rho^{b_1}h^{c_1} \qquad \{\Pi_1\} = \left\{\left(L^1t^{-1}\right)\left(L^1t^{-1}\right)^{a_1}\left(m^1L^{-3}\right)^{b_1}\left(L^1\right)^{c_1}\right\}$$

mass: $\left\{m^0\right\} = \left\{m^{b_1}\right\}$ $\qquad 0 = b_1 \qquad b_1 = 0$

time: $\left\{t^0\right\} = \left\{t^{-1}t^{-a_1}\right\}$ $\qquad 0 = -1 - a_1 \qquad a_1 = -1$

length: $\left\{L^0\right\} = \left\{L^1L^{a_1}L^{-3b_1}L^{c_1}\right\}$ $\qquad 0 = 1 + a_1 - 3b_1 + c_1 \qquad c_1 = 0$

The dependent Π is thus

Π_1: $$\Pi_1 = \frac{u}{V}$$

The second Pi (the first independent Π in this problem) is generated:

$$\Pi_2 = \mu V^{a_2}\rho^{b_2}h^{c_2} \qquad \{\Pi_2\} = \left\{\left(m^1L^{-1}t^{-1}\right)\left(L^1t^{-1}\right)^{a_2}\left(m^1L^{-3}\right)^{b_2}\left(L^1\right)^{c_2}\right\}$$

mass: $\left\{m^0\right\} = \left\{m^1m^{b_2}\right\}$ $\qquad 0 = 1 + b_2 \qquad b_2 = -1$

time: $\{t^0\} = \{t^{-1} t^{-a_2}\}$ $0 = -1 - a_2$ $a_2 = -1$

length: $\{L^0\} = \{L^{-1} L^{a_2} L^{-3b_2} L^{c_2}\}$ $0 = -1 + a_2 - 3b_2 + c_2$ $c_2 = -1$

$0 = -1 - 1 + 3 + c_2$

which yields

Π_2: $\Pi_2 = \dfrac{\mu}{\rho V h}$

We recognize this Π as the inverse of the Reynolds number. So, after inverting,

Modified Π_2: $\Pi_2 = \dfrac{\rho V h}{\mu} = \text{Reynolds number} = \text{Re}$

The third Pi (the second independent Π in this problem) is generated:

$\Pi_3 = y V^{a_3} \rho^{b_3} h^{c_3}$ $\{\Pi_3\} = \left\{(L^1)(L^1 t^{-1})^{a_3} (m^1 L^{-3})^{b_3} (L^1)^{c_3}\right\}$

mass: $\{m^0\} = \{m^{b_3}\}$ $0 = b_3$ $b_3 = 0$

time: $\{t^0\} = \{t^{-a_3}\}$ $0 = -a_3$ $a_3 = 0$

length: $\{L^0\} = \{L^1 L^{a_3} L^{-3b_3} L^{c_3}\}$

$$0 = 1 + a_3 - 3b_3 + c_3$$
$$0 = 1 + c_3$$

$c_3 = -1$

which yields

Π_3: $\Pi_3 = \dfrac{y}{h}$

Step 6 **We write the final functional relationship as**

Relationship between Πs: $$\boxed{\frac{u}{V} = f\left(\mathrm{Re}, \frac{y}{h}\right)} \qquad (2)$$

End of Chapter 5

Next Lecture

Chapter 6: Boundary Layer Concept