Mechanical Engineering Department

# Fluid Mechanics

## (MEng 2113)

## Chapter 4

## Diffrential Relations For A Fluid Flow

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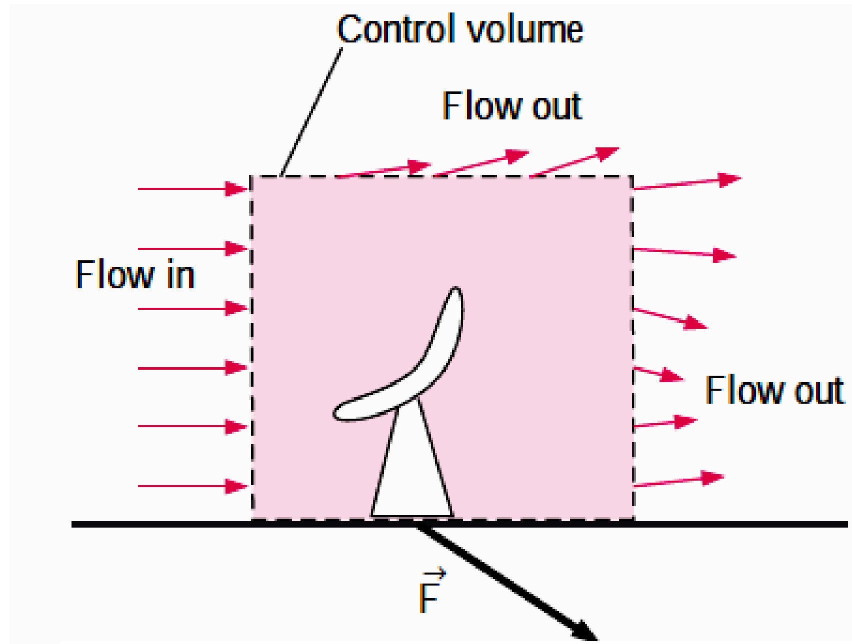
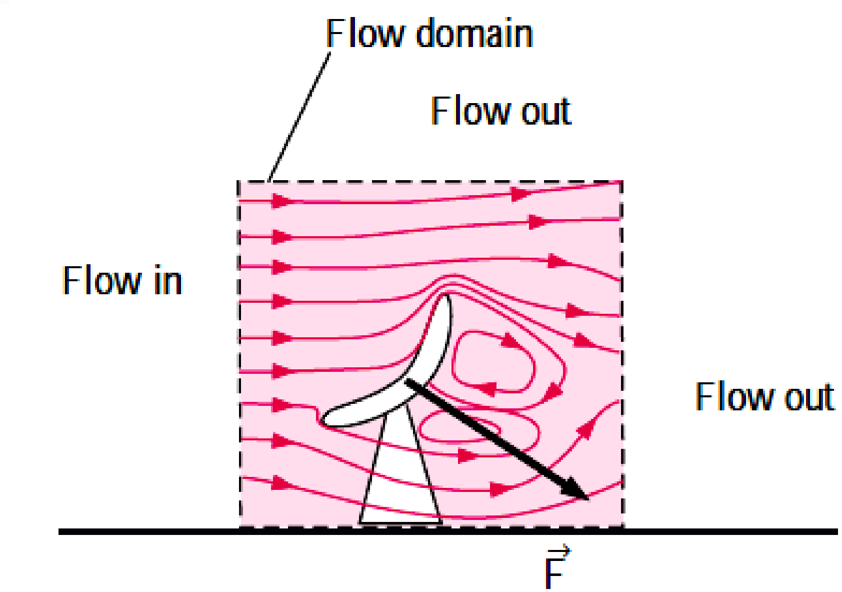
# Introduction

- In analyzing fluid motion, we might take one of two paths:

1. Seeking an estimate of gross effects (mass flow, induced force, energy change) over a finite region or control volume or
2. Seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow.



# Introduction

- The control volume technique is useful when we are interested in the overall features of a flow, such as mass flow rate into and out of the control volume or net forces applied to bodies.
- Differential analysis, on the other hand, involves application of differential equations of fluid motion to *any and every point in the flow field over a region* called the flow domain.
- When solved, these differential equations yield details about the velocity, density, pressure, etc., at *every point* throughout the *entire flow domain*.

# The Acceleration Field of a Fluid

- Velocity is a vector function of position and time and thus has three components *u*, *v*, and *w*, each a scalar field in itself.

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t)$$

- This is the most important variable in fluid mechanics: Knowledge of the velocity vector field is nearly equivalent to *solving a fluid flow problem.*
- The acceleration vector field a of the flow is derived from Newton's second law by computing the total time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt}$$

# The Acceleration Field of a Fluid

- Since each scalar component ($u$, $v$ , $w$) is a function of the four variables ($x$, $y$, $z$, $t$), we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

- But, by definition, $dx/dt$ is the local velocity component $u$, *and* $dy/dt = v$ , *and* $dz/dt = w$.
- The total time derivative of $u$ may thus be written as follows, with exactly similar expressions for the time derivatives of $v$ *and* $w$:

# The Acceleration Field of a Fluid

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V} \cdot \nabla)u$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V} \cdot \nabla)v$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V} \cdot \nabla)w$$

- Summing these into a vector, we obtain the total

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

# The Acceleration Field of a Fluid

- The term δV/δt is called the **local acceleration**, which vanishes if the flow is steady-that is, independent of time.
- The three terms in parentheses are called the **convective acceleration**, which arises when the particle moves through regions of spatially varying velocity, as in a nozzle or diffuser.
- The gradient operator $\nabla$ is given by:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \mathbf{V} \cdot \nabla$$

# The Acceleration Field of a Fluid

- The total time derivative—sometimes called the ***substantial or material derivative***— concept may be applied to any variable, such as the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \upsilon\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})p$$

- Wherever convective effects occur in the basic laws involving mass, momentum, or energy, the basic differential equations become nonlinear and are usually more complicated than flows that do not involve convective changes.

## Example 1. Acceleration field

Given the eulerian velocity vector field

$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the total acceleration of a particle.

**Solution**

- *Assumptions:* Given three known unsteady velocity components, $u = 3t$, $v = xz$, and $w = ty^2$.

- *Solution step 1:* First work out the local acceleration $\partial\mathbf{V}/\partial t$:

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i}\frac{\partial u}{\partial t} + \mathbf{j}\frac{\partial v}{\partial t} + \mathbf{k}\frac{\partial w}{\partial t} = \mathbf{i}\frac{\partial}{\partial t}(3t) + \mathbf{j}\frac{\partial}{\partial t}(xz) + \mathbf{k}\frac{\partial}{\partial t}(ty^2) = 3\mathbf{i} + 0\mathbf{j} + y^2\mathbf{k}$$

Solution step 2: In a similar manner, the convective acceleration terms, are

Solution step 2: In a similar manner, the convective acceleration terms, are

$$u\frac{\partial \mathbf{V}}{\partial x} = (3t)\frac{\partial}{\partial x}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (3t)(0\mathbf{i} + z\mathbf{j} + 0\mathbf{k}) = 3tz\,\mathbf{j}$$

$$v\frac{\partial \mathbf{V}}{\partial y} = (xz)\frac{\partial}{\partial y}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (xz)(0\mathbf{i} + 0\mathbf{j} + 2ty\mathbf{k}) = 2txyz\,\mathbf{k}$$

$$w\frac{\partial \mathbf{V}}{\partial z} = (ty^2)\frac{\partial}{\partial z}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (ty^2)(0\mathbf{i} + x\mathbf{j} + 0\mathbf{k}) = txy^2\,\mathbf{j}$$

- *Solution step 3:* Combine all four terms above into the single "total" or "substantial" derivative:

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (3\mathbf{i} + y^2\mathbf{k}) + 3tz\mathbf{j} + 2txyz\mathbf{k} + txy^2\mathbf{j}$$

$$= 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (y^2 + 2txyz)\mathbf{k} \qquad \textit{Ans.}$$

- *Comments:* Assuming that **V** is valid everywhere as given, this total acceleration vector $d\mathbf{V}/dt$ applies to all positions and times within the flow field.

# Example 2. Acceleration field

- An idealized velocity field is given by the formula

$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

- Is this flow field steady or unsteady? Is it two- or three dimensional? At the point $(x, y, z) = (1, 1, 0)$, compute the acceleration vector.

***Solution***

- The flow is unsteady because time *t* appears explicitly in the components.
- The flow is three-dimensional because all three velocity components are nonzero.
- Evaluate, by differentiation, the acceleration vector at (x, y, z) = (−1, +1, 0).

# Example 2. Acceleration field

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=4x+4tx(4t)-2t^2y(0)+4xz(0)=4x+16t^2x$$

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-4ty+4tx(0)-2t^2y(-2t^2)+4xz(0)=-4ty+4t^4y$$

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0+4tx(4z)-2t^2y(0)+4xz(4x)=16txz+16x^2z$$

or: $$\frac{d\mathbf{V}}{dt}=(4x+16t^2x)\mathbf{i}+(-4ty+4t^4y)\mathbf{j}+(16txz+16x^2z)\mathbf{k}$$

at $(x, y, z) = (-1, +1, 0)$, we obtain $$\frac{d\mathbf{V}}{dt}=-4(1+4t^2)\mathbf{i}-4t(1-t^3)\mathbf{j}+0\mathbf{k}$$

## *Exercise 1*

- The velocity in a certain two-dimensional flow field is given by the equation

$$\mathbf{V} = 2xt\hat{\mathbf{i}} - 2yt\hat{\mathbf{j}}$$

where the velocity is in m/s when $x$, $y$, and $t$ are in meter and seconds, respectively.

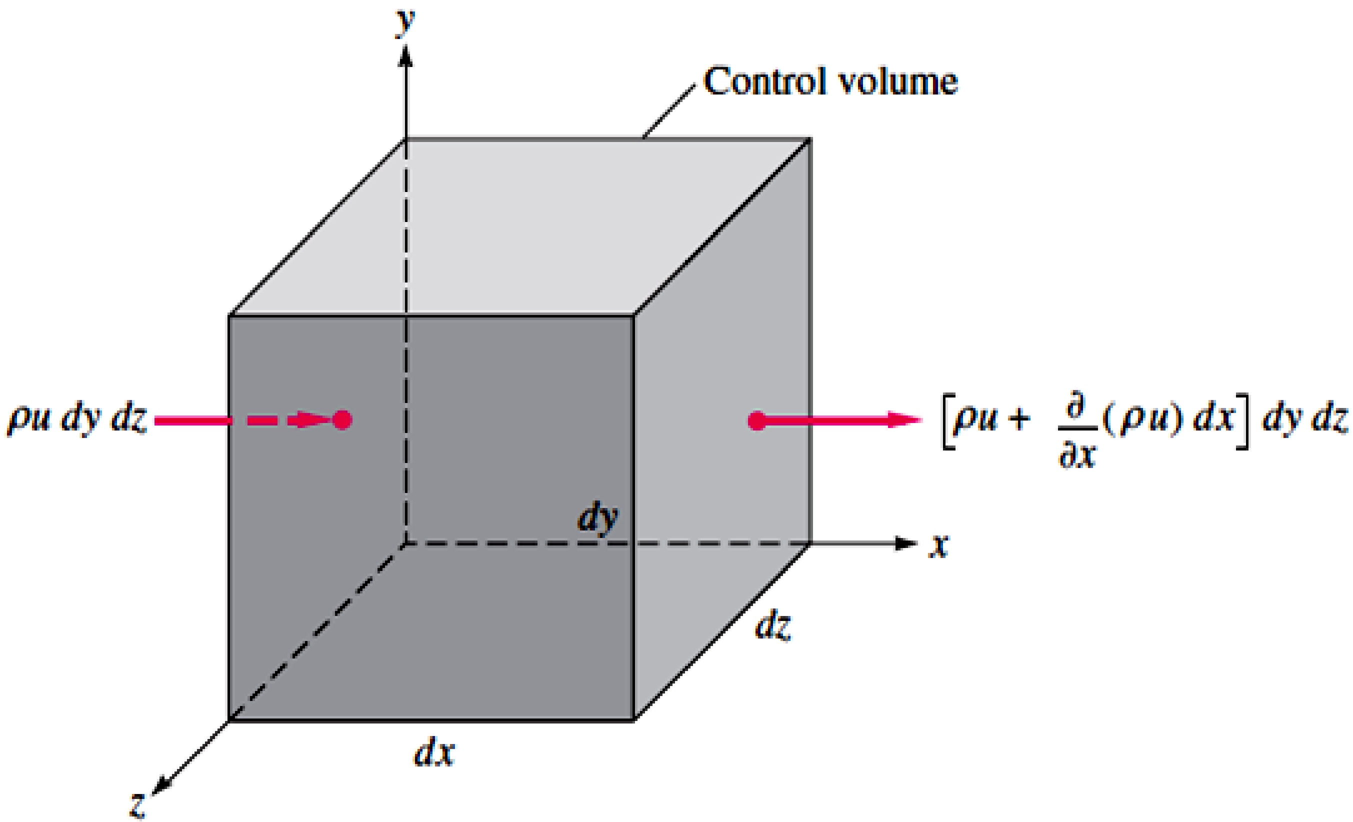
1. Determine expressions for the local and convective components of acceleration in the x and y directions.
2. What is the magnitude and direction of the velocity and the acceleration at the point $x = y = 2\ m$ at the time $t = 0$?

## The Differential Equation of Mass Conservation

- Conservation of mass, often called the ***continuity relation***, states that the fluid mass cannot change.
- We apply this concept to a very small region. All the basic differential equations can be derived by considering either an elemental control volume or an elemental system.
- We choose an infinitesimal fixed control volume (*dx*, *dy*, *dz)*, as in shown in fig below, and use basic control volume relations.
- The flow through each side of the element is approximately one-dimensional, and so the appropriate mass conservation relation to use here is

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

# The Differential Equation of Mass Conservation




- The element is so small that the volume integral simply reduces to a differential term:

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

## The Differential Equation of Mass Conservation

- The mass flow terms occur on all six faces, three inlets and three outlets.
- Using the field or continuum concept where all fluid properties are considered to be uniformly varying functions of time and position, such as $\rho = \rho\,(x, y, z, t)$.
- Thus, if T is the temperature on the left face of the element, the right face will have a slightly different temperature $T + (\partial T/\partial x)\,dx$.
- For mass conservation, if $\rho u$ *is* known on the left face, the value of this product on the right face is $\rho u + (\partial \rho u/\partial x)\,dx$.

# The Differential Equation of Mass Conservation

| Face | Inlet mass flw | Outlet mass flw |
|---|---|---|
| $x$ | $\rho u \, dy \, dz$ | $\left[\rho u + \frac{\partial}{\partial x}(\rho u) \, dx\right] dy \, dz$ |
| $y$ | $\rho v \, dx \, dz$ | $\left[\rho v + \frac{\partial}{\partial y}(\rho v) \, dy\right] dx \, dz$ |
| $z$ | $\rho w \, dx \, dy$ | $\left[\rho w + \frac{\partial}{\partial z}(\rho w) \, dz\right] dx \, dy$ |

- Introducing these terms into the main relation

$$\frac{\partial \rho}{\partial t} dx \, dy \, dz + \frac{\partial}{\partial x}(\rho u) \, dx \, dy \, dz + \frac{\partial}{\partial y}(\rho v) \, dx \, dy \, dz + \frac{\partial}{\partial z}(\rho w) \, dx \, dy \, dz = 0$$

- Simplifying gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

*equation of continuity*

## The Differential Equation of Mass Conservation

- The vector gradient operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

- enables us to rewrite the equation of continuity in a compact form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho \mathbf{V})$$

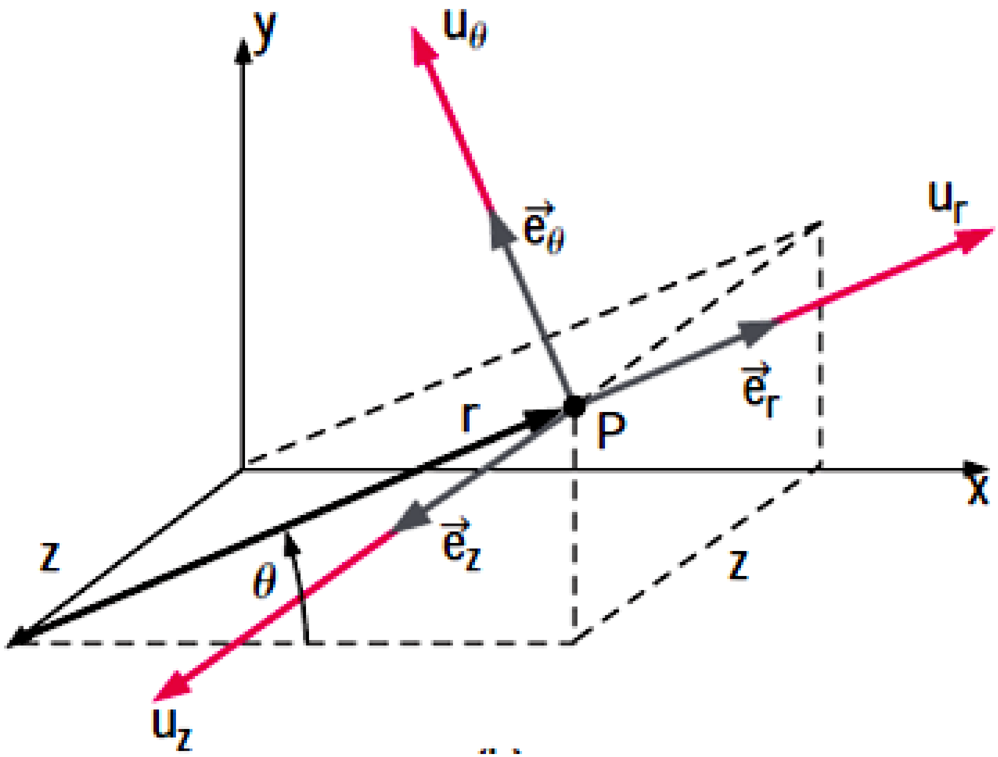
- so that the compact form of the continuity relation is

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0}$$

## The Differential Equation of Mass Conservation

### Continuity Equation in Cylindrical Coordinates

- Many problems in fluid mechanics are more conveniently solved in cylindrical coordinates ($r$, $\theta$, $z$) (often called cylindrical polar coordinates), rather than in Cartesian coordinates.



## The Differential Equation of Mass Conservation

- Continuity equation in cylindrical coordinates is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

**Steady Compressible Flow**

- If the flow is steady $\partial/\partial t \equiv 0$ and all properties are functions of position only and the continuity equation reduces to

Cartesian:
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

## The Differential Equation of Mass Conservation

### Incompressible Flow

- A special case that affords great simplification is incompressible flow, where the density changes are negligible. Then $\partial\rho/\partial t \approx 0$ regardless of whether the flow is steady or unsteady,
- The result

$$\nabla \cdot \mathbf{V} = 0$$

- is valid for steady or unsteady incompressible flow. The two coordinate forms are

Cartesian: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

## The Differential Equation of Mass Conservation

- The criterion for incompressible flow is $\mathrm{Ma} \leq 0.3$
- where Ma = *V/a* is the dimensionless Mach number of the flow.
- For air at standard conditions, a flow can thus be considered incompressible if the velocity is less than about 100 m/s.

# Example 3

- Consider the steady, two-dimensional velocity field given by

$$\vec{V} = (u, v) = (1.3 + 2.8x)\vec{i} + (1.5 - 2.8y)\vec{j}$$

- Verify that this flow field is incompressible.

## *Solution*

- **Analysis.** The flow is two-dimensional, implying no z component of velocity and no variation of u or v with z.
- The components of velocity in the x and y directions respectively are

$$u = 1.3 + 2.8x \qquad v = 1.5 - 2.8y$$

- To check if the flow is incompressible, we see if the incompressible continuity equation is satisfied:

$$\underbrace{\frac{\partial u}{\partial x}}_{2.8} + \underbrace{\frac{\partial v}{\partial y}}_{-2.8} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0 \text{ since 2-D}} = 0 \qquad \text{or} \qquad 2.8 - 2.8 = 0$$

- So we see that the incompressible continuity equation is indeed satisfied. Hence the flow field is incompressible.

## Example 4

- Consider the following steady, three-dimensional velocity field in Cartesian coordinates:

$$\vec{V} = (u, v, w) = (axy^2 - b)\vec{i} + cy^3\vec{j} + dxy\vec{k}$$

where *a*, *b*, *c*, *and d* are constants. Under what conditions is this flow field incompressible?

**Solution**

*Condition for incompressibility:*

$$\underbrace{\frac{\partial u}{\partial x}}_{ay^2} + \underbrace{\frac{\partial v}{\partial y}}_{3cy^2} + \underbrace{\frac{\partial w}{\partial z}}_{0} = 0 \qquad ay^2 + 3cy^2 = 0$$

- Thus to guarantee incompressibility, constants *a and c* must satisfy the following relationship:

$$a = -3c$$

## Example 5

- An idealized incompressible flow has the proposed three-dimensional velocity distribution

$$\mathbf{V} = 4xy^2\mathbf{i} + f(y)\mathbf{j} - zy^2\mathbf{k}$$

- Find the appropriate form of the function *f(y)* which satisfies the continuity relation.
- **Solution:** Simply substitute the given velocity components into the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}(4xy^2) + \frac{\partial f}{\partial y} + \frac{\partial}{\partial z}(-zy^2) = 4y^2 + \frac{df}{dy} - y^2 = 0$$

$$or: \quad \frac{df}{dy} = -3y^2. \quad \text{Integrate: } f(y) = \int (-3y^2)dy = \mathbf{-y^3 + constant} \quad \textit{Ans.}$$

## Example 6

- For a certain incompressible flow field it is suggested that the velocity components are given by the equations

$$u = 2xy \quad v = -x^2y \quad w = 0$$

Is this a physically possible flow field? Explain.

Any physically possible incompressible flow field must satisfy conservation of mass as expressed by the relationship

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

For the velocity distribution given,

$$\frac{\partial u}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = -x^2 \qquad \frac{\partial w}{\partial z} = 0$$

Substitution into Eq. (1) shows that

$$2y - x^2 + 0 \neq 0$$

Thus, this is not a physically possible flow field. <u>No.</u>

## *Example 7*

- For a certain incompressible, two-dimensional flow field the velocity component in the *y* direction is given by the equation

$$v = x^2 + 2xy$$

- Determine the velocity in the x direction so that the continuity equation is satisfied.

# Example 7 - solution

To satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Since

$$\frac{\partial v}{\partial y} = 2x$$

Then from Eq. (1)

$$\frac{\partial u}{\partial x} = -2x \qquad (2)$$

Equation (2) can be integrated with respect to x to obtain

$$\int du = -\int 2x\, dx + f(y)$$

or

$$u = \underline{\underline{-x^2 + f(y)}}$$

where $f(y)$ is an undetermined function of $y$.

## *Example 8*

- The radial velocity component in an incompressible, two dimensional flow field $(v_z = 0)$ is

$$v_r = 2r + 3r^2 \sin\theta$$

- Determine the corresponding tangential velocity component, $v_\theta$, required to satisfy conservation of mass.

**Solution.**

- The continuity equation for incompressible steady flow in cylindrical coordinates is given by

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Since $v_z = 0$,

$$\frac{\partial v_\theta}{\partial \theta} = -\frac{\partial (r v_r)}{\partial r} \quad (1)$$

# Example 8

and with

$$r v_r = 2r^2 + 3r^3 \sin\theta$$

it follows that

$$\frac{\partial (r v_r)}{\partial r} = 4r + 9r^2 \sin\theta$$

Thus, Eq.(1) becomes

$$\frac{\partial v_\theta}{\partial \theta} = -(4r + 9r^2 \sin\theta) \qquad (2)$$

Equation(2) can be integrated with respect to $\theta$ to obtain

$$\int d v_\theta = -\int (4r + 9r^2 \sin\theta)\, d\theta + f(r)$$

or

$$v_\theta = \underline{\underline{-4r\theta - 9r^2 \cos\theta + f(r)}}$$

where $f(r)$ is an undetermined function of $r$.

# The Stream Function

- Consider the simple case of incompressible, two-dimensional flow in the *xy*-plane.
- The continuity equation in Cartesian coordinates reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

- A clever variable transformation enables us to rewrite this equation (Eq. 1) in terms of *one* dependent variable ($\psi$) instead of two dependent variables ($u$ and $v$).
- We define the **stream function** $\boldsymbol{\psi}$ as

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} \qquad (2)$$

# The Stream Function

- Substitution of Eq. 2 into Eq. 1 yields

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$$

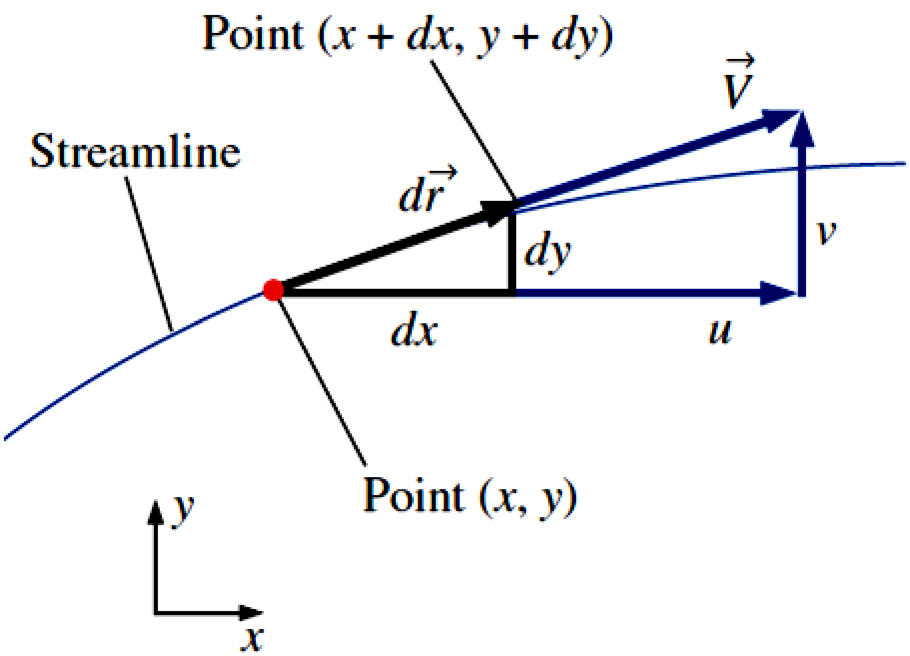
- which is identically satisfied for any smooth function $\psi(x, y)$.

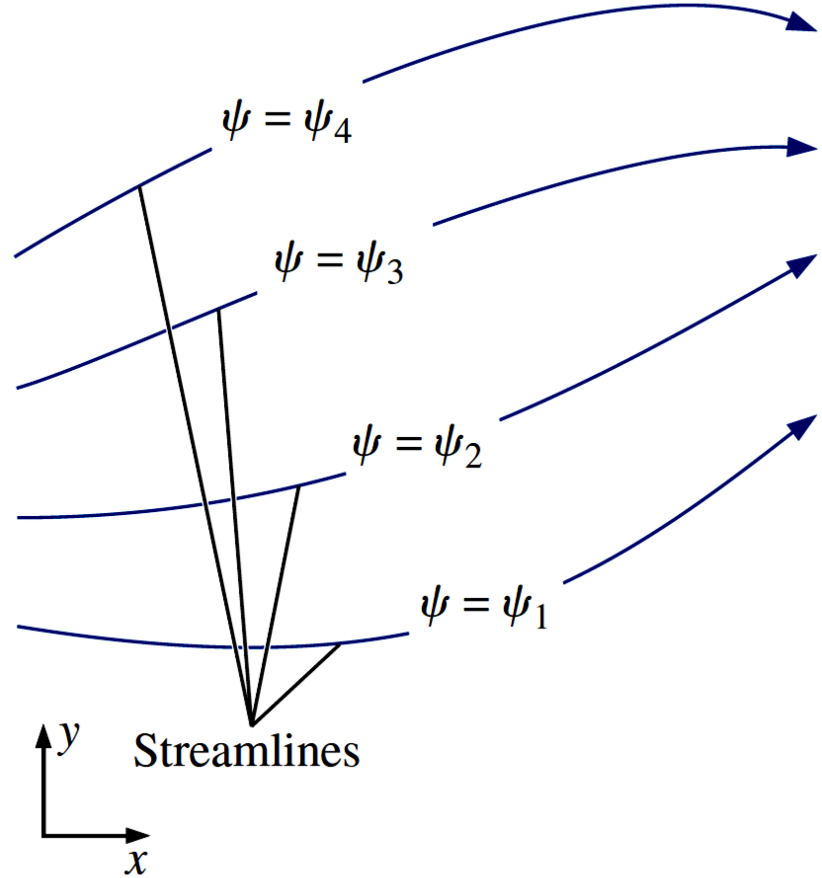
What have we gained by this transformation?

- First, as already mentioned, a single variable *($\psi$)* replaces two variables *(u and v)*—once $\psi$ *is* known, we can generate both *u* and *v* via *Eq. 2* and we are guaranteed that the solution satisfies continuity, Eq. 1.
- Second, it turns out that the stream function has useful physical significance . Namely, Curves of constant $\psi$ are **streamlines of the flow.**

# The Stream Function

- This is easily proven by considering a streamline in the *xy*-plane




*Curves of constant stream function represent streamlines of the flow*

## The Stream Function

- The change in the value of ψ as we move from one point (*x*, *y*) to a nearby point (*x* + *dx*, *y* + *dy*) is given by the relationship:

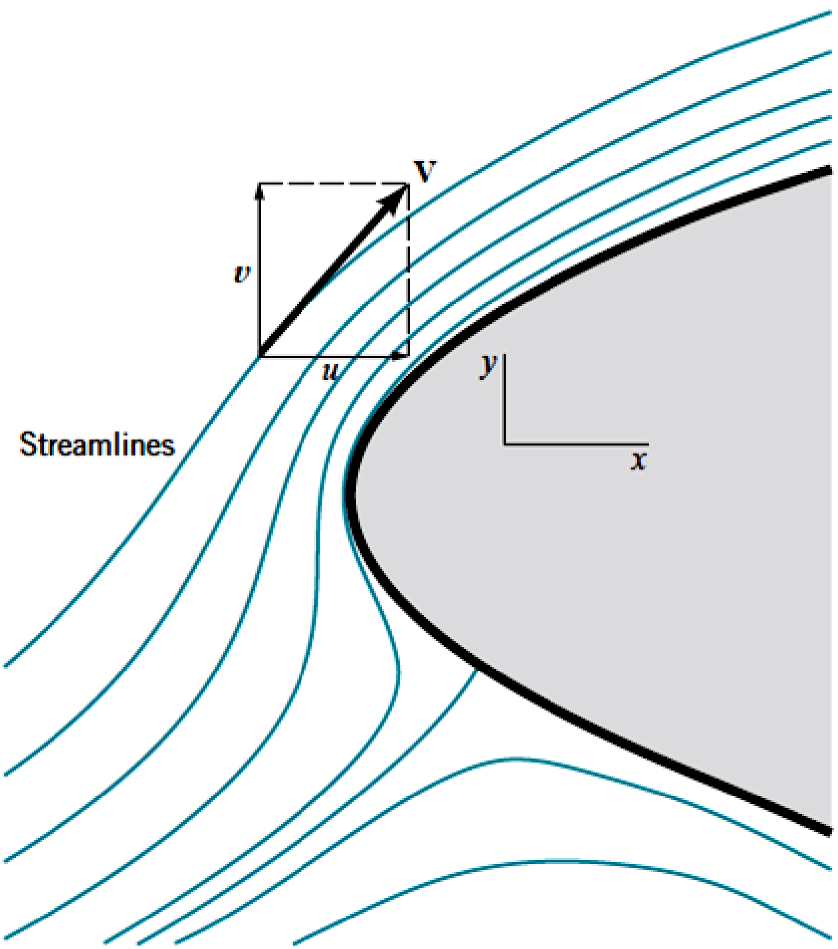
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v\, dx + u\, dy$$

- Along a line of constant ψ we have dψ = 0 so that

$$-v\, dx + u\, dy = 0$$

- and, therefore, along a line of constant ψ

$$\frac{dy}{dx} = \frac{v}{u}$$



# The Stream Function

- *Along a streamline:*

$$\frac{dy}{dx} = \frac{v}{u} \quad \rightarrow \quad \underbrace{-v}_{\partial\psi/\partial x}\, dx + \underbrace{u}_{\partial\psi/\partial y}\, dy = 0$$

- where we have applied Eq. 2, the definition of $\psi$. Thus along a streamline:

$$\frac{\partial\psi}{\partial x}\, dx + \frac{\partial\psi}{\partial y}\, dy = 0$$

- But for any smooth function $\psi$ of two variables *x and y*, we know by the chain rule of mathematics that the total change of $\psi$ from point *(x, y)* to another point *(x + dx, y + dy)* some infinitesimal distance away is

# The Stream Function

- Total change of $\psi$:

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy$$

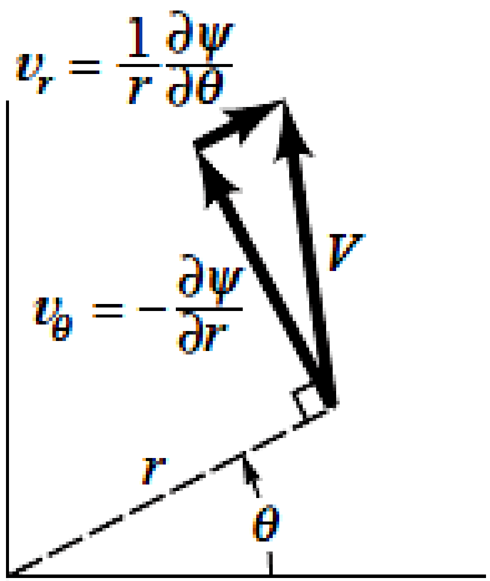
- By comparing the above two equations we see that $d\psi = 0$ *along a streamline;*

## The Stream Function

- In cylindrical coordinates the continuity equation for incompressible, plane, two dimensional flow reduces to

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

- and the velocity components, $v_r$ and $v_\theta$ can be related to the stream function, $\psi(r, \theta)$, through the equations

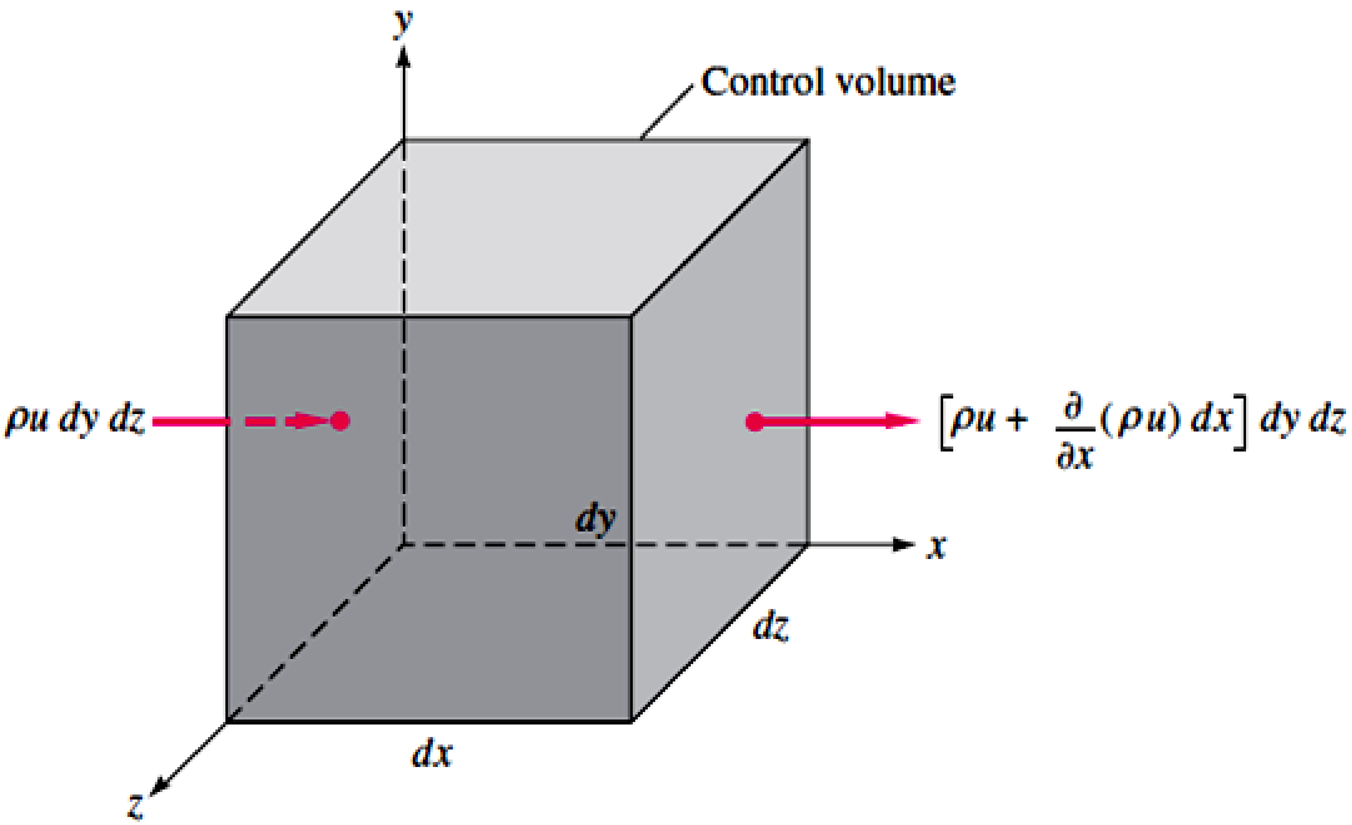



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

# The Differential Equation of Linear Momentum

- Using the same elemental control volume as in mass conservation, for which the appropriate form of the linear momentum relation is

$$\sum \mathbf{F} = \frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum (\dot{m}_i \mathbf{V}_i)_{out} - \sum (\dot{m}_i \mathbf{V}_i)_{in}$$



# The Differential Equation of Linear Momentum

- Again the element is so small that the volume integral simply reduces to a derivative term:

$$\frac{\partial}{\partial t}(\mathbf{V}\rho \, d\mathcal{V}) \approx \frac{\partial}{\partial t}(\rho \mathbf{V}) \, dx \, dy \, dz$$

- The momentum fluxes occur on all six faces, three inlets and three outlets.

| Faces | Inlet momentum flx | Outlet momentum flx |
|---|---|---|
| $x$ | $\rho u \mathbf{V} \, dy \, dz$ | $\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V}) \, dx\right] dy \, dz$ |
| $y$ | $\rho v \mathbf{V} \, dx \, dz$ | $\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V}) \, dy\right] dx \, dz$ |
| $z$ | $\rho w \mathbf{V} \, dx \, dy$ | $\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V}) \, dz\right] dx \, dy$ |

## The Differential Equation of Linear Momentum

- Introducing these terms

$$\sum \mathbf{F} = dx\,dy\,dz\left[\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho v \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V})\right]$$

- A simplification occurs if we split up the term in brackets as follows:

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho v \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V})$$
$$= \mathbf{V}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})\right] + \rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)$$

- The term in brackets on the right-hand side is seen to be the equation of continuity, which vanishes identically

## The Differential Equation of Linear Momentum

- The long term in parentheses on the right-hand side is the total acceleration of a particle that instantaneously occupies the control volume:

$$\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{d\mathbf{V}}{dt}$$

- Thus now we have

$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt}\, dx\, dy\, dz$$

- This equation points out that the net force on the control volume must be of differential size and proportional to the element volume.

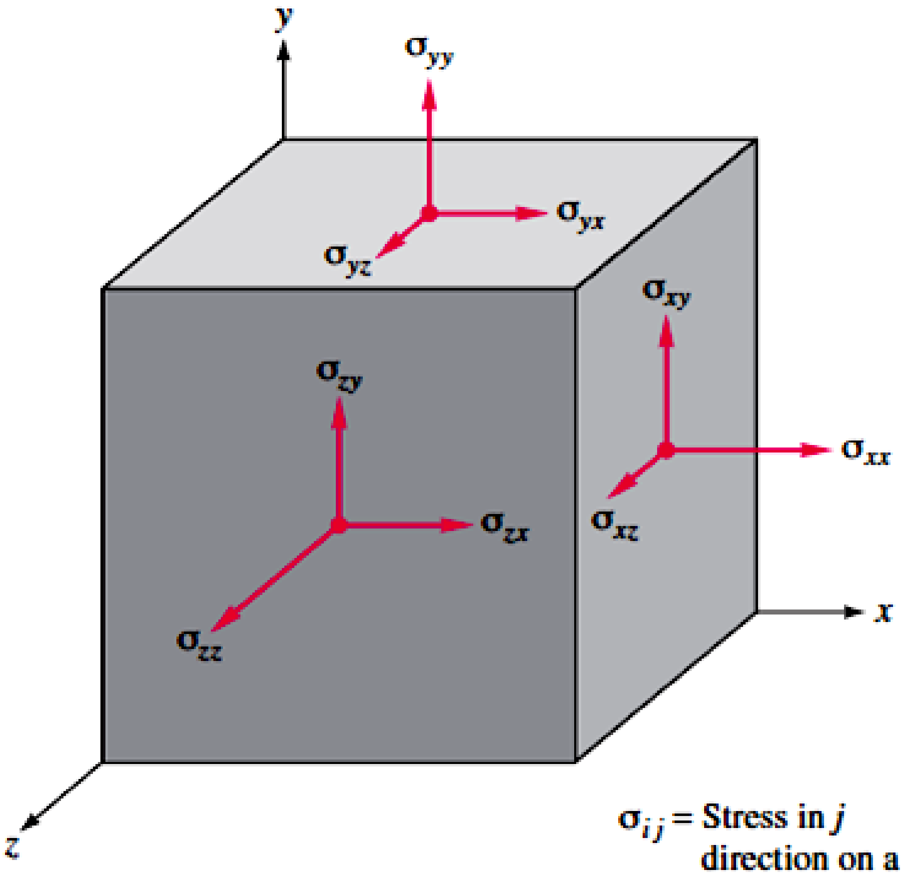
## The Differential Equation of Linear Momentum

- These forces are of two types, ***body forces and surface forces.***
- Body forces are due to external fields (gravity, magnetism, electric potential) that act on the entire mass within the element.
- The only body force we shall consider is gravity.
- The gravity force on the differential mass $\rho\, dx\, dy\, dz$ within the control volume is

$$d\mathbf{F}_{\text{grav}} = \rho \mathbf{g}\, dx\, dy\, dz$$

- The surface forces are due to the stresses on the sides of the control surface. These stresses are the sum of hydrostatic pressure plus viscous stresses $\tau_{ij}$ that arise from motion with velocity gradients

# The Differential Equation of Linear Momentum




$\sigma_{ij}$ = Stress in $j$ direction on a face normal to $i$ axis

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

# The Differential Equation of Linear Momentum

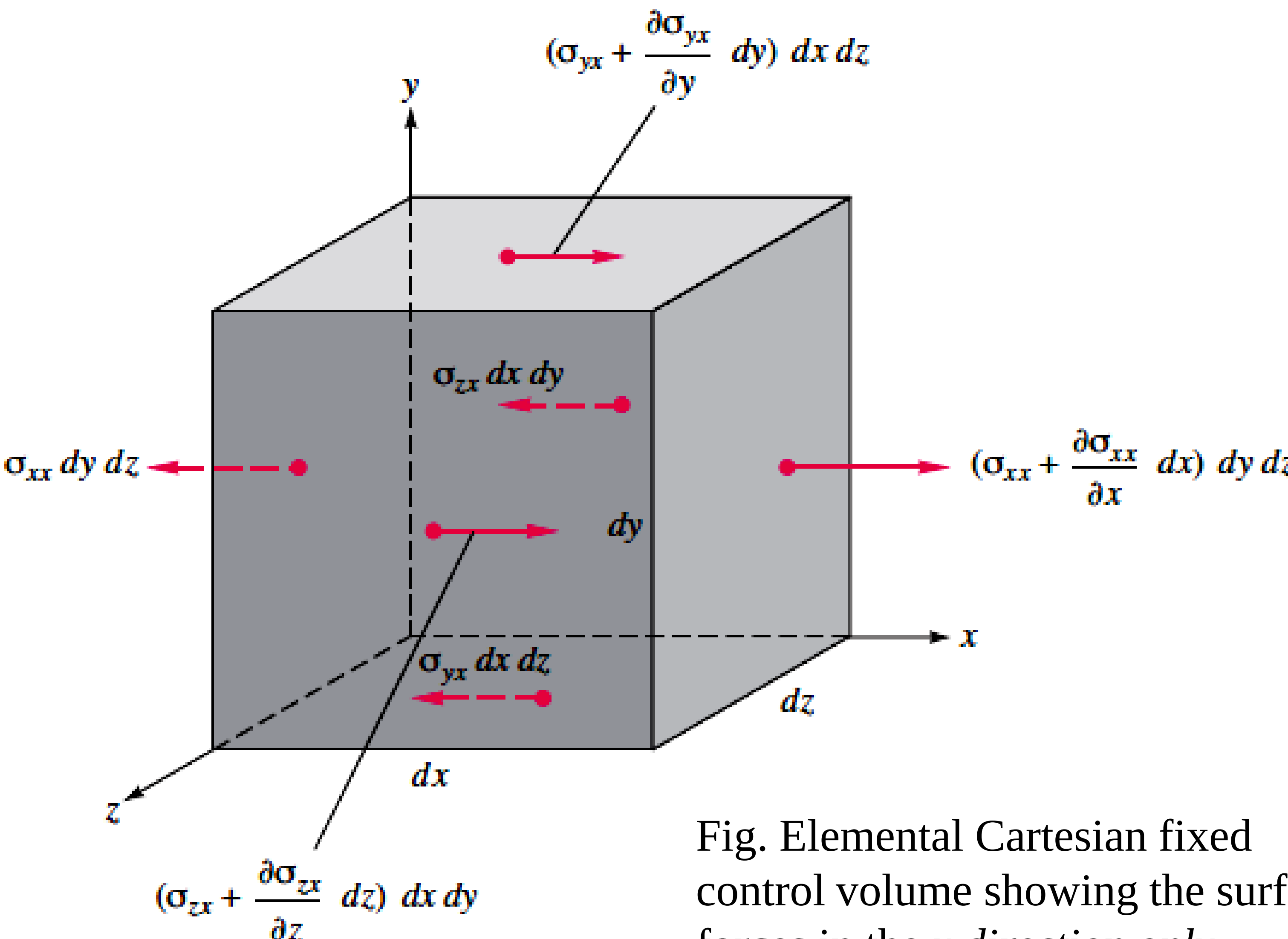



Fig. Elemental Cartesian fixed control volume showing the surface forces in the *x direction only.*

## The Differential Equation of Linear Momentum

- The net surface force in the *x direction is given by*

$$dF_{x,\text{surf}} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\, dy\, dz$$

- Splitting into pressure plus viscous stresses

$$\frac{dF_x}{d\mathcal{V}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})$$

- where *dv* = *dx dy dz.*
- *Similarly* we can derive the *y and z forces* per unit volume on the control surface

$$\frac{dF_y}{d\mathcal{V}} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$

$$\frac{dF_z}{d\mathcal{V}} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})$$

## The Differential Equation of Linear Momentum

- The net vector surface force can be written as

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{surf}} = -\nabla p + \left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}}$$

where the viscous force has a total of nine terms:

$$\begin{aligned}\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} &= \mathbf{i}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \\ &+ \mathbf{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \\ &+ \mathbf{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\end{aligned}$$

# The Differential Equation of Linear Momentum

- In divergence form

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} = \boldsymbol{\nabla} \cdot \tau_{ij}$$

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

- is the viscous stress tensor acting on the element
- The surface force is thus the sum of the pressure gradient vector and the divergence of the viscous stress tensor

## The Differential Equation of Linear Momentum

- The basic differential momentum equation for an infinitesimal element is thus

$$\rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho \frac{d\mathbf{V}}{dt}$$

where

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}$$

- In words

Gravity force per unit volume + pressure force per unit volume + viscous force per unit volume = density × acceleration

## The Differential Equation of Linear Momentum

- the component equations are

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

- This is the differential momentum equation in its full glory, and it is valid for any fluid in any general motion, particular fluids being characterized by particular viscous stress terms.

## Inviscid Flow: Eulers' Equation

- For Frictionless flow $\tau_{ij} = 0$, for which

$$\rho \mathbf{g} - \nabla p = \rho \frac{d\mathbf{V}}{dt}$$

- This is *Eulers' equation for inviscid flow*

# Newtonian Fluid: Navier-Stokes Equations

- For a newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

- where μ *is the viscosity coefficient*
- Substitution gives the differential momentum equation for a newtonian fluid with constant density and viscosity:

## Newtonian Fluid: Navier-Stokes Equations

- These are the incompressible flow *Navier-Stokes equations* named after C. L. M. H. Navier (1785–1836) and Sir George G. Stokes (1819–1903), who are credited with their derivation.

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho \frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$

# Inviscid Flow

- Shearing stresses develop in a moving fluid because of the viscosity of the fluid.
- We know that for some common fluids, such as air and water, the viscosity is small, therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity (and thus shearing stresses).
- Flow fields in which the shearing stresses are assumed to be negligible are said to be ***inviscid, nonviscous, or frictionless.***
- For fluids in which there are no shearing stresses the normal stress at a point is independent of direction—that is $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$.

# Inviscid Flow

## Euler's Equations of Motion

- For an inviscid flow in which all the shearing stresses are zero and the Euler's equation of motion is written as

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

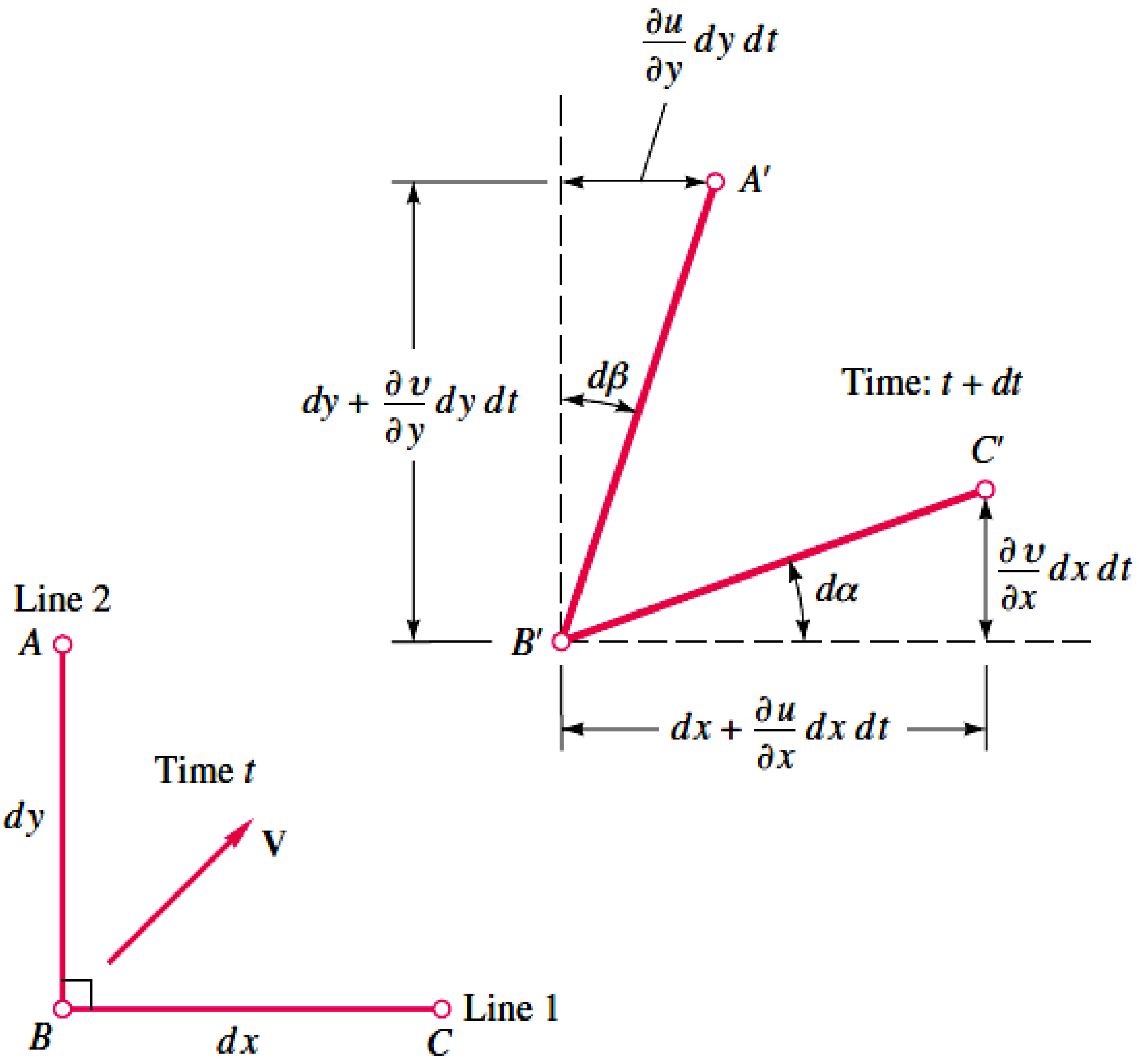
- In vector notation Euler's equations can be expressed as

$$\rho \mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]$$

## Vorticity and Irrotationality

- The assumption of zero fluid angular velocity, or irrotationality, is a very useful simplification.
- Here we show that angular velocity is associated with the curl of the local velocity vector.
- The differential relations for deformation of a fluid element can be derived by examining the Fig. below.
- Two fluid lines AB and BC, initially perpendicular at time t, move and deform so that at t + dt they have slightly different lengths A'B' and B'C' and are slightly off the perpendicular by angles $d\alpha$ and $d\beta$.

# Vorticity and Irrotationality



## Vorticity and Irrotationality

- We define the angular velocity $\omega_z$ about the z axis as the average rate of counterclockwise turning of the two lines:

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$$

- But from the fig. *dα and dβ* are each directly related to velocity derivatives in the limit of small dt:

$$d\alpha = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial \upsilon/\partial x)\,dx\,dt}{dx + (\partial u/\partial x)\,dx\,dt}\right] = \frac{\partial \upsilon}{\partial x}dt$$

$$d\beta = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial u/\partial y)\,dy\,dt}{dy + (\partial \upsilon/\partial y)\,dy\,dt}\right] = \frac{\partial u}{\partial y}dt$$

- Substitution results

$$\omega_z = \frac{1}{2}\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right)$$

# Vorticity and Irrotationality

In exactly similar manner we determine the other two rates:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

- The vector $\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$ is thus one-half the curl of the velocity vector

$$\boldsymbol{\omega} = \frac{1}{2}(\text{curl } \mathbf{V}) = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

- A vector twice as large is called the ***vorticity***

$$\boldsymbol{\zeta} = 2\boldsymbol{\omega} = \text{curl } \mathbf{V}$$

## Vorticity and Irrotationality

- Many flows have negligible or zero vorticity and are called ***irrotational.***

$$\text{curl } \mathbf{V} \equiv 0$$

- **Example.** For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

- Is this flow irrotational?

**Solution.**

- For the prescribed velocity field

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

# Vorticity and Irrotationality

and therefore

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[(-2y) - (-2y)] = 0$$

Thus, the flow is irrotational.

## Velocity Potential

- The velocity components of irrotational flow can be expressed in terms of a scalar function $\phi(x, y, z, t)$ as

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

- where $\phi$ is called the ***velocity potential.***
- In vector form, it can be written as

$$\boxed{\mathbf{V} = \nabla \phi}$$

- so that for an irrotational flow the velocity is expressible as the gradient of a scalar function $\phi$.
- The velocity potential is a consequence of the irrotationality of the flow field, whereas the stream function is a consequence of conservation of mass

## Velocity Potential

- It is to be noted, however, that the velocity potential can be defined for a general three-dimensional flow, whereas the stream function is restricted to two-dimensional flows.
- For an incompressible fluid we know from conservation of mass that

$$\nabla \cdot \mathbf{V} = 0$$

- and therefore for incompressible, irrotational flow (with $\mathbf{V} = \nabla \phi$) it follows that

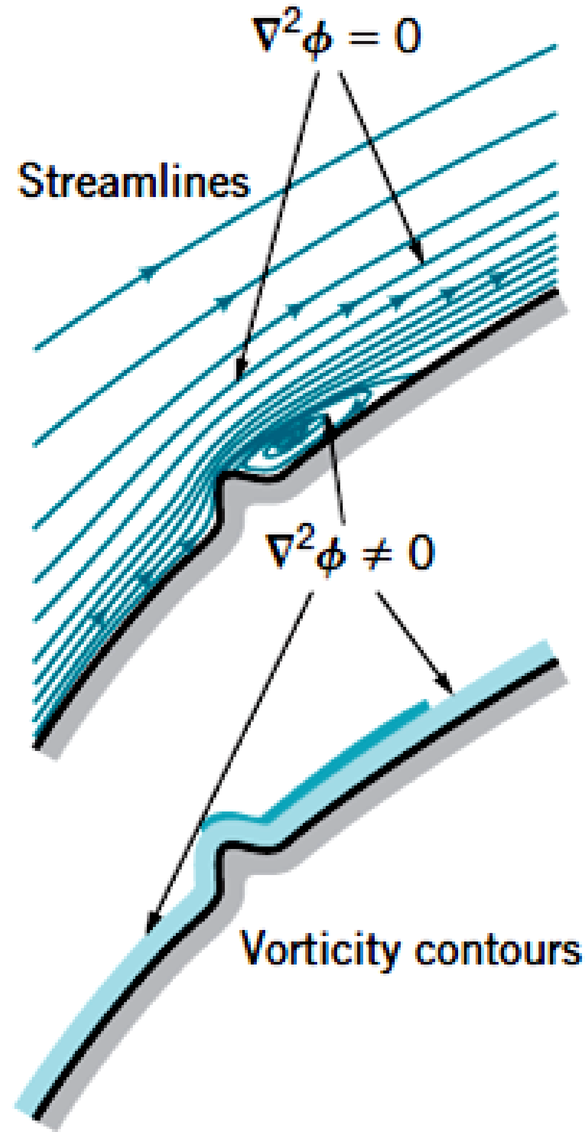
$$\nabla^2 \phi = 0$$

where $\nabla^2(\ ) = \nabla \cdot \nabla(\ )$ is the *Laplacian operator*. In Cartesian coordinates

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

## Velocity Potential

- This differential equation arises in many different areas of engineering and physics and is called ***Laplace's equation.*** Thus, inviscid, incompressible, irrotational flow fields are governed by Laplace's equation.
- This type of flow is commonly called a ***potential flow.***
- Potential flows are irrotational flows. That is, the vorticity is zero throughout. If vorticity is present (e.g., boundary layer, wake), then the flow cannot be described by Laplace's equation.



# Velocity Potential

- For some problems it will be convenient to use cylindrical coordinates, r,θ, and z. In this coordinate system the gradient operator is

$$\nabla(\ ) = \frac{\partial(\ )}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(\ )}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial(\ )}{\partial z}\hat{\mathbf{e}}_z$$

so that

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z$$

where $\phi = \phi(r, \theta, z)$. Since

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z$$

it follows for an irrotational flow (with $\mathbf{V} = \nabla\phi$)

$$v_r = \frac{\partial\phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad v_z = \frac{\partial\phi}{\partial z}$$

## Velocity Potential

Also, Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

## Example 1

- The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of the corner of Fig. is described by the stream function

$$\psi = 2r^2 \sin 2\theta$$

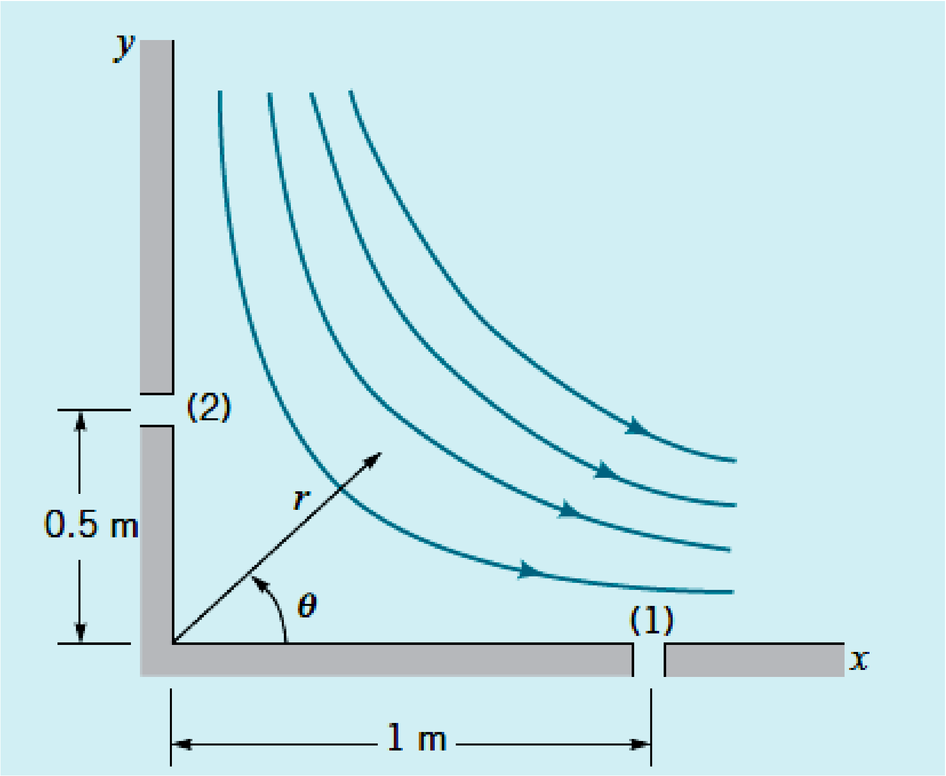
- where ψ has units of $m^2/s$ when $r$ is in meters. Assume the fluid density is $10^3\ kg/m^3$ and the x–y plane is horizontal that is, there is no difference in elevation between points (1) and (2).

**FIND**

a) Determine, if possible, the corresponding velocity potential.

b) If the pressure at point (1) on the wall is 30 kPa, what is the pressure at point (2)?

# Example 1




## Solution

- The radial and tangential velocity components can be obtained from the stream function as

## Solution

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r\cos 2\theta$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -4r\sin 2\theta$$

Since

$$v_r = \frac{\partial \phi}{\partial r}$$

it follows that

$$\frac{\partial \phi}{\partial r} = 4r\cos 2\theta$$

and therefore by integration

$$\phi = 2r^2 \cos 2\theta + f_1(\theta) \tag{1}$$

where $f_1(\theta)$ is an arbitrary function of $\theta$. Similarly

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r \sin 2\theta$$

and integration yields

$$\phi = 2r^2 \cos 2\theta + f_2(r) \tag{2}$$

where $f_2(r)$ is an arbitrary function of $r$. To satisfy both Eqs. 1 and 2, the velocity potential must have the form

$$\phi = 2r^2 \cos 2\theta + C \qquad \textbf{(Ans)}$$

where $C$ is an arbitrary constant. As is the case for stream functions, the specific value of $C$ is not important, and it is customary to let $C = 0$ so that the velocity potential for this corner flow is

$$\phi = 2r^2 \cos 2\theta \qquad \textbf{(Ans)}$$

**(b)** Since we have an irrotational flow of a nonviscous, incompressible fluid, the Bernoulli equation can be applied between any two points. Thus, between points (1) and (2) with no elevation change

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

or

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) \qquad \textbf{(3)}$$

Since

$$V^2 = v_r^2 + v_\theta^2$$

it follows that for any point within the flow field

$$\begin{aligned} V^2 &= (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2 \\ &= 16r^2(\cos^2 2\theta + \sin^2 2\theta) \\ &= 16r^2 \end{aligned}$$

This result indicates that the square of the velocity at any point depends only on the radial distance, $r$, to the point. Note that the constant, 16, has units of $s^{-2}$. Thus,

$$V_1^2 = (16\ \text{s}^{-2})(1\ \text{m})^2 = 16\ \text{m}^2/\text{s}^2$$

and

$$V_2^2 = (16\ \text{s}^{-2})(0.5\ \text{m})^2 = 4\ \text{m}^2/\text{s}^2$$

Substitution of these velocities into Eq. 3 gives

$$p_2 = 30 \times 10^3\ \text{N/m}^2 + \frac{10^3\ \text{kg/m}^3}{2}(16\ \text{m}^2/\text{s}^2 - 4\ \text{m}^2/\text{s}^2)$$

$$= 36\ \text{kPa}$$

**(Ans)**

## Basic Plane Potential Flows

- For simplicity, only plane (two-dimensional) flows will be considered. In this case, by using Cartesian coordinates

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

or by using cylindrical coordinates

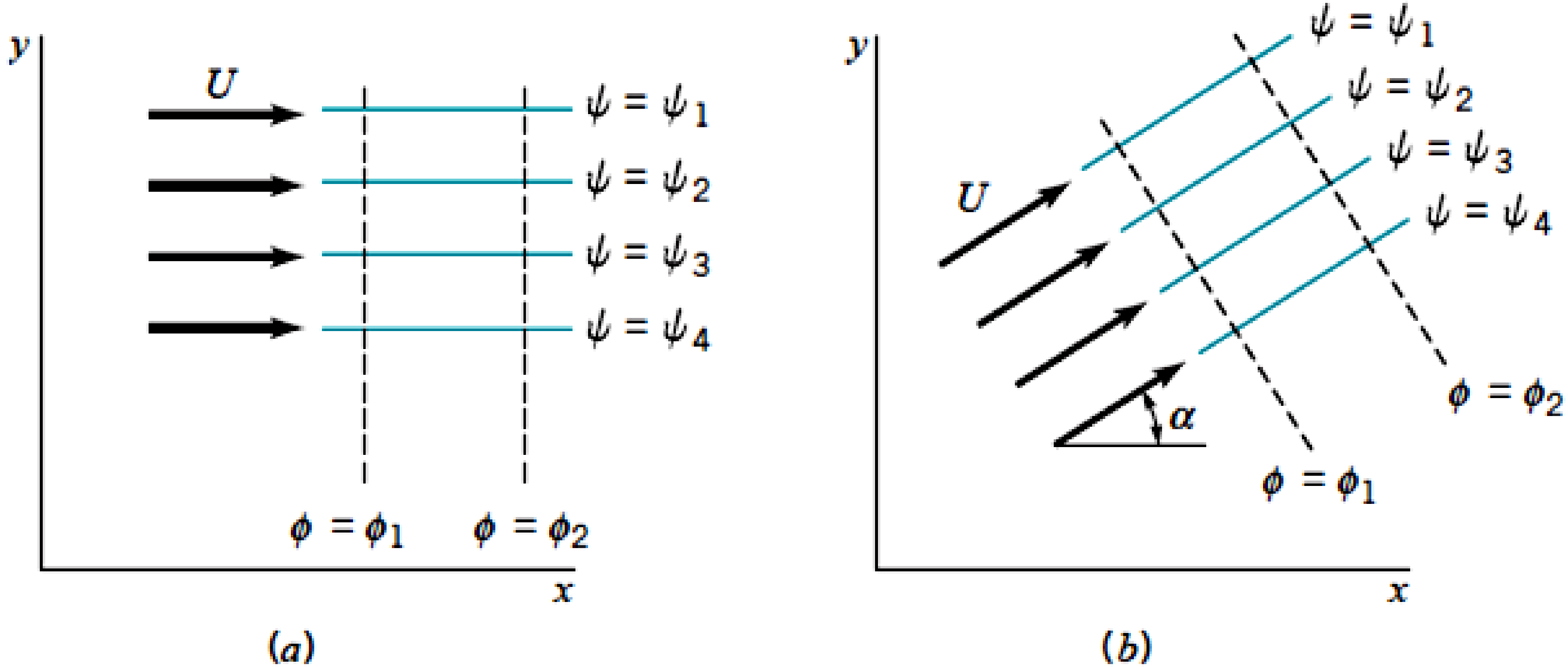
$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

- Since we can define a stream function for plane flow, we can also let

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

# Uniform Flow

- The simplest plane flow is one for which the streamlines are all straight and parallel, and the magnitude of the velocity is constant. This type of flow is called a ***uniform flow.***
- For example, consider a uniform flow in the positive *x direction* as is illustrated in Fig a.



## Uniform Flow

- In this instance, $u = U$ *and* $v = 0$, and in terms of the velocity potential

$$\frac{\partial \phi}{\partial x} = U \qquad \frac{\partial \phi}{\partial y} = 0$$

- These two equations can be integrated to yield

$$\phi = Ux + C$$

- where C is an arbitrary constant, which can be set equal to zero.
- Thus, for a uniform flow in the positive x direction

$$\phi = Ux$$

## Uniform Flow

- The corresponding stream function can be obtained in a similar manner, since

$$\frac{\partial \psi}{\partial y} = U \qquad \frac{\partial \psi}{\partial x} = 0$$

- and, therefore,

$$\psi = Uy$$

- These results can be generalized to provide the velocity potential and stream function for a uniform flow at an angle α with the *x axis, as in Fig. b. For this case*

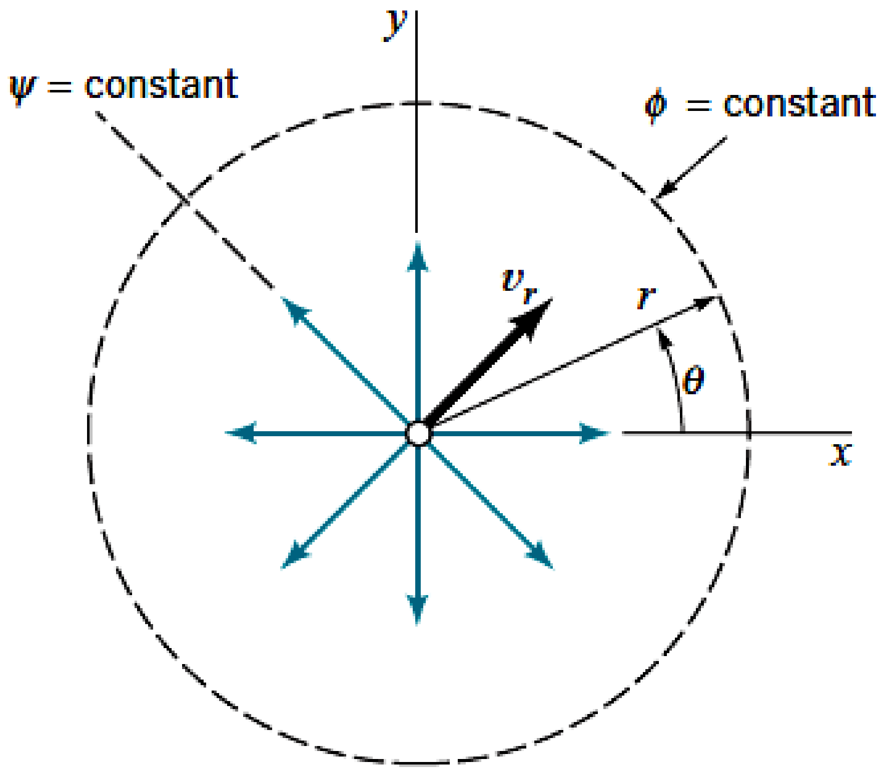
$$\phi = U(x \cos \alpha + y \sin \alpha)$$

- *and*

$$\psi = U(y \cos \alpha - x \sin \alpha)$$

## Source and Sink

- Consider a fluid flowing radially outward from a line through the origin perpendicular to the *x*–*y* plane as is shown in Fig. Let m be the volume rate of flow emanating from the line (per unit length), and therefore to satisfy conservation of mass




$$(2\pi r)v_r = m$$

or

$$v_r = \frac{m}{2\pi r}$$

## Source and Sink

- A source or sink represents a purely radial flow.
- Since the flow is a purely radial flow, $v_\theta = 0$, the corresponding velocity potential can be obtained by integrating the equations

$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

- It follows that

$$\phi = \frac{m}{2\pi} \ln r$$

- If m is positive, the flow is radially outward, and the flow is considered to be a **source flow.** If m is negative, the flow is toward the origin, and the flow is considered to be a **sink flow.** The flowrate, m, is the strength of the source or sink.

## Source and Sink

- The stream function for the source can be obtained by integrating the relationships

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \qquad v_\theta = -\frac{\partial \psi}{\partial r} = 0$$

- To yield

$$\psi = \frac{m}{2\pi}\theta$$

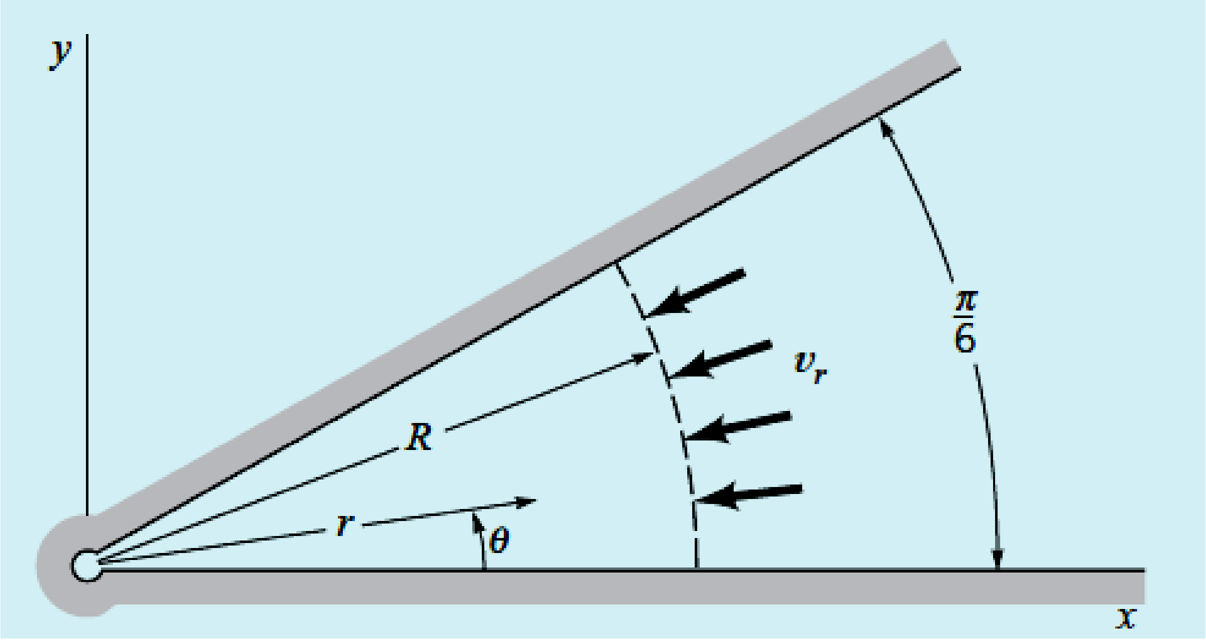
- The streamlines (lines of $\psi$ = constant ) are radial lines, and the equipotential lines (lines of $\phi$ = constant) are concentric circles centered at the origin.

## Example 2

- A nonviscous, incompressible fluid flows between wedge-shaped walls into a small opening as shown in Fig. The velocity potential (in ft/s$^2$), which approximately describes this flow is

$$\phi = -2 \ln r$$

- Determine the volume rate of flow (per unit length) into the opening.



# Solution

The components of velocity are

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

which indicates we have a purely radial flow. The flowrate per unit width, $q$, crossing the arc of length $R\pi/6$ can thus be obtained by integrating the expression

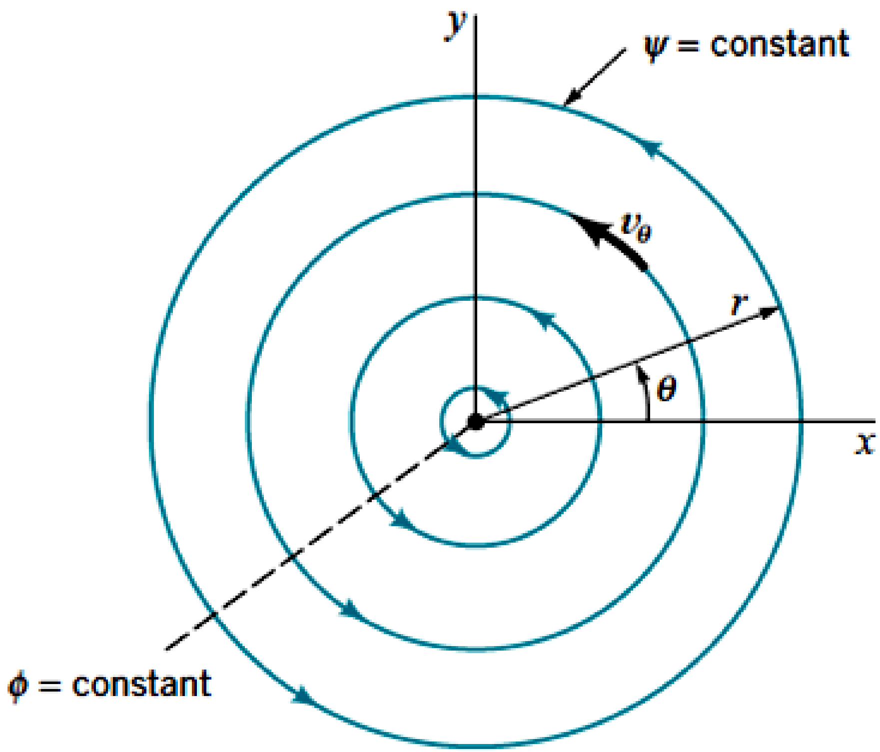
$$q = \int_0^{\pi/6} v_r R \, d\theta = -\int_0^{\pi/6} \left(\frac{2}{R}\right) R \, d\theta$$

$$= -\frac{\pi}{3} = -1.05 \text{ ft}^2/\text{s} \qquad \textbf{(Ans)}$$

The negative sign indicates that the flow is toward the opening, that is, in the negative radial direction

## Vortex

- We next consider a flow field in which the streamlines are concentric circles—that is, we interchange the velocity potential and stream function for the source. Thus, let




$$\phi = K\theta$$

and

$$\psi = -K \ln r$$

where K is a constant. In this case the streamlines are concentric circles with $v_r = 0$ and

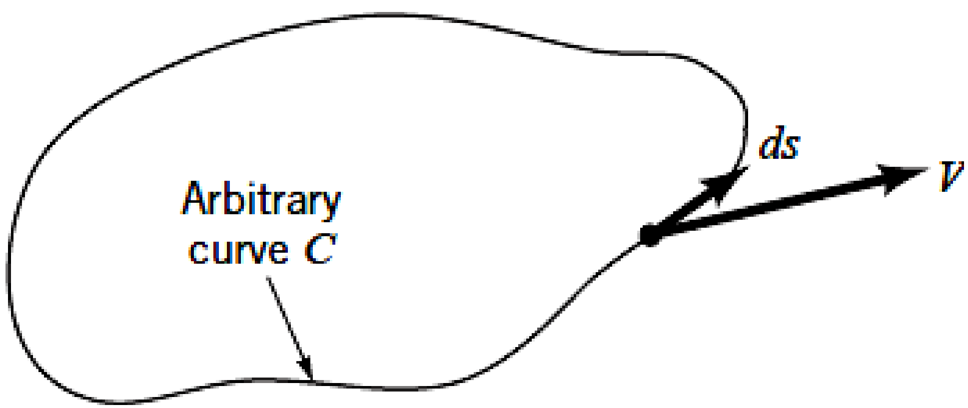
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$

This result indicates that the tangential velocity varies inversely with the distance from the origin

## Circulation

- A mathematical concept commonly associated with vortex motion is that of ***circulation.*** The circulation, Γ, is defined as the line integral of the tangential component of the velocity taken around a closed curve in the flow field. In equation form, Γ, can be expressed as

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$




where the integral sign means that the integration is taken around a closed curve, *C, in the* counterclockwise direction, and *ds is a* differential length along the curve

## Circulation

- For an irrotational flow

$$\mathbf{V} = \nabla\phi \text{ so that } \mathbf{V} \cdot d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = d\phi \text{ and, therefore,}$$

$$\Gamma = \oint_C d\phi = 0$$

- This result indicates that for an irrotational flow the circulation will generally be zero.
- However, for the free vortex with $v_\theta = K/r$ , the circulation around the circular path of radius *r is*

$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r\,d\theta) = 2\pi K$$

- which shows that the circulation is nonzero.
- However, for irrotational flows the circulation around any path that does not include a singular point will be zero.

## Circulation

- The velocity potential and stream function for the free vortex are commonly expressed in terms of the circulation as

$$\phi = \frac{\Gamma}{2\pi}\theta$$

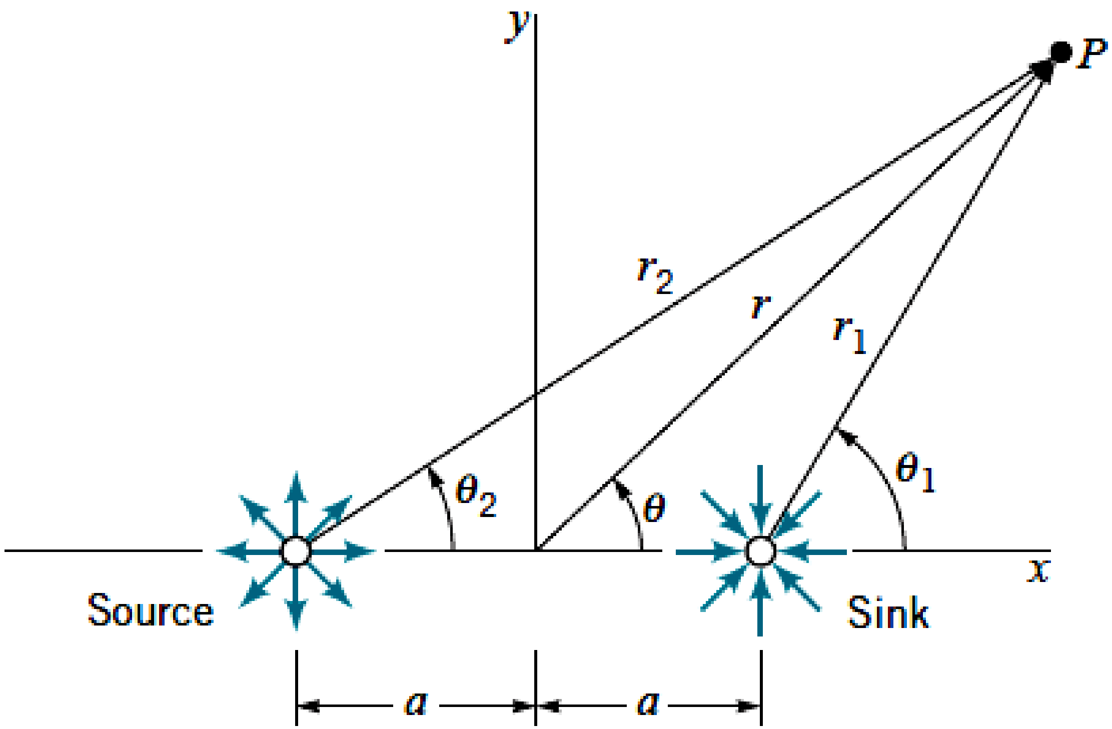
and

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

# Doublet

- Doublet is formed by combining a source and sink in a special way. Consider the equal strength, source–sink pair shown. The combined stream function for the pair is

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$



# Doublet

- which can be rewritten as

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

- From the Fig. above it follows that

$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$

- And

$$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$

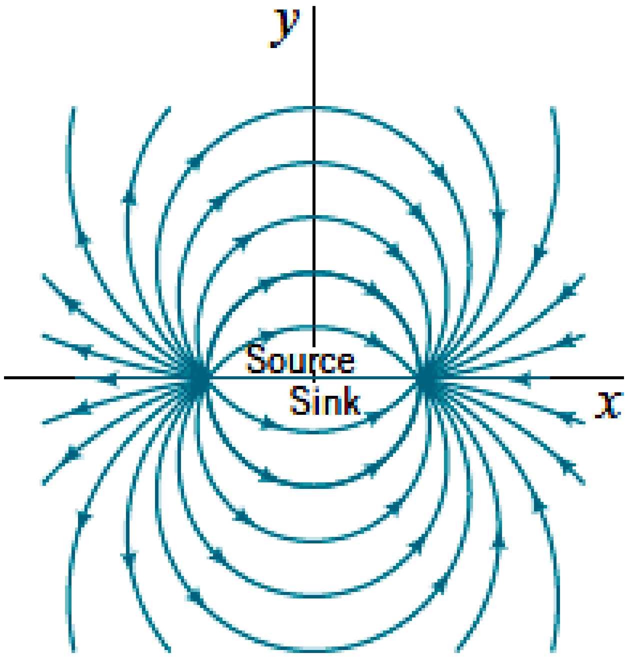
- These results substitution gives

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$

# Doublet

- So that $$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$
- For small values of the distance *a*

$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$




since the tangent of an angle approaches the value of the angle for small angles

A doublet is formed by letting a source and sink approach one another.

## Doublet

- The so-called **doublet** is formed by letting the source and sink approach one another $(a \rightarrow 0)$ while increasing the strength $m$ $(m \rightarrow \infty)$ so that the product $ma/\pi$ remains constant. In this case, since $r/(r^2 - a^2) \rightarrow 1/r$,

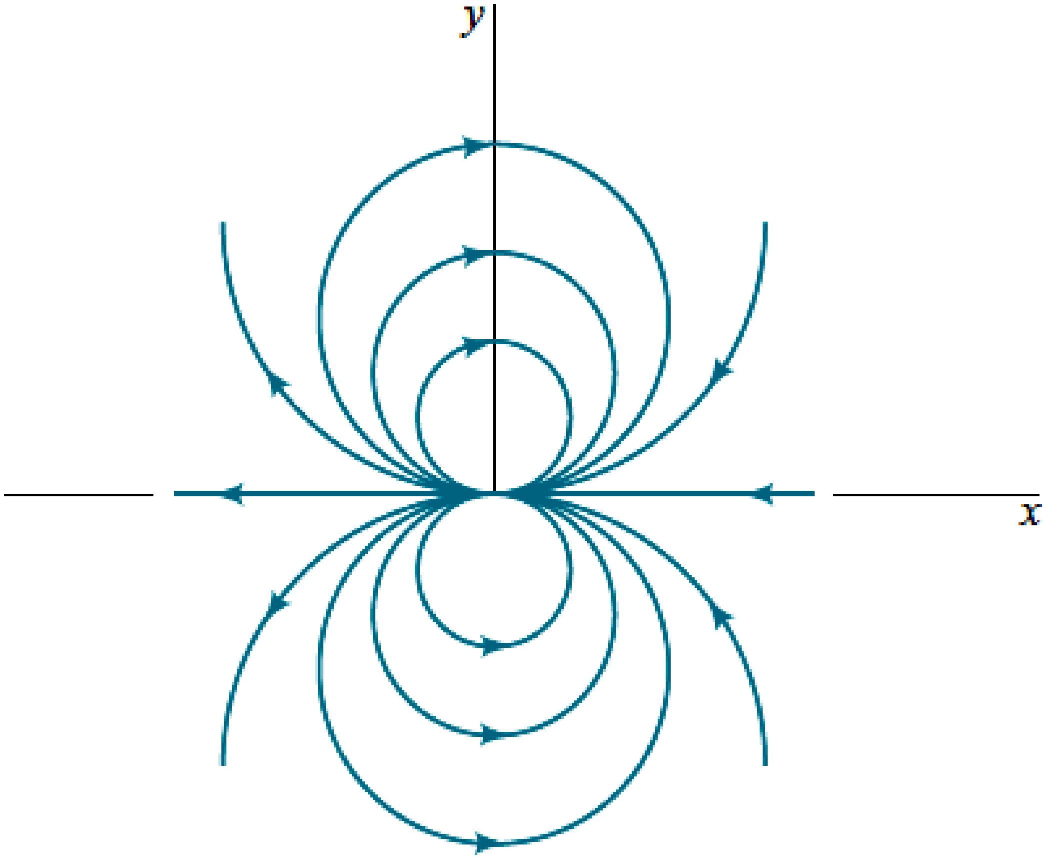
$$\psi = -\frac{K \sin \theta}{r}$$

- where K, a constant equal to $ma/\pi$, is called the strength of the doublet.
- The corresponding velocity potential for the doublet is

$$\phi = \frac{K \cos \theta}{r}$$

# Doublet

- Plots of lines of constant ψ reveal that the streamlines for a doublet are circles through the origin tangent to the *x* axis as shown in fig below.

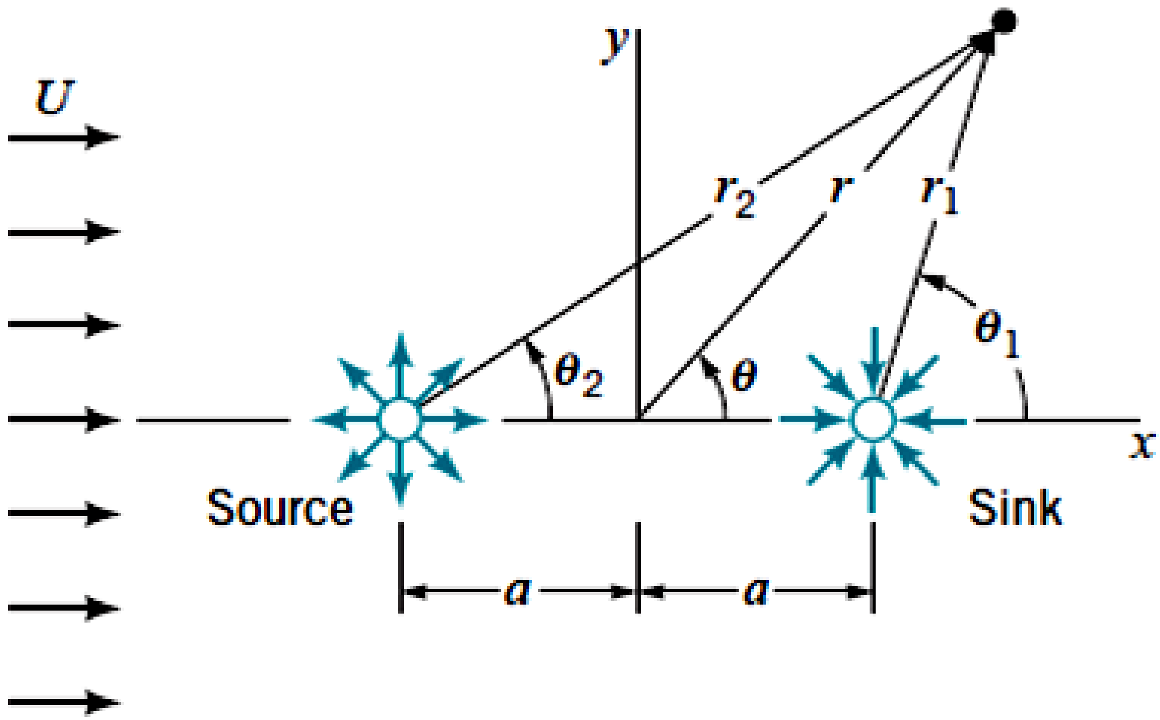


## Summary of Basic, Plane Potential Flows

| Description of Flow Field | Velocity Potential | Stream Function | Velocity Components |
|---|---|---|---|
| Uniform flow at angle $\alpha$ with the $x$ axis | $\phi = U(x\cos\alpha + y\sin\alpha)$ | $\psi = U(y\cos\alpha - x\sin\alpha)$ | $u = U\cos\alpha$<br>$v = U\sin\alpha$ |
| Source or sink<br>$m > 0$ source<br>$m < 0$ sink | $\phi = \dfrac{m}{2\pi}\ln r$ | $\psi = \dfrac{m}{2\pi}\theta$ | $v_r = \dfrac{m}{2\pi r}$<br>$v_\theta = 0$ |
| Free vortex<br>$\Gamma > 0$ counterclockwise motion<br>$\Gamma < 0$ clockwise motion | $\phi = \dfrac{\Gamma}{2\pi}\theta$ | $\psi = -\dfrac{\Gamma}{2\pi}\ln r$ | $v_r = 0$<br>$v_\theta = \dfrac{\Gamma}{2\pi r}$ |
| Doublet | $\phi = \dfrac{K\cos\theta}{r}$ | $\psi = -\dfrac{K\sin\theta}{r}$ | $v_r = -\dfrac{K\cos\theta}{r^2}$<br>$v_\theta = -\dfrac{K\sin\theta}{r^2}$ |

## Rankine Ovals

- To study the flow around a closed body, a source and a sink of equal strength can be combined with a uniform flow as shown in Fig. below.




(a)

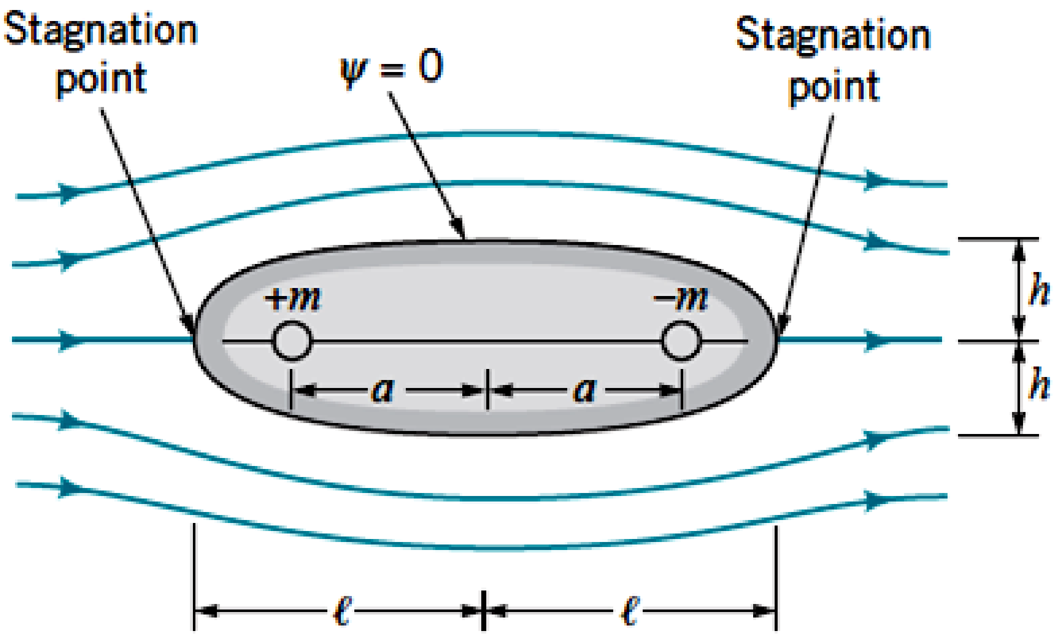
# Rankine Ovals

- The stream function for this combination is

$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2)$$

and the velocity potential is

$$\phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2)$$



## Rankine Ovals

- Using the stream function for the source–sink pair, the stream function for Rankine Ovals can be written as

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

- Or

$$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2 + y^2 - a^2}\right)$$

- The corresponding streamlines for this flow field are obtained by setting ψ = constant. If several of these streamlines are plotted, it will be discovered that the streamline ψ =0 forms a closed body as shown in fig. above.

## Rankine Ovals

- **Stagnation points** occur at the upstream and downstream ends of the body. These points can be located by determining where along the *x axis the velocity is* zero.
- The stagnation points correspond to the points where the uniform velocity, the source velocity, and the sink velocity all combine to give a zero velocity.
- The locations of the stagnation points depend on the value of *a*, *m*, *and U*.
- The body half-length, $\ell$ (the value of $|x|$ that gives $\mathbf{V} = 0$
- When y= 0), can be expressed as

$$\ell = \left(\frac{ma}{\pi U} + a^2\right)^{1/2} \quad \text{or} \quad \frac{\ell}{a} = \left(\frac{m}{\pi U a} + 1\right)^{1/2}$$

## Rankine Ovals

- The body half-width, h, can be obtained by determining the value of y where the y axis intersects the ψ = 0 streamline.

with $\psi = 0$, $x = 0$, and $y = h$, it follows that

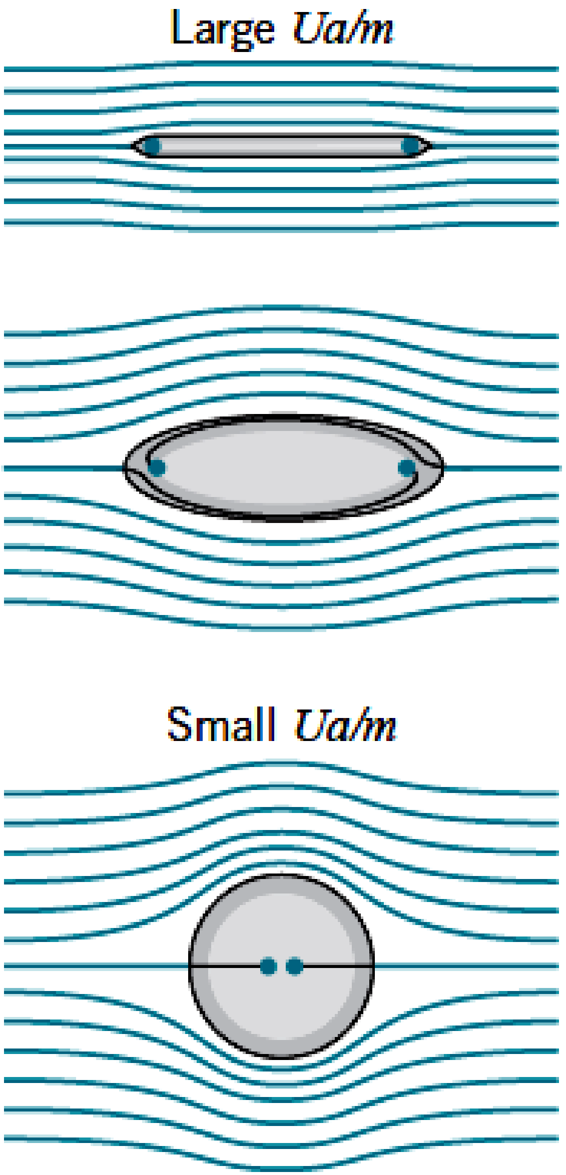
$$h = \frac{h^2 - a^2}{2a} \tan \frac{2\pi U h}{m}$$

- Or

$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right]$$

- both $\ell/a$ and $h/a$ are functions of the dimensionless parameter, $\pi Ua/m$. A large variety of body shapes with different length to width ratios can be obtained by using different values of *Ua/m*,

# Rankine Ovals




- As this parameter becomes large, flow around a long slender body is described, whereas for small values of the parameter, flow around a more blunt shape is obtained

# Flow around a Circular Cylinder

- When the distance between the source–sink pair approaches zero, the shape of the Rankine oval becomes more blunt and in fact approaches a circular shape.
- Since the Doublet was developed by letting a source–sink pair approach one another, it might be expected that a uniform flow in the positive x direction combined with a doublet could be used to represent flow around a circular cylinder.
- This combination gives for the stream function

$$\psi = Ur \sin\theta - \frac{K \sin\theta}{r}$$

- and for the velocity potential

$$\phi = Ur \cos\theta + \frac{K \cos\theta}{r}$$

# Flow around a Circular Cylinder

- In order for the stream function to represent flow around a circular cylinder it is necessary that $\psi = \text{constant for } r = a$, where $a$ is the radius of the cylinder.

$$\psi = \left(U - \frac{K}{r^2}\right) r \sin\theta$$

it follows that $\psi = 0$ for $r = a$ if

$$U - \frac{K}{a^2} = 0$$

- which indicates that the doublet strength, $K$, must be equal to $Ua^2$. Thus, the stream function for flow around a circular cylinder can be expressed as

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

## Flow around a Circular Cylinder

- and the corresponding velocity potential is

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

- The velocity components are

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

- On the surface of the cylinder $(r = a)$ it follows $v_r = 0$ and

$$v_{\theta s} = -2U\sin\theta$$

# Flow around a Circular Cylinder

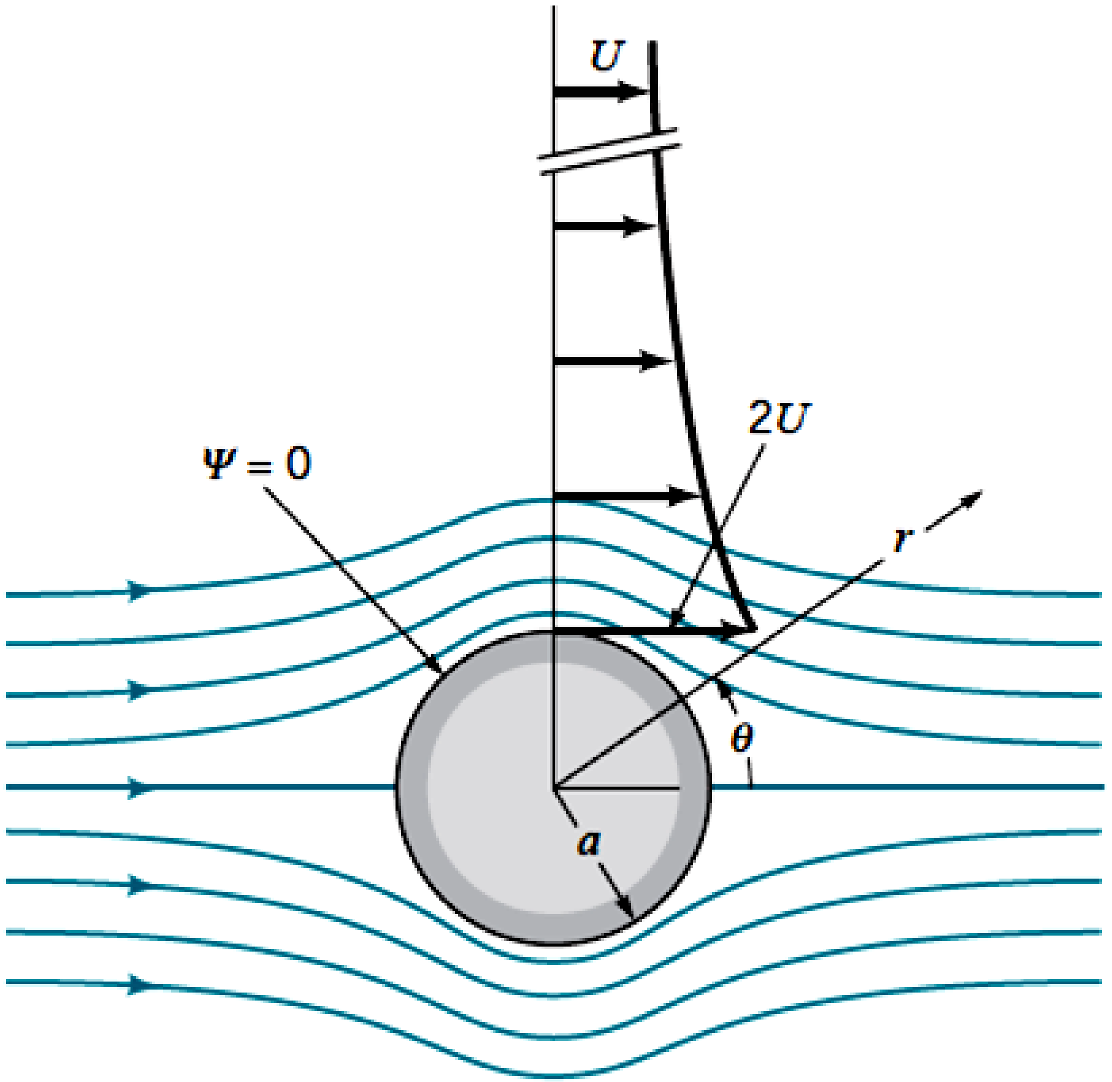



Fig. The flow around a circular cylinder.

# Solved Problems

1. The velocity potential for a certain flow field is $\phi = 4xy$. Determine the corresponding stream function.

For the given velocity potential,

$$u = \frac{\partial \phi}{\partial x} = 4y \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = 4x$$

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} = 4y \qquad (1)$$

Integrate Eq. (1) with respect to y to obtain

$$\int d\psi = \int 4y \, dy$$

or

$$\psi = 2y^2 + f_1(x) \qquad (2)$$

where $f_1(x)$ is an arbitrary function of x.

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = 4x$$

and

$$\int d\psi = -\int 4x\,dx$$

or

$$\psi = -2x^2 + f_2(y) \quad \text{where } f_2(y) \text{ is an arbitrary function of } y. \qquad (3)$$

To satisfy both Eqs. (2) and (3) $f_1(x) = f_2(y)$ for all x and y.
Thus, $f_1 = f_2 = \text{constant}$.

$$\psi = \underline{\underline{2(y^2 - x^2) + C}}$$

where C is a constant.

2. The stream function for an incompressible, two dimensional flow field is

$$\psi = ay^2 - bx$$

Where a and b are constants. Is this an irrotational flow? Explain.

For the flow to be irrotational (:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

and for the stream function given,

$$u = \frac{\partial \psi}{\partial y} = 2ay$$

$$v = -\frac{\partial \psi}{\partial x} = b$$

Thus,

$$\frac{\partial u}{\partial y} = 2a \qquad \frac{\partial v}{\partial x} = 0$$

so that

$$\omega_z = \frac{1}{2}\left[0 - (2a)\right] = -a$$

Since $\omega_z \neq 0$ flow is not irrotational (unless $a = 0$). NO.

3. The stream function for a given two dimensional flow filed is

$$\psi = 5x^2 y - (5/3) y^3$$

Determine the corresponding velocity potential.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 5x^2 - 5y^2 \quad (1)$$

Integrate with respect to x to obtain

$$\int d\phi = \int (5x^2 - 5y^2)\, dx$$

or

$$\phi = \frac{5}{3} x^3 - 5xy^2 + f_1(y) \quad (2)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -10xy \quad (3)$$

and

$$\int d\phi = -\int 10xy\, dy$$

or

$$\phi = -5xy^2 + f_2(x)$$

To satisfy both Eqs. (2) and (4)

$$\phi = \left(\frac{5}{3}\right)x^3 - 5xy^2 + C$$

where $C$ is an arbitrary constant.

4. Determine the stream function corresponding to the velocity potential $\phi = x^3 - 3xy^2$. Sketch the streamline $\psi = 0$, which passes through the origin.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} = 3x^2 - 3y^2$$

Integrate with respect to y to obtain

$$\int d\psi = \int (3x^2 - 3y^2)\, dy$$

or

$$\psi = 3\left(x^2 y - \frac{y^3}{3}\right) + f_1(x) \qquad (1)$$

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} = -6xy$$

and integrating with respect to x yields

$$\int d\psi = \int 6xy\, dx$$

or

$$\psi = 3x^2 y + f_2(y) \qquad (2)$$

To satisfy both Eqs. (1) and (2)

$$\psi = 3x^2y - y^3 + C$$

where $C$ is an arbitrary constant. Since the streamline $\psi = 0$ passes through the origin $(x=0, y=0)$ it follows that $C = 0$ and

$$\psi = \underline{\underline{3x^2y - y^3}} \qquad (3)$$

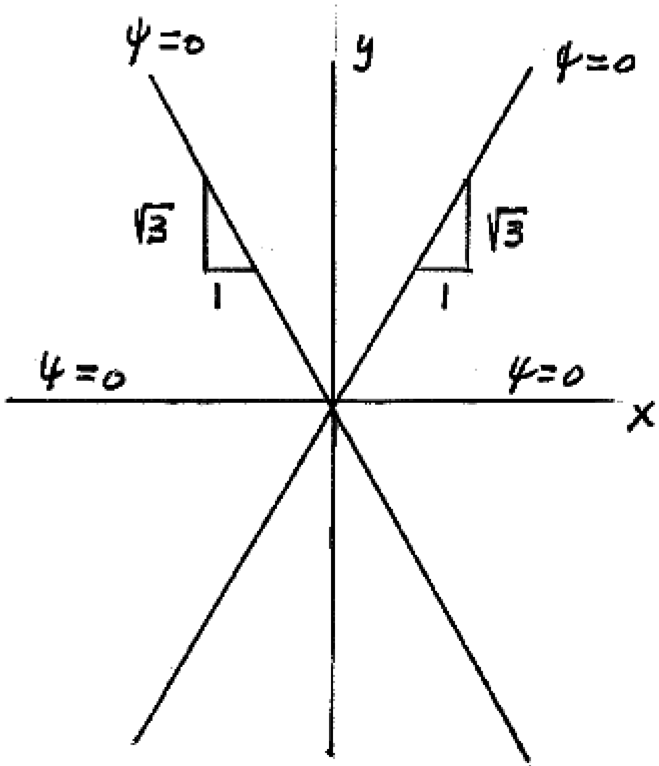
The equation of the streamline passing through the origin is found by setting $\psi = 0$ in Eq. (3) to yield

$$y(3x^2 - y^2) = 0$$

which is satisfied for $y = 0$ and

$$y = \pm\sqrt{3}\,x$$

A sketch of the $\psi = 0$ streamlines are shown in the figure.



5. The velocity potential for a certain inviscid, incompressible flow field is given by the equation

$$\phi = 2x^2 y - (\frac{2}{3}) y^3$$

Where $\phi$ has the units of $m^2/s$ when x and y are in meters. Determine the pressure at the point x = 2 m, y = 2m if the pressure at x = 1 m, y = 1 m is 200 kPa. Elevation changes can be neglected and the fluid is water.

Since the flow is irrotational,

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

with $V^2 = u^2 + v^2$. For the velocity potential given,

$$u = \frac{\partial \phi}{\partial x} = 4xy \qquad v = \frac{\partial \phi}{\partial y} = 2x^2 - 2y^2$$

At point 1 let $x = 1\,m$ and $y = 1\,m$ so that

$$u_1 = 4(1)(1) = 4\ \frac{m}{s} \qquad v_1 = 2(1)^2 - 2(1)^2 = 0$$

and $$V_1^2 = \left(4\ \frac{m}{s}\right)^2 = 16\ \frac{m^2}{s^2}$$

At point 2 $x = 2\,m$ and $y = 2\,m$ so that

$$u_2 = 4(2)(2) = 16\ \frac{m}{s} \qquad v_2 = 2(2)^2 - 2(2)^2 = 0$$

and $$V_2^2 = \left(16\ \frac{m}{s}\right)^2 = 256\ \frac{m^2}{s^2}$$

Thus, from Eq. (1)

$$p_2 = p_1 + \frac{\gamma}{2g}\left(V_1^2 - V_2^2\right)$$

$$= 200 \times 10^3\ \frac{N}{m^2} + \frac{\left(9.80 \times 10^3\ \frac{N}{m^3}\right)}{2\left(9.81\ \frac{m}{s^2}\right)}\left(16\ \frac{m^2}{s^2} - 256\ \frac{m^2}{s^2}\right)$$

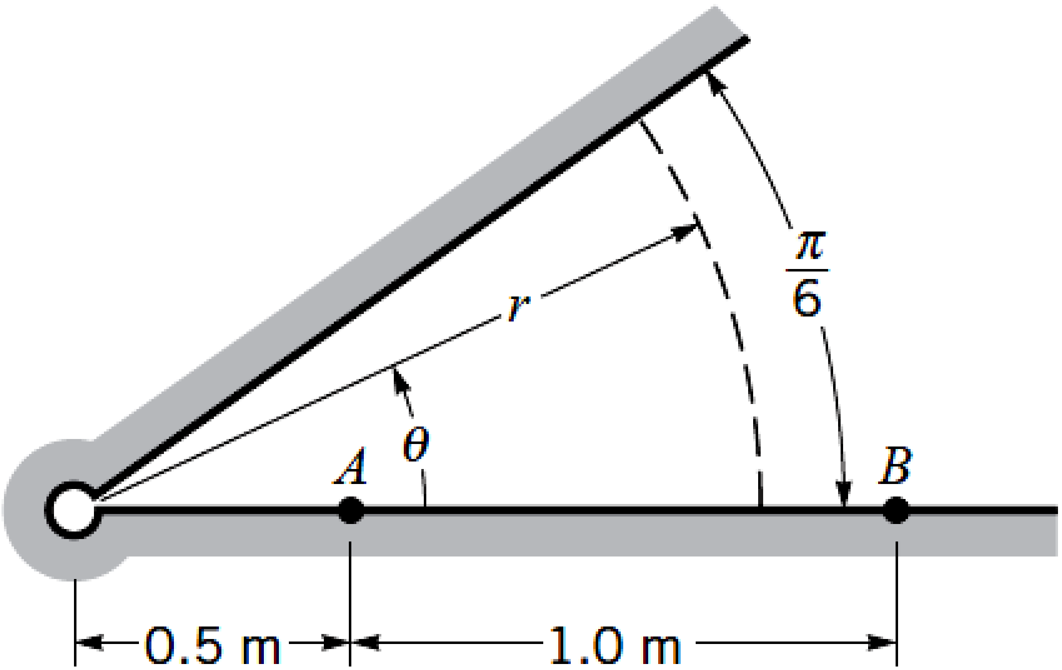
$$= \underline{\underline{80.1\ kPa}}$$

6. Water is flowing between wedge-shaped walls into a small opening as shown in the Fig. below.

The velocity potential with units $m^2/s$ for this flow is

$$\phi = -2 \ln r \qquad \text{with r in meters.}$$

Determine the pressure differential between points A and B.



$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} \qquad (1)$$

Along the horizontal surface, $v_\theta = 0$, and

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r}$$

so that

$$V = v_r = -\frac{2}{r}$$

Thus,

$$V_A = -\frac{2}{0.5} = -4\ \frac{m}{s} \qquad V_B = -\frac{2}{1.5} = -\frac{4}{3}\ \frac{m}{s}$$

and from Eq.(1)

$$p_A - p_B = \frac{\gamma}{2g}\left[V_B^2 - V_A^2\right]$$

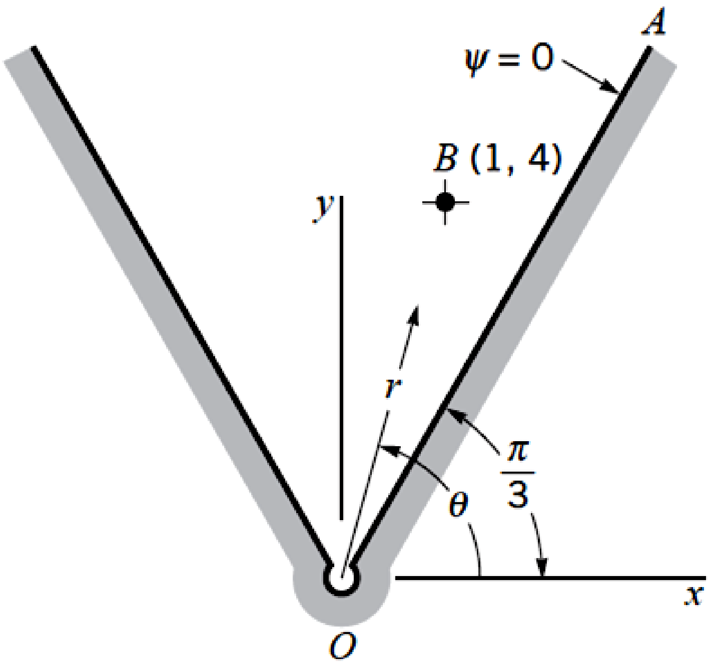
$$= \frac{9.80 \times 10^3\ \frac{N}{m^3}}{2\left(9.81\ \frac{m}{s^2}\right)}\left[\left(-\frac{4}{3}\ \frac{m}{s}\right)^2 - \left(-4\ \frac{m}{s}\right)^2\right]$$

$$= \underline{\underline{-7.10\ kPa}}$$

7. An ideal fluid flows between the inclined walls of a two dimensional channel into a sink located at origin. The velocity potential for this flow field is

$$\phi = \frac{m}{2\pi} \ln r$$

where $m$ is a constant. **(a)** Determine the corresponding stream function. Note that the value of the stream function along the wall $OA$ is zero. **(b)** Determine the equation of the streamline passing through the point $B$, located at $x = 1$, $y = 4$.



(a) $$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad (1)$$

Integrate Eq.(1) with respect to $\theta$ to obtain

$$\int d\psi = \int \frac{m}{2\pi}\, d\theta$$

or

$$\psi = \frac{m\theta}{2\pi} + f_1(r)$$

Since

$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0 \qquad (2)$$

$\psi$ is not a function of $r$ so Eq.(2) becomes

$$\psi = \frac{m\theta}{2\pi} + C$$

where $C$ is a constant. Also, $\psi = 0$ for $\theta = \frac{\pi}{3}$ so that

$$C = -\frac{m}{6}$$

and

$$\psi = m\left(\frac{\theta}{2\pi} - \frac{1}{6}\right) \qquad (3)$$

(b) At B $\tan\theta = \frac{4}{1}$ so that $\theta = 1.33$ rad. From Eq. (3) the value of $\psi$ passing through this point is

$$\psi = m\left(\frac{1.33}{2\pi} - \frac{1}{6}\right) = 0.0450\,m$$

and therefore the equation of the streamline passing through B is

$$0.0450\,m = m\left(\frac{\theta}{2\pi} - \frac{1}{6}\right)$$

or

$$\underline{\underline{\theta = 1.33 \text{ rad}}}$$

(Note: It can be seen from Eq. (3) that the streamlines are all straight lines passing through the origin.)

# End of Chapter 4

***Next Lecture***

**Chapter 5: Dimensional Analysis And Similitude**