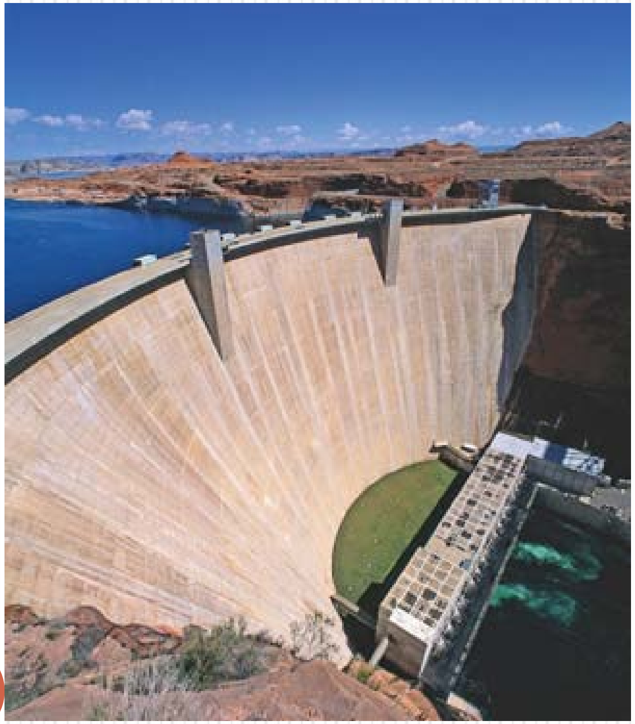
DMiT

School of Mechanical and Industrial Engineering

Fluid Mechanics

(MEng 2113)



Chapter 2

Fluid Statics

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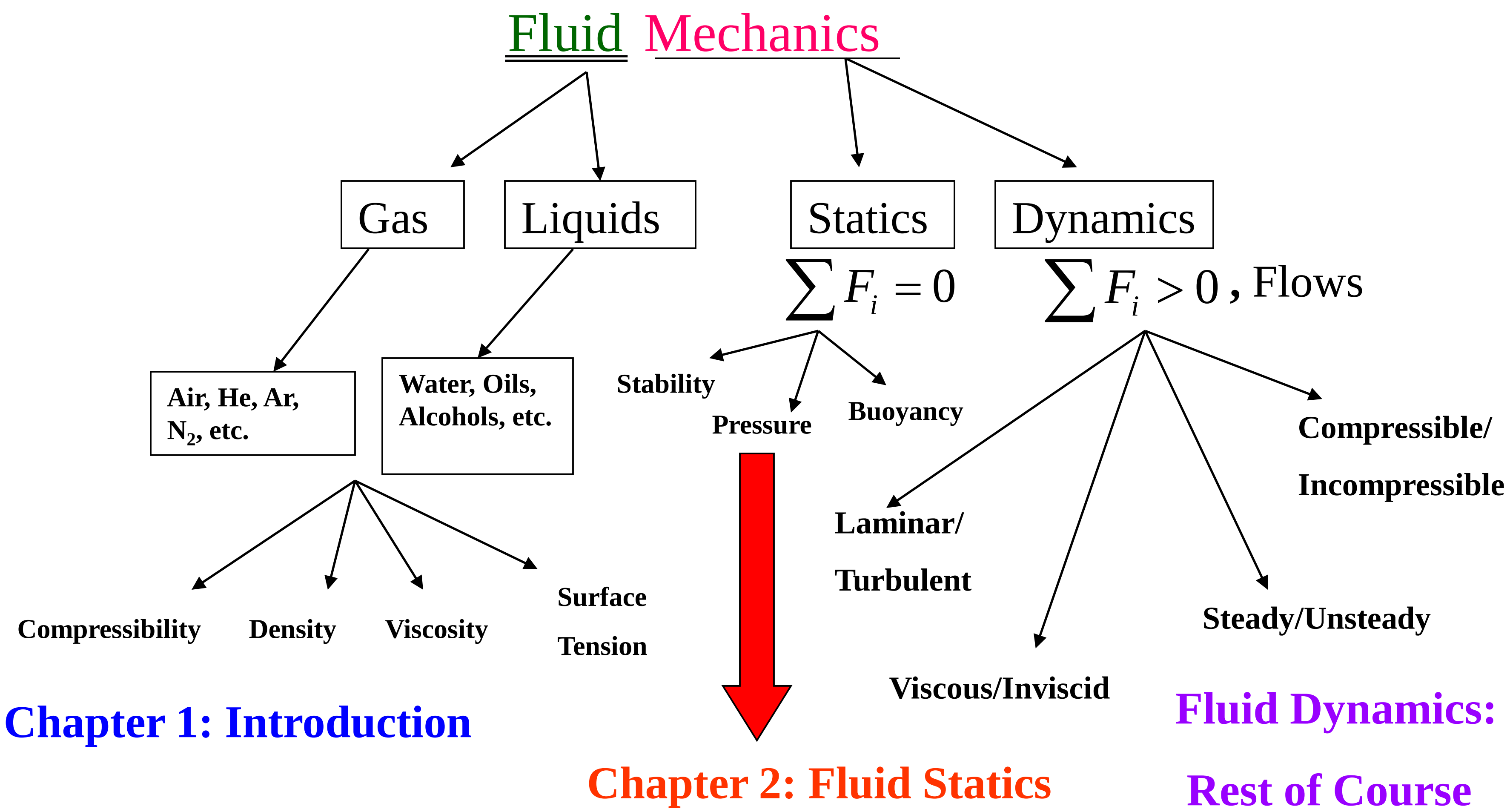
Introduction



Introduction

- This chapter deals with forces applied by fluids at rest or in rigid-body motion (there is no relative motion between adjacent layers).
- In both instances there will be no shearing stresses in the fluid, and the only forces that develop on the surfaces of the particles will be due to the pressure.
- The fluid property responsible for those forces is **pressure**, which is a normal force exerted by a fluid per unit area.
- Thus, our principal concern is to investigate pressure and its variation throughout a fluid and the effect of pressure on submerged surfaces.

Pressure

- **Pressure** is defined as a normal force exerted by a fluid per unit area.
- We speak of pressure only when we deal with a gas or a liquid. The counterpart of pressure in solids is normal stress.
- Since pressure is defined as force per unit area, it has the unit of newtons per square meter (N/m^2), which is called a **pascal (Pa).** That is, $1\ Pa = 1\ N/m^2$
- The pressure unit pascal is too small for pressures encountered in practice. Therefore, its multiples kilopascal ($1\ kPa = 10^3\ Pa$) and megapascal ($1\ MPa = 10^6\ Pa$) are commonly used.
- Other pressure units commonly used in practice, especially in Europe, are bar and standard atmosphere

$$\boxed{\begin{aligned} 1\ \text{bar} &= 10^5\ \text{Pa} = 0.1\ \text{MPa} = 100\ \text{kPa} \\ 1\ \text{atm} &= 101{,}325\ \text{Pa} = 101.325\ \text{kPa} = 1.01325\ \text{bars} \end{aligned}}$$

Pressure

- The actual pressure at a given position is called the **absolute pressure,** and it is measured relative to absolute vacuum (i.e., absolute zero pressure).
- Most pressure-measuring devices, however, are calibrated to read zero in the atmosphere , and so they indicate the difference between the absolute pressure and the local atmospheric pressure.
- This difference is called the **gage pressure.**

Fig. Some basic pressure gages.

Pressure

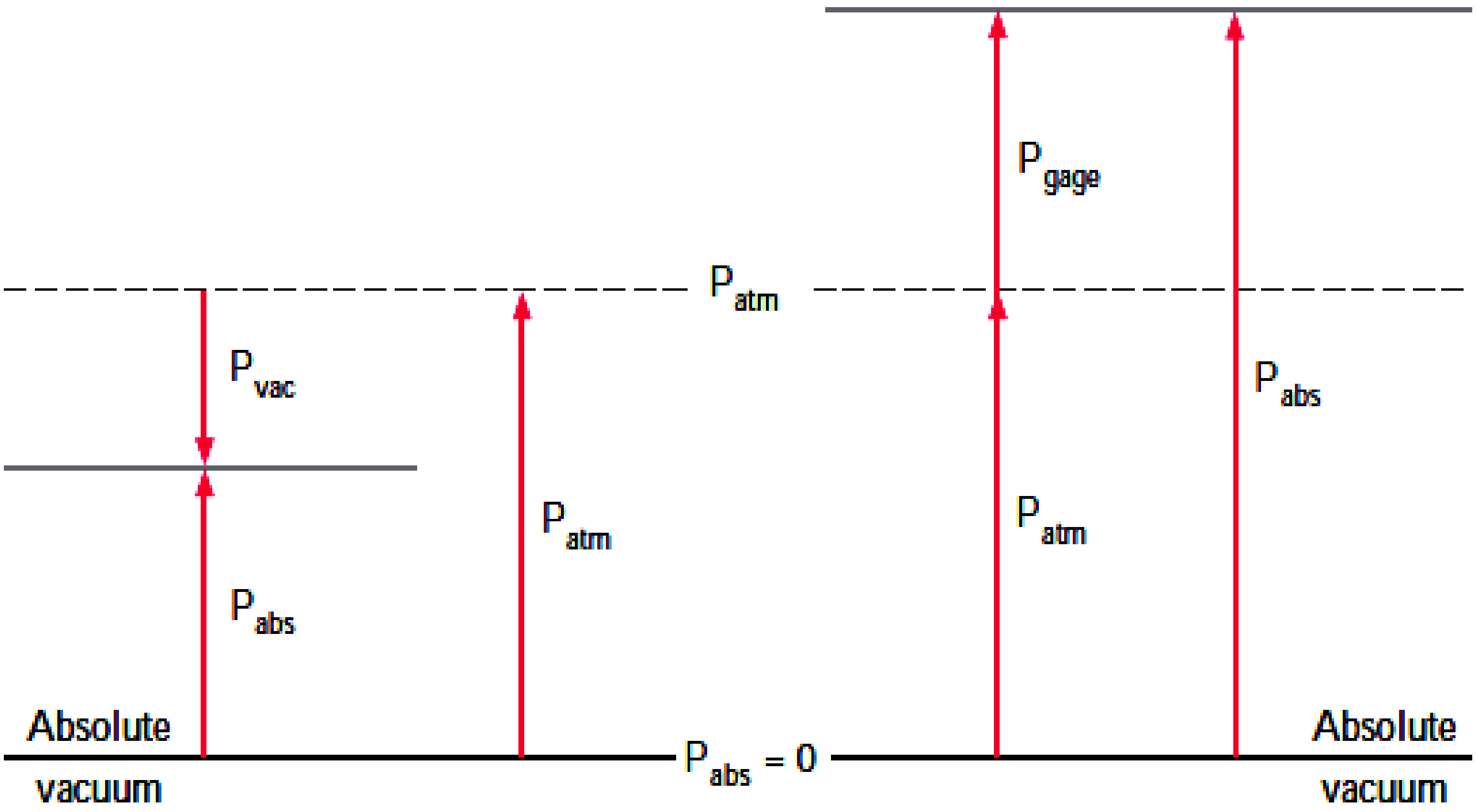
- Pressures below atmospheric pressure are called **vacuum pressures** and are measured by vacuum gages that indicate the difference between the atmospheric pressure and the absolute pressure.
- Absolute, gage, and vacuum pressures are all positive quantities and are related to each other by

$$P_{gage} = P_{abs} - P_{atm}$$

$$P_{vac} = P_{atm} - P_{abs}$$

- In thermodynamic relations and tables, absolute pressure is almost always used. Throughout this course, the pressure P will denote absolute pressure unless specified otherwise.

Pressure



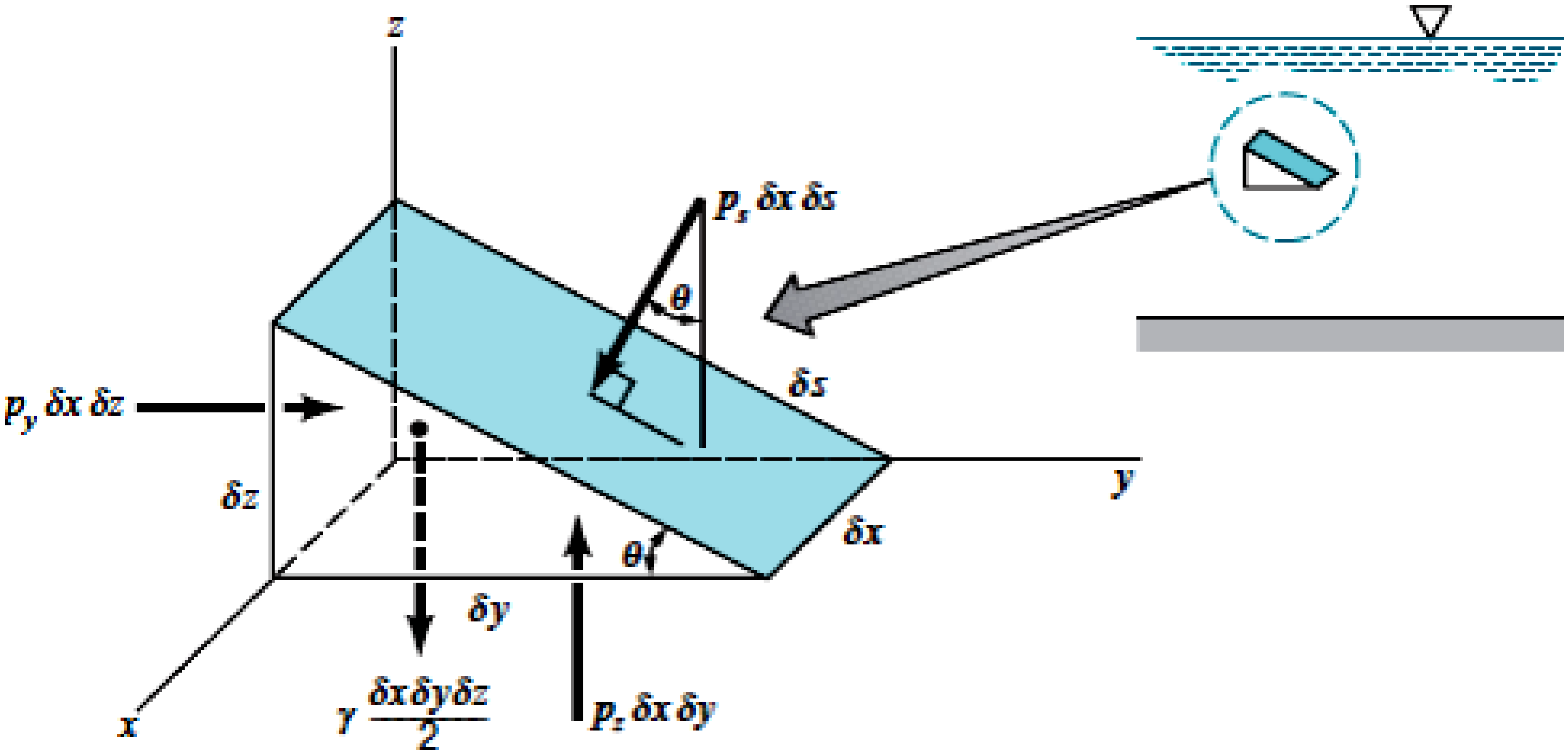
P_{gage}
P_{atm}
P_{vac}
P_{abs}
P_{atm}
P_{atm}
P_{abs}
Absolute
vacuum
$P_{abs} = 0$
Absolute
vacuum

Pressure at a point

- Pressure is the compressive force per unit area, and it gives the impression of being a vector. However, pressure at any point in a fluid is the same in all directions. That is, it has magnitude but not a specific direction, and thus it is a scalar quantity.
- This can be demonstrated by considering a small wedge-shaped fluid element that was obtained by removing a small triangular wedge of fluid from some arbitrary location within a fluid mass.
- Since we are considering the situation in which there are no shearing stresses, the only external forces acting on the wedge are due to the pressure and the weight.

Pressure at a point

- For simplicity the forces in the x direction are not shown, and the z axis is taken as the vertical axis so the weight acts in the negative z direction.



Pressure at a point

- From Newton's second law, a force balance in the y- and z directions gives

$$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \sin\theta = 0 \qquad \text{(a)}$$

$$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \cos\theta - \gamma \frac{\delta x \delta y \delta z}{2} = 0 \qquad \text{(b)}$$

- where p_s, p_y and p_z are the average pressures on the faces, γ and ρ are the fluid specific weight and density
- From the geometry

$$\delta y = \delta s \cos\theta \qquad \delta z = \delta s \sin\theta$$

- The last term in Eq. b drops out as δx ,δy and $\delta z \rightarrow 0$ and the wedge becomes infinitesimal, and thus the fluid element shrinks to a point.

Pressure at a point

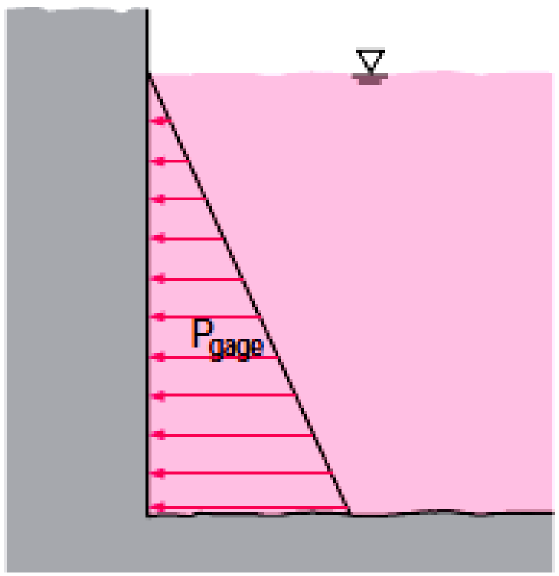
- Thus substituting and simplifying results

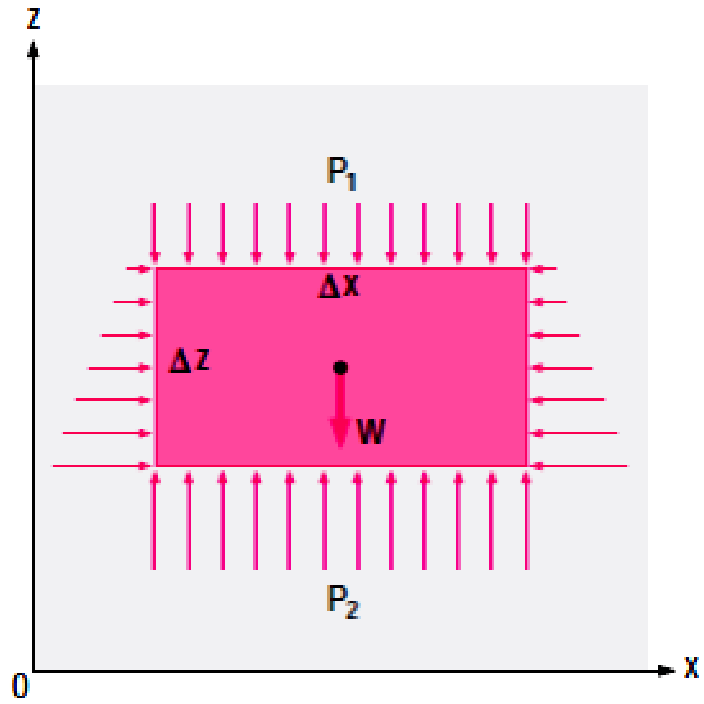
$$p_y = p_s \quad p_z = p_s \quad or \quad p_y = p_s = p_z$$

- Thus we conclude that the pressure at a point in a fluid has the same magnitude in all directions.
- It can be shown in the absence of shear forces that this result is applicable to fluids in motion (rigid body motion, no relative motion between layers) as well as fluids at rest.
- This important result is known as **Pascal's law**

Variation of pressure with depth

- Pressure in a fluid increases with depth because more fluid rests on deeper layers, and the effect of this "extra weight" on a deeper layer is balanced by an increase in pressure.
- To obtain a relation for the variation of pressure with depth, consider a rectangular fluid element of height Δz, length Δx, and unit depth (into the page) in equilibrium.
- Assuming the density of the fluid ρ to be constant, a force balance in the vertical z-direction gives





Variation of pressure with depth

$$\sum F_z = ma_z = 0: \qquad P_2\,\Delta x - P_1\,\Delta x - \rho g\,\Delta x\,\Delta z = 0$$

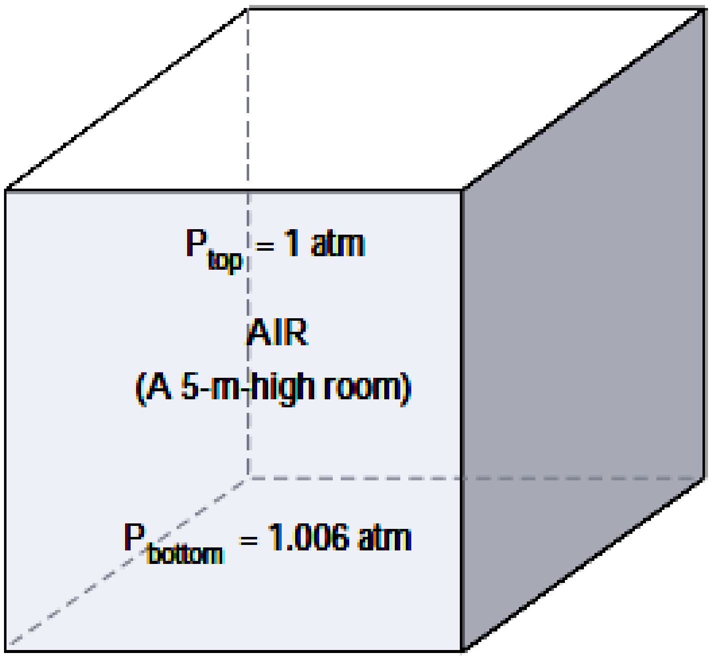
- Where W =mg = $\rho g \Delta x \Delta z$ is the weight of the fluid element.
- Dividing by Δx and rearranging gives

$$\Delta P = P_2 - P_1 = \rho g\,\Delta z = \gamma_s\,\Delta z$$

- $\gamma_s = \rho g$ is the specific weight of the fluid.
- Thus, we conclude that the pressure difference between two points in a constant density fluid is proportional to the vertical distance Δz between the points and the density ρ of the fluid.
- In other words, pressure in a fluid increases linearly with depth

Variation of pressure with depth

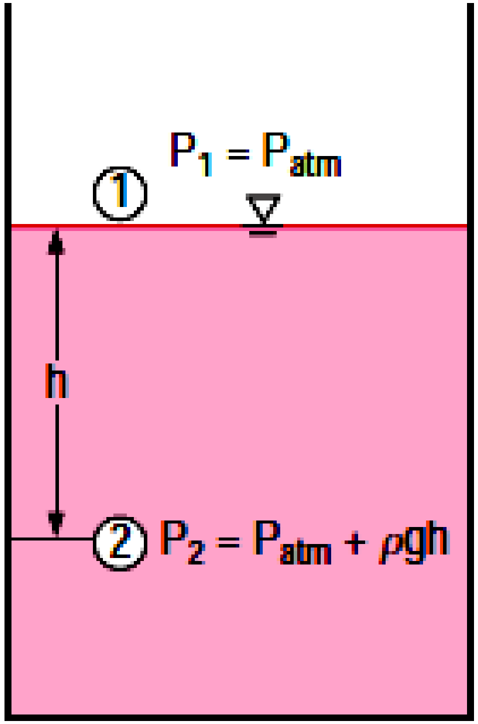
- For small to moderate distances, the variation of pressure with height is negligible for gases because of their low density.
- The pressure in a tank containing a gas, for example, can be considered to be uniform since the weight of the gas is too small to make a significant difference.
- Also, the pressure in a room filled with air can be assumed to be constant



Variation of pressure with depth

- If we take point 1 to be at the free surface of a liquid open to the atmosphere, where the pressure is the atmospheric pressure P_{atm} then the pressure at a depth h from the free surface becomes

$$P = P_{atm} + \rho g h \quad \text{or} \quad P_{gage} = \rho g h$$



- Liquids are essentially incompressible substances, and thus the variation of density with depth is negligible.
- This is also the case for gases when the elevation change is not very large.

Variation of pressure with depth

- For fluids whose density changes significantly with elevation, a relation for the variation of pressure with elevation can be written as

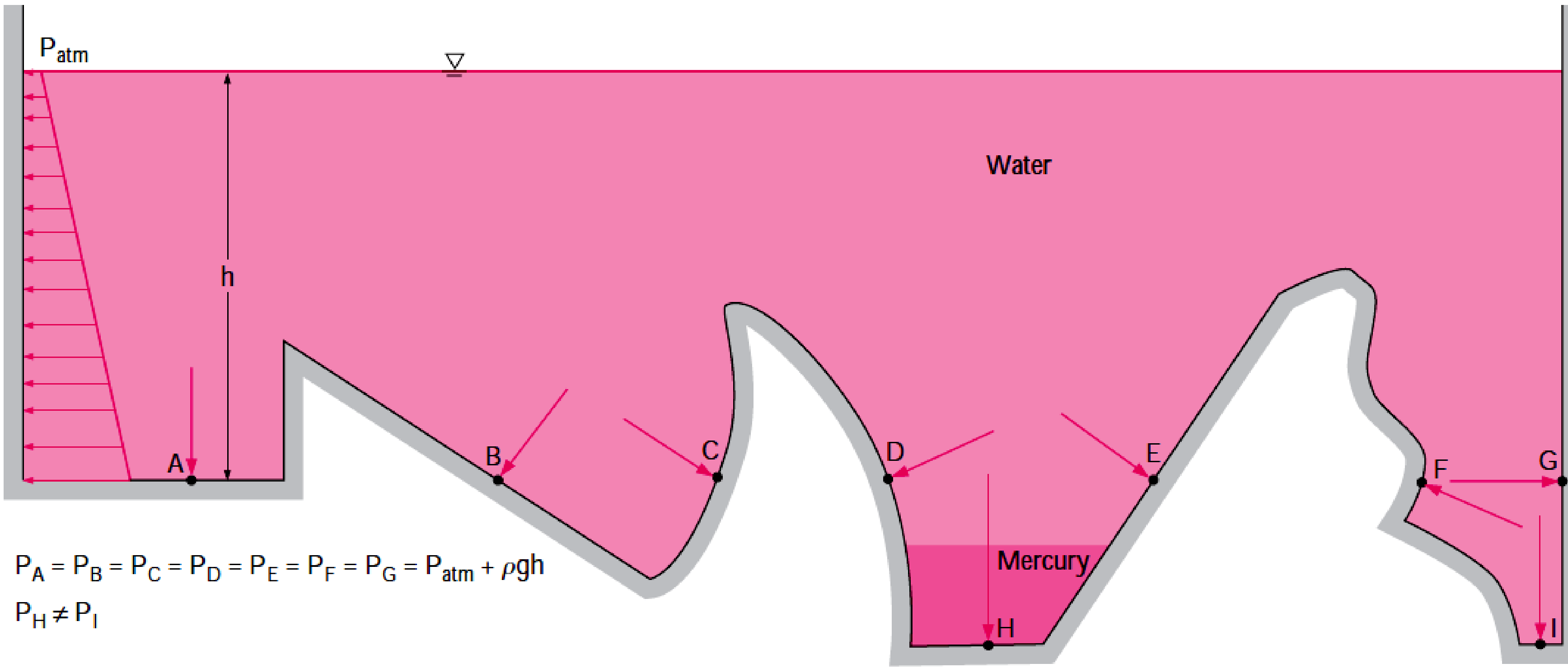
$$\boxed{\frac{dP}{dz} = -\rho g}$$

- The negative sign indicates that pressure decreases in an upward direction.
- When the variation of density with elevation is known the pressure difference between points 1 and 2 can be determined by integration to be

$$\Delta P = P_2 - P_1 = -\int_1^2 \rho g \, dz$$

Variation of pressure with depth

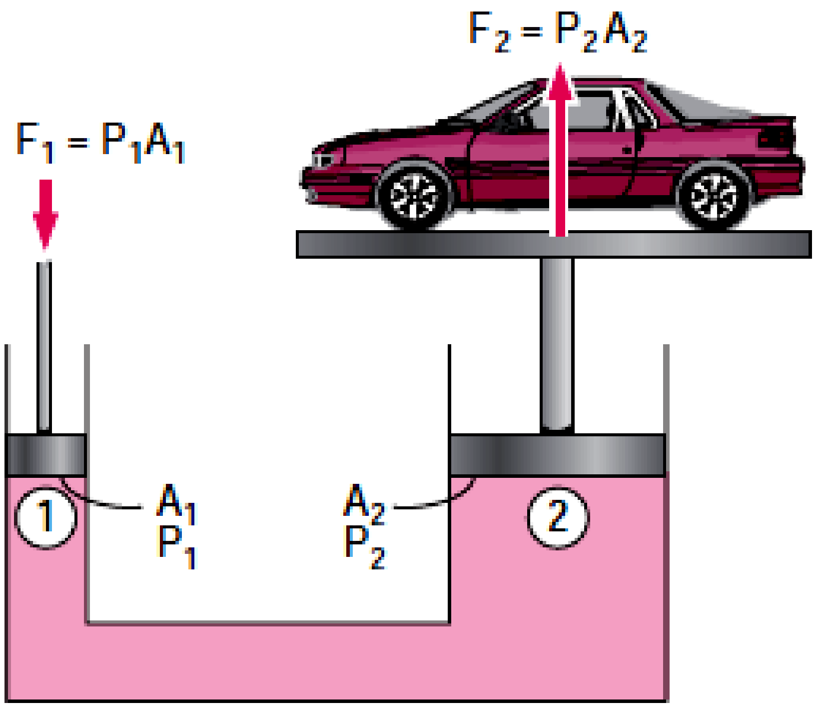
- Pressure is independent of the shape of the container.
- The pressure is the same at all points on a given horizontal plane in the same fluid.



Variation of pressure with depth

Application of Pascal's law

- Two hydraulic cylinders of different areas could be connected, and the larger could be used to exert a proportionally greater force than that applied to the smaller.
- Noting that $P_1 = P_2$ since both pistons are at the same level.



The area ratio A_2 / A_1 is called the ideal mechanical advantage of the hydraulic lift.

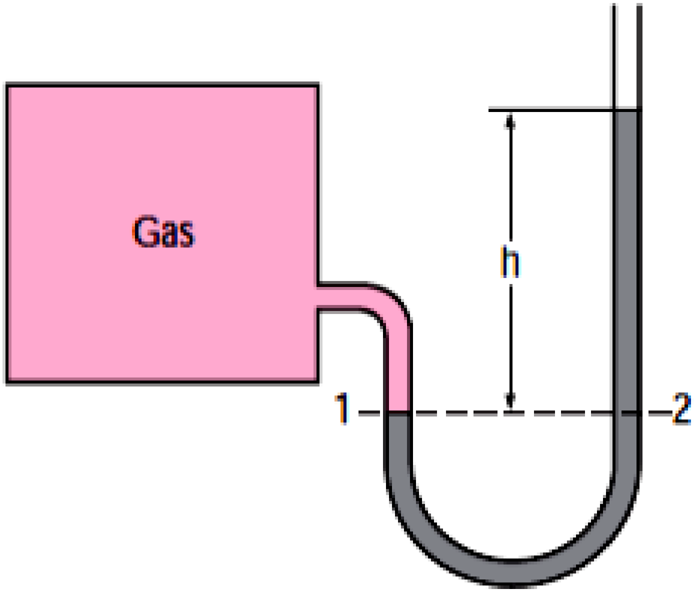
$$P_1 = P_2 \quad \rightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \rightarrow \quad \frac{F_2}{F_1} = \frac{A_2}{A_1}$$

The Manometer

- An elevation change of Δz in a fluid at rest corresponds to $\Delta P/\rho g$, which suggests that a fluid column can be used to measure pressure differences.
- A device based on this principle is called a **manometer,** and it is commonly used to measure small and moderate pressure differences.
- A manometer mainly consists of a glass or plastic U-tube containing one or more fluids such as mercury, water, alcohol, or oil.
- To keep the size of the manometer to a manageable level, heavy fluids such as mercury are used if large pressure differences are anticipated.

U-tube Manometer

- Consider the manometer that is used to measure the pressure in the tank.
- Since the gravitational effects of gases are negligible, the pressure anywhere in the tank and at position 1 has the same value.
- Furthermore, since pressure in a fluid does not vary in the horizontal direction within a fluid, the pressure at point 2 is the same as the pressure at point 1, $P_2=P_1$.

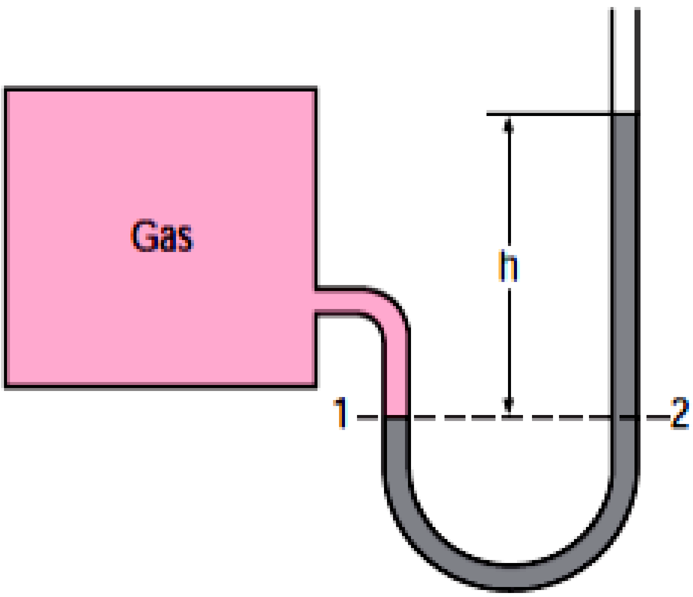


The differential fluid column of height h is in static equilibrium, and it is open to the atmosphere. Then the pressure at point 2 is determined directly by

$$P_2 = P_{atm} + \rho g h$$

U-tube Manometer

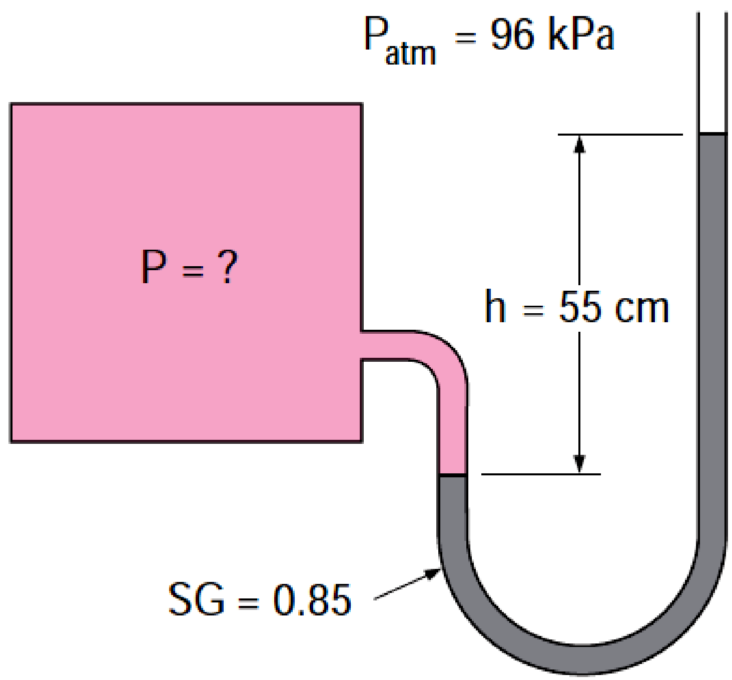
- where ρ is the density of the fluid in the tube. Note that the cross-sectional area of the tube has no effect on the differential height h, and thus the pressure exerted by the fluid.
- However, the diameter of the tube should be large enough (more than a few millimeters) to ensure that the surface tension effect and thus the capillary rise is negligible.



$$P_2 = P_{atm} + \rho g h$$

EXAMPLE 1. Measuring Pressure with a Manometer

- A manometer is used to measure the pressure in a tank. The fluid used has a specific gravity of 0.85, and the manometer column height is 55 cm, as shown in the Fig. If the local atmospheric pressure is 96 kPa, determine the absolute pressure within the tank.



Solution

Assumptions The fluid in the tank is a gas whose density is much lower than the density of manometer fluid.

Properties The specific gravity of the manometer fluid is given to be 0.85. We take the standard density of water to be 1000 kg/m^3.

Analysis The density of the fluid is obtained by multiplying its specific gravity by the density of water, which is taken to be 1000 kg/m^3:

$$\rho = \text{SG}\,(\rho_{H_2O}) = (0.85)(1000\ \text{kg/m}^3) = 850\ \text{kg/m}^3$$

Then from Eq. 3–12,

$$\begin{aligned} P &= P_{atm} + \rho g h \\ &= 96\ \text{kPa} + (850\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.55\ \text{m})\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right) \\ &= \mathbf{100.6\ kPa} \end{aligned}$$

Discussion Note that the gage pressure in the tank is 4.6 kPa.

Multifluid Manometer

- Many engineering problems and some manometers involve multiple immiscible fluids of different densities stacked on top of each other.
- Such systems can be analyzed easily by remembering that

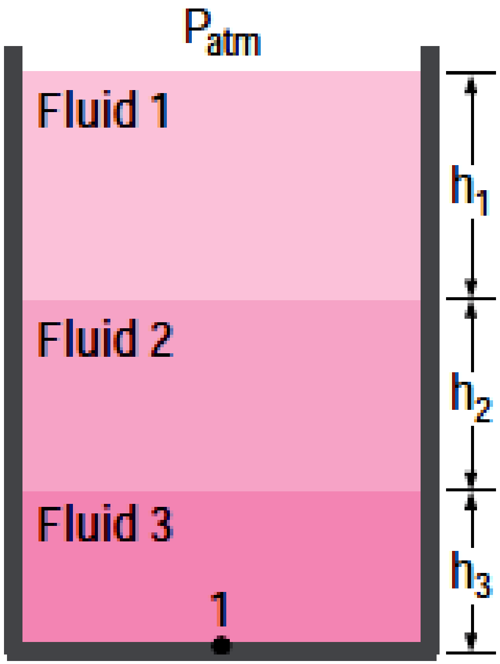
  (1) the pressure change across a fluid column of height h is $\Delta P = \rho gh$,

  (2) pressure increases downward in a given fluid and decreases upward (i.e., $P_{bottom} > P_{top}$), and

  (3) two points at the same elevation in a continuous fluid at rest are at the same pressure.

Multifluid Manometer

- The last principle, which is a result of Pascal's law, allows us to "jump" from one fluid column to the next in manometers without worrying about pressure change as long as we don't jump over a different fluid, and the fluid is at rest.
- Then the pressure at any point can be determined by starting with a point of known pressure and adding or subtracting ρgh terms as we advance toward the point of interest.



$$P_{atm} + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3 = P_1$$

Differential Manometer

- Manometers are particularly well-suited to measure pressure drops across a horizontal flow section between two specified points due to the presence of a device such as a valve or heat exchanger or any resistance to flow.
- This is done by connecting the two legs of the manometer to these two points, as shown in the Fig.
- The working fluid can be either a gas or a liquid whose density is ρ_1. The density of the manometer fluid is ρ_2, and the differential fluid height is *h*.

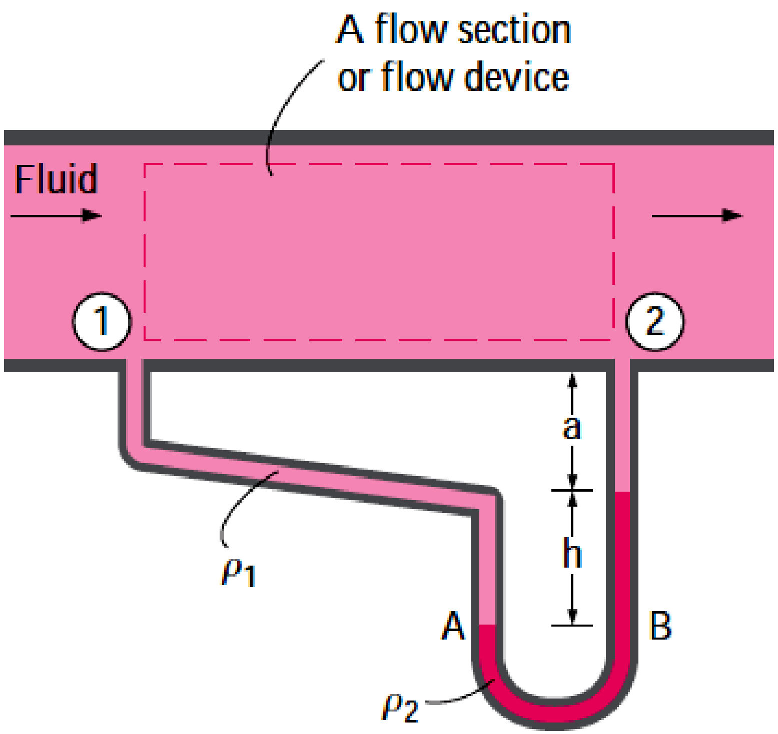
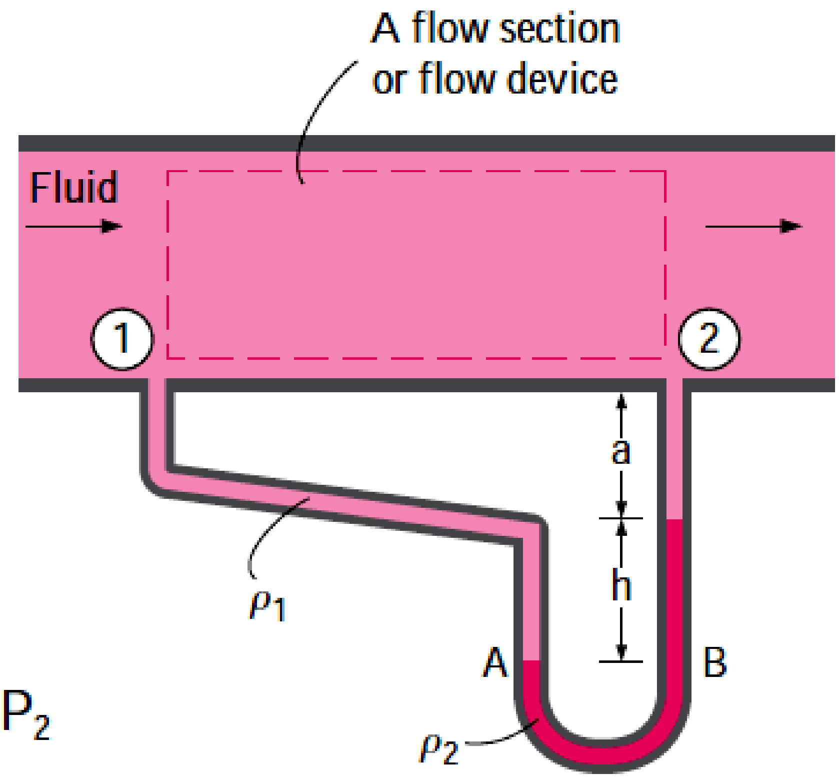


Fig. Measuring the pressure drop across a flow section or a flow device by a differential manometer.

Differential Manometer

- A relation for the pressure difference P_1-P_2 can be obtained by starting at point 1 with P_1, moving along the tube by adding or subtracting the ρgh terms until we reach point 2, and setting the result equal to P_2:



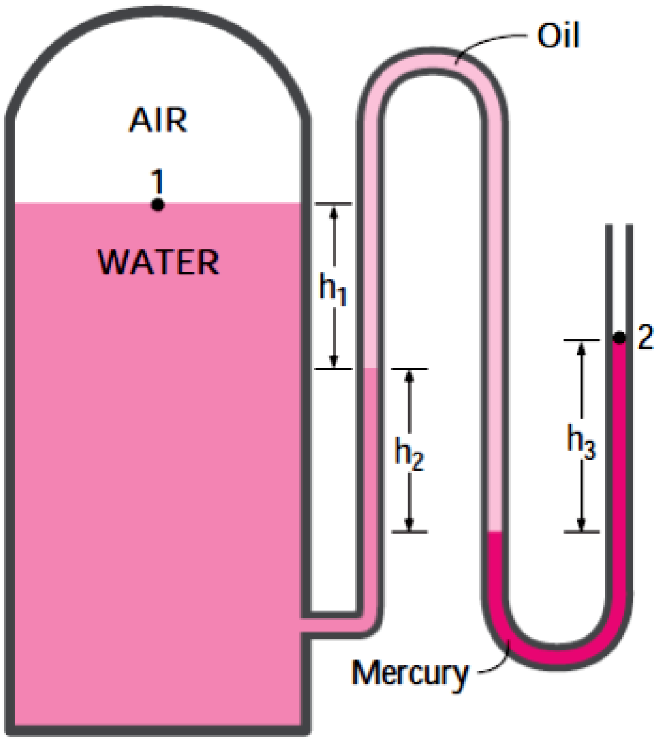
$$P_1 + \rho_1 g(a + h) - \rho_2 gh - \rho_1 ga = P_2$$

Note that we jumped from point A horizontally to point B and ignored the part underneath since the pressure at both points is the same. Simplifying

$$P_1 - P_2 = (\rho_2 - \rho_1)gh$$

EXAMPLE 2. Measuring Pressure with a Multifluid Manometer

- The water in a tank is pressurized by air, and the pressure is measured by a multifluid manometer as shown in Fig. below. The tank is located on a mountain at an altitude of 1400 m where the atmospheric pressure is 85.6 kPa. Determine the air pressure in the tank if h_1 =0.1 m, h_2 =0.2 m, and h_3 = 0.35 m. Take the densities of water, oil, and mercury to be 1000 kg/m^3, 850 kg/m^3, and 13,600 kg/m^3, respectively.



Solution

Analysis Starting with the pressure at point 1 at the air–water interface, moving along the tube by adding or subtracting the ρgh terms until we reach point 2, and setting the result equal to P_{atm} since the tube is open to the atmosphere gives

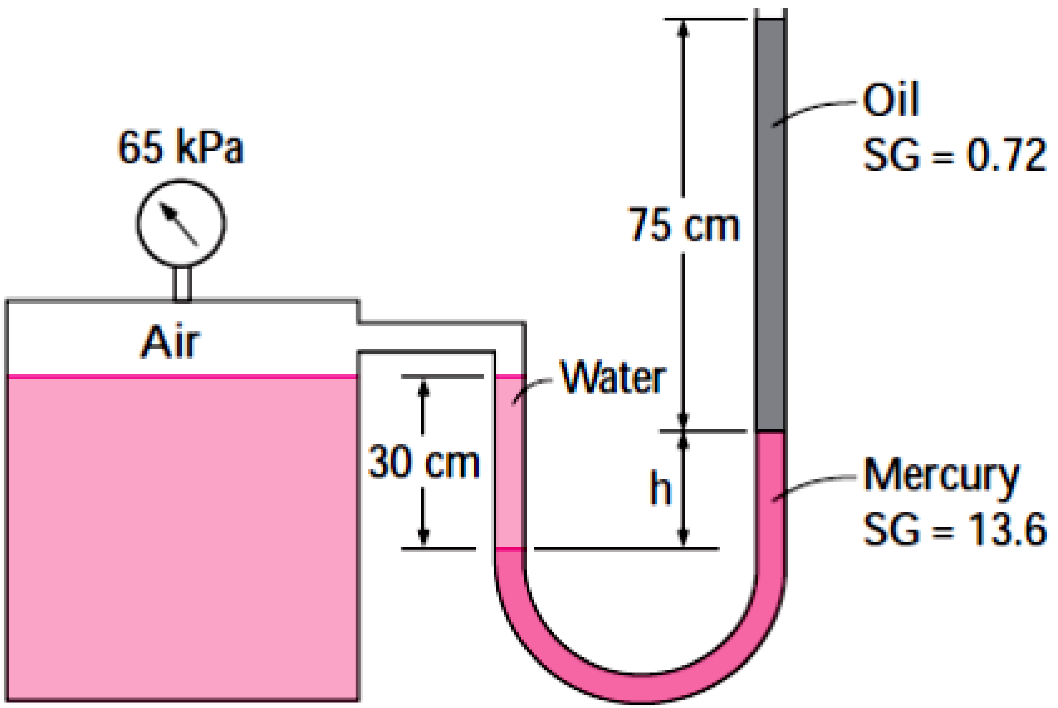
$$P_1 + \rho_{water} g h_1 + \rho_{oil} g h_2 - \rho_{mercury} g h_3 = P_{atm}$$

Solving for P_1 and substituting,

$$\begin{aligned} P_1 &= P_{atm} - \rho_{water} g h_1 - \rho_{oil} g h_2 + \rho_{mercury} g h_3 \\ &= P_{atm} + g(\rho_{mercury} h_3 - \rho_{water} h_1 - \rho_{oil} h_2) \\ &= 85.6 \text{ kPa} + (9.81 \text{ m/s}^2)[(13{,}600 \text{ kg/m}^3)(0.35 \text{ m}) - (1000 \text{ kg/m}^3)(0.1 \text{ m}) \\ &\quad - (850 \text{ kg/m}^3)(0.2 \text{ m})]\left(\frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m/s}^2}\right)\left(\frac{1 \text{ kPa}}{1000 \text{ N/m}^2}\right) \\ &= \mathbf{130 \text{ kPa}} \end{aligned}$$

Exercise 1

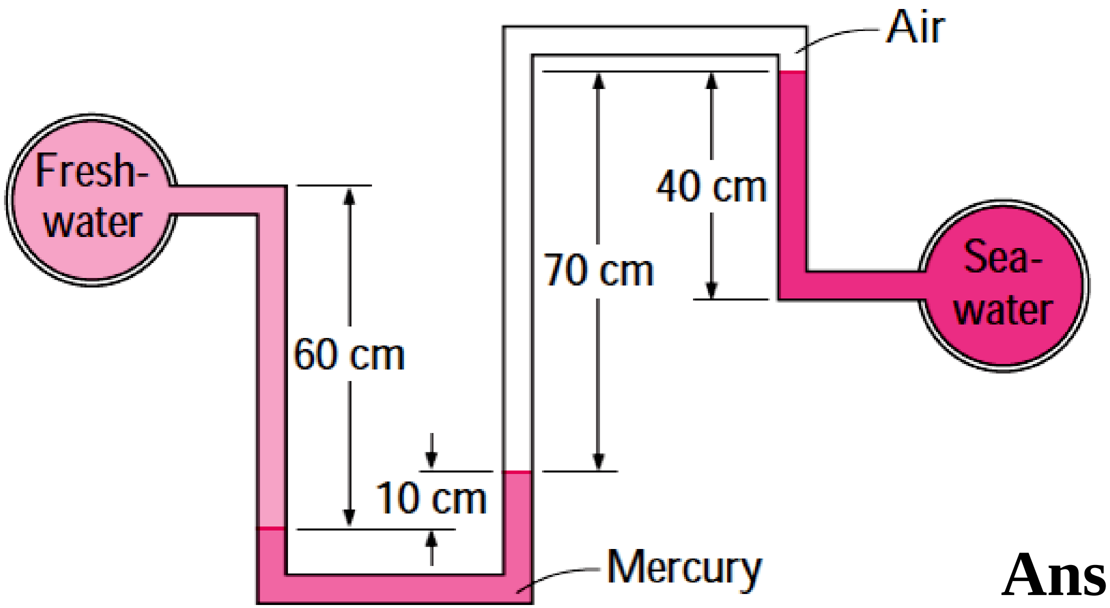
- The gage pressure of the air in the tank shown in Fig. below is measured to be 65 kPa. Determine the differential height h of the mercury column.



Ans. ***h*= 47 cm**

Exercise 2

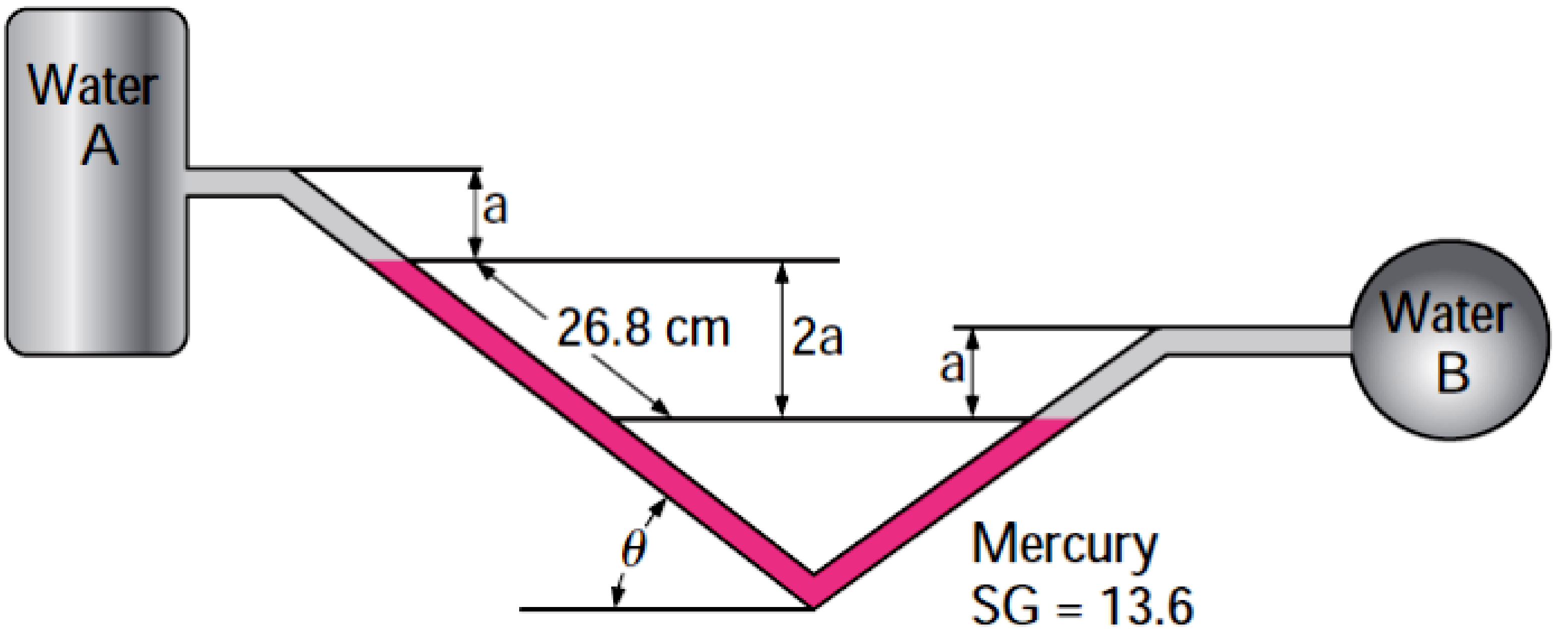
- Freshwater and seawater flowing in parallel horizontal pipelines are connected to each other by a double U-tube manometer, as shown in the Fig. Determine the pressure difference between the two pipelines. Take the density of seawater at that location to be ρ = 1035 kg/m^3. Can the air column be ignored in the analysis?



Ans. 3.39 kPa

Exercise 3

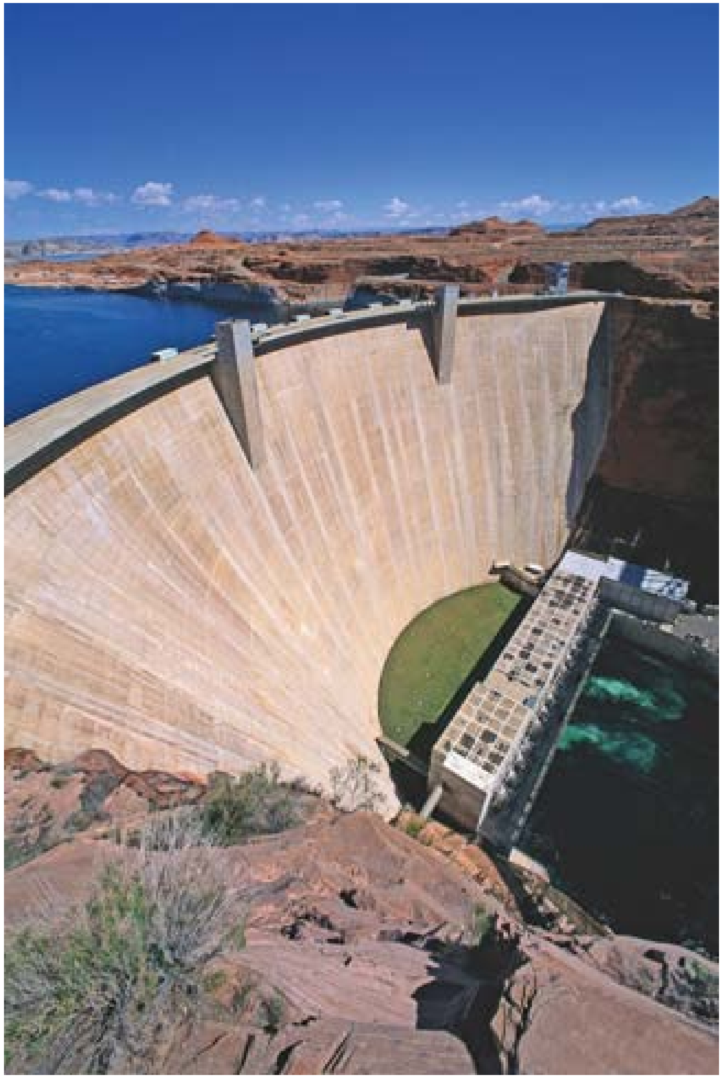
- Two water tanks are connected to each other through a mercury manometer with inclined tubes, as shown in the Fig. below. If the pressure difference between the two tanks is 20 kPa, calculate a and θ.



Ans a= 7.50 cm θ = 34.0º

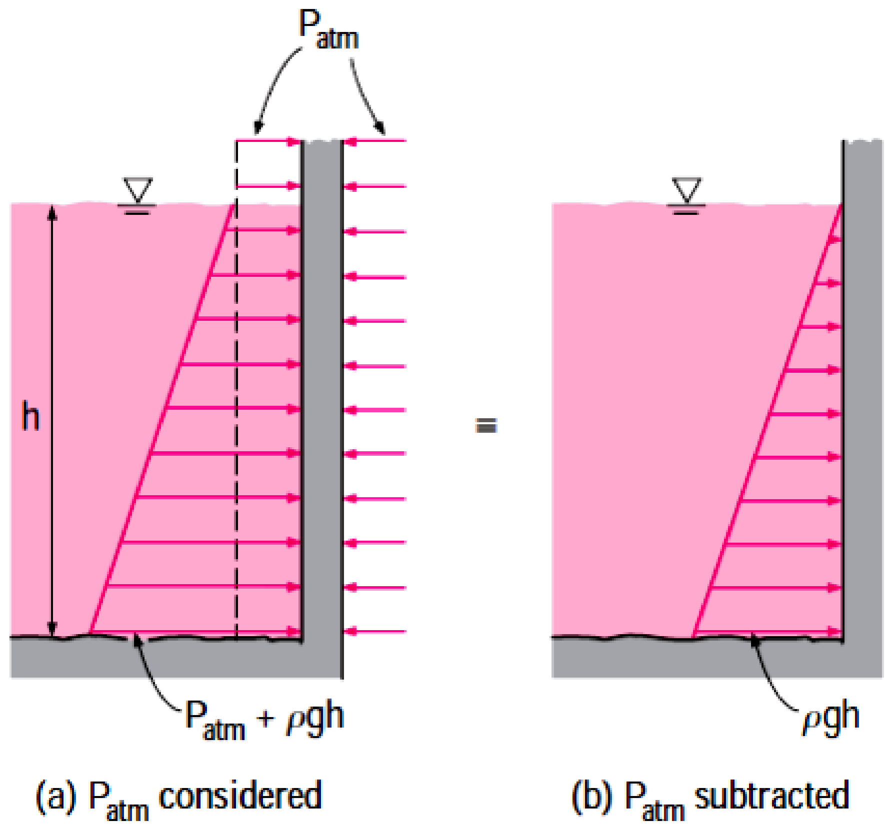
Hydrostatic Forces on Submerged Plane surfaces

- A plate exposed to a liquid, such as a gate valve in a dam, the wall of a liquid storage tank is subjected to fluid pressure distributed over its surface
- On a plane surface, the hydrostatic forces form a system of parallel forces, and we often need to determine the magnitude of the force and its point of application, which is called the **center of pressure.**



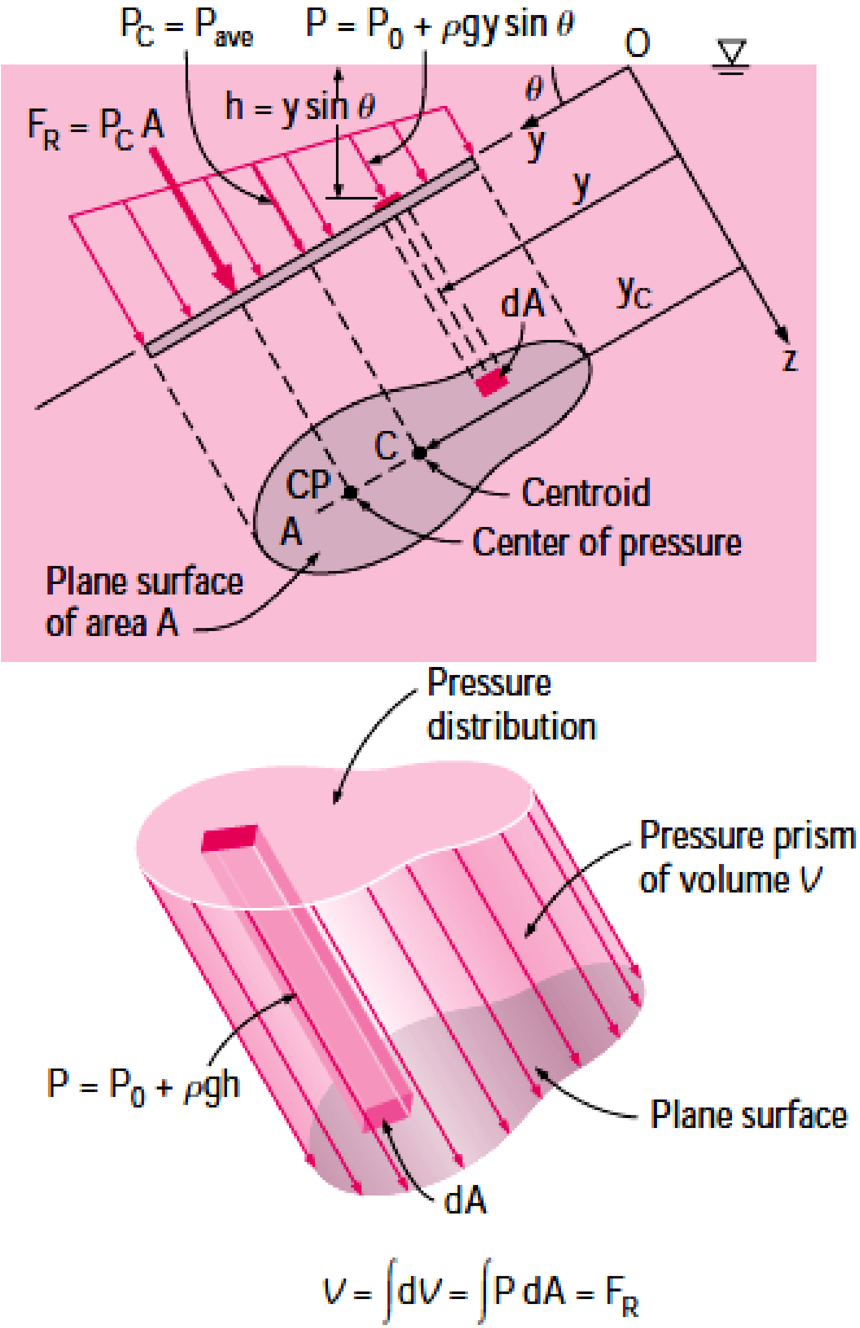
Hydrostatic Forces on Submerged Plane surfaces

- In most cases, the other side of the plate is open to the atmosphere (such as the dry side of a gate), and thus atmospheric pressure acts on both sides of the plate, yielding a zero resultant.
- In such cases, it is convenient to subtract atmospheric pressure and work with the gage pressure only



Hydrostatic Forces on Submerged Plane surfaces

- Consider the top surface of a flat plate of arbitrary shape completely submerged in a liquid.
- The plane of this surface (normal to the page) intersects the horizontal free surface with an angle θ, and we take the line of intersection to be the *x*-*axis*.



Hydrostatic Forces on Submerged Plane surfaces

- The absolute pressure above the liquid is P_0, which is the local atmospheric pressure P_{atm} if the liquid is open to the atmosphere (but P_0 may be different than P_{atm} if the space above the liquid is evacuated or pressurized).
- Then the absolute pressure at any point on the plate is

$$\boxed{P = P_0 + \rho g h = P_0 + \rho g y \sin\theta}$$

- where h is the vertical distance of the point from the free surface and y is the distance of the point from the x-axis.
- The resultant hydrostatic force F_R acting on the surface is determined by integrating the force P dA acting on a differential area dA over the entire surface area,

$$\boxed{F_R = \int_A P\,dA = \int_A (P_0 + \rho g y \sin\theta)\,dA = P_0 A + \rho g \sin\theta \int_A y\,dA}$$

Hydrostatic Forces on Submerged Plane surfaces

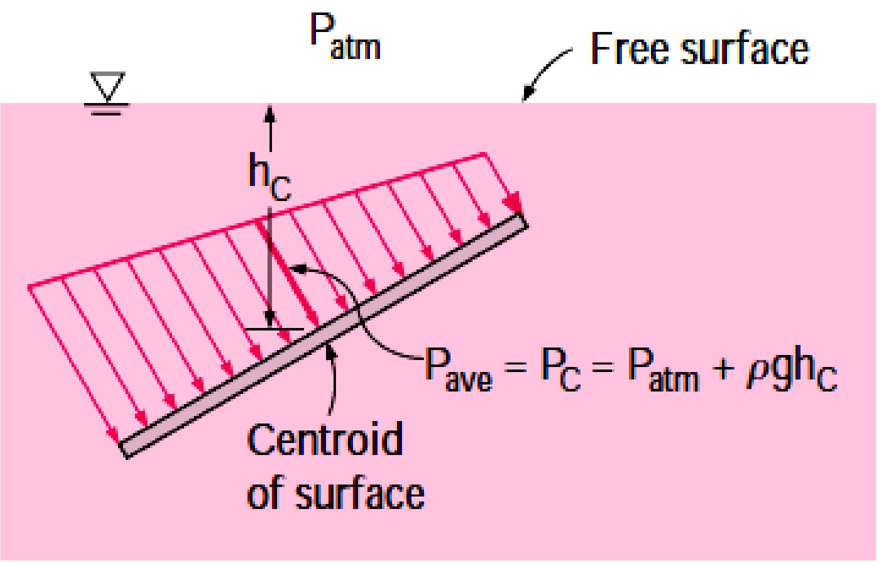
- But the first moment of area $\int_A y\, dA$ is related to the y-coordinate of the centroid (or center) of the surface by

$$y_C = \frac{1}{A}\int_A y\, dA$$

- Substituting

$$F_R = (P_0 + \rho g y_C \sin\theta)A = (P_0 + \rho g h_C)A = P_C A = P_{ave} A$$

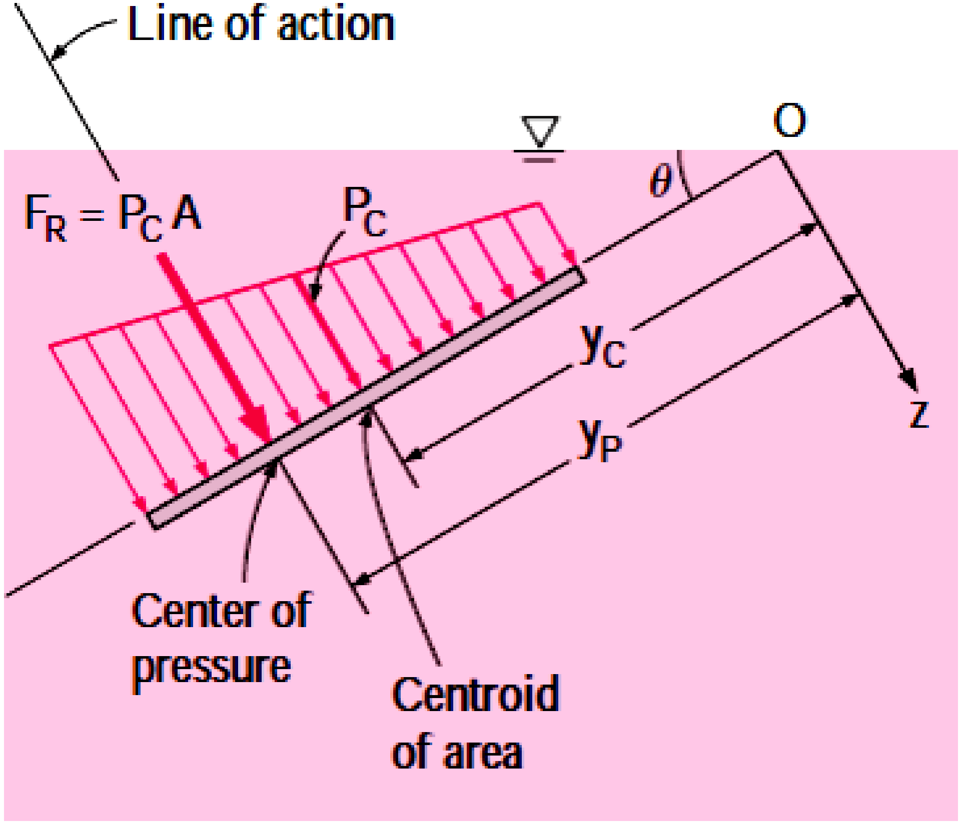
Where $P_C = P_0 + \rho g h_C$ is the pressure at the centroid of the surface, which is equivalent to the average pressure on the surface, and $h_C = y_C \sin\theta$ is the vertical distance of the centroid from the free surface of the liquid .



Hydrostatic Forces on Submerged Plane surfaces

Thus we conclude that

- The magnitude of the resultant force acting on a plane surface of a completely submerged plate in a homogeneous (constant density) fluid is equal to the product of the pressure P_C at the centroid of the surface and the area A of the surface.



Hydrostatic Forces on Submerged Plane surfaces

Center of pressure

- The line of action of the resultant hydrostatic force, in general, does not pass through the centroid of the surface—it lies underneath where the pressure is higher.
- The point of intersection of the line of action of the resultant force and the surface is the **center of pressure.**
- The vertical location of the line of action is determined by equating the moment of the resultant force to the moment of the distributed pressure force about the *x-axis*. *It gives*

$$y_P F_R = \int_A yP\,dA = \int_A y(P_0 + \rho g y \sin\theta)\,dA = P_0 \int_A y\,dA + \rho g \sin\theta \int_A y^2\,dA$$

- or

$$y_P F_R = P_0 y_C A + \rho g \sin\theta\, I_{xx,\,O}$$

Hydrostatic Forces on Submerged Plane surfaces

Center of pressure

- where y_P is the distance of the center of pressure from the x-axis and $I_{xx,\,O} = \int_A y^2\, dA$ is the second moment of area

  (also called the area moment of inertia) about the x-axis.
- The second moments of area are widely available for common shapes in engineering handbooks, but they are usually given about the axes passing through the centroid of the area.
- Fortunately, the second moments of area about two parallel axes are related to each other by the **parallel axis theorem,** which in this case is expressed as

$$I_{xx,\,O} = I_{xx,\,C} + y_C^2 A$$

Hydrostatic Forces on Submerged Plane surfaces

- where $I_{xx,\,C}$ is the second moment of area about the x-axis passing through the centroid of the area and y_C (the y coordinate of the centroid) is the distance between the two parallel axes.
- Substituting the F_R relation and the $I_{xx,\,O}$ relation and solving for y_P gives

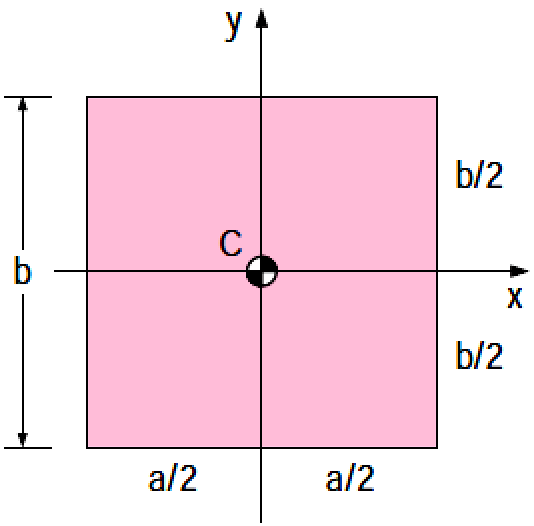
$$y_P = y_C + \frac{I_{xx,\,C}}{[y_C + P_0/(\rho g \sin\theta)]A}$$

- For $P_0 = 0$, which is usually the case when the atmospheric pressure is ignored, it simplifies to

$$y_P = y_C + \frac{I_{xx,\,C}}{y_C A}$$

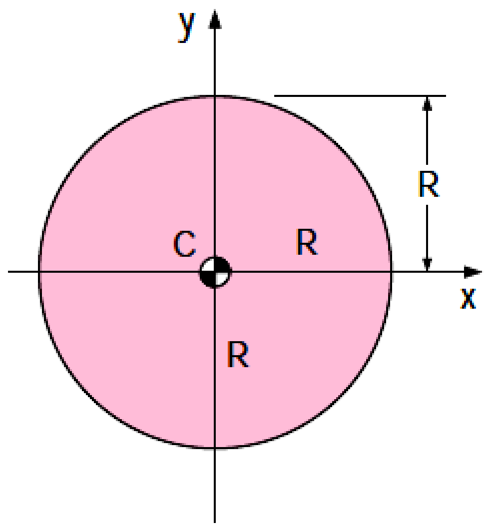
Hydrostatic Forces on Submerged Plane surfaces

- Knowing y_P, the vertical distance of the center of pressure from the free surface is determined from $h_P = y_P \sin \theta$.
- The $I_{xx,\,C}$ values for some common areas are given below. For these and other areas that possess symmetry about the y-axis, the center of pressure lies on the y-axis directly below the centroid.
- The location of the center of pressure in such cases is simply the point on the surface of the vertical plane of symmetry at a distance h_P from the free surface.



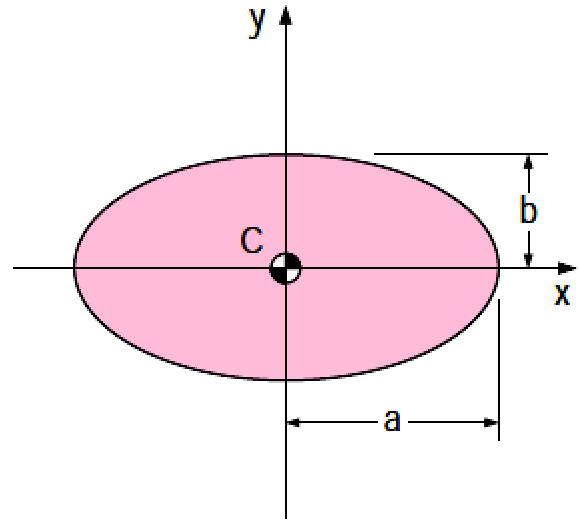
$A = ab,\ I_{xx,C} = ab^3/12$

(a) Rectangle



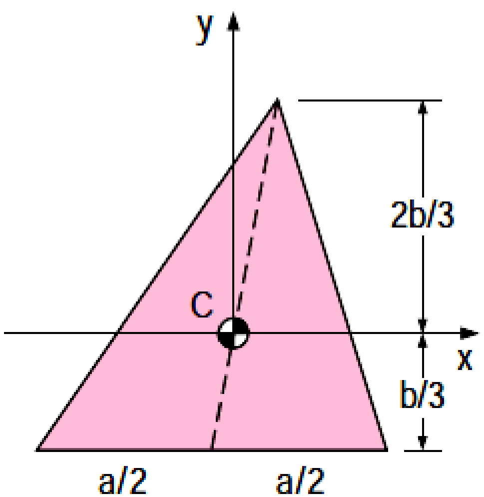
$A = \pi R^2,\ I_{xx,C} = \pi R^4/4$

(b) Circle



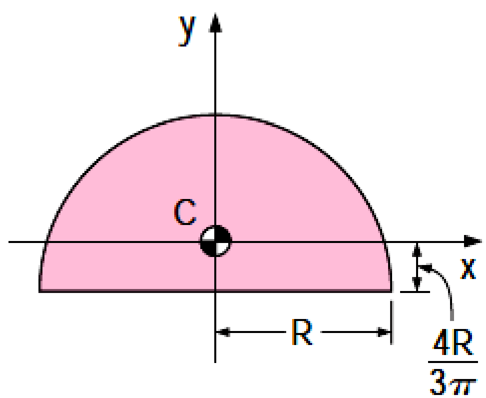
$A = \pi ab,\ I_{xx,C} = \pi ab^3/4$

(c) Ellipse



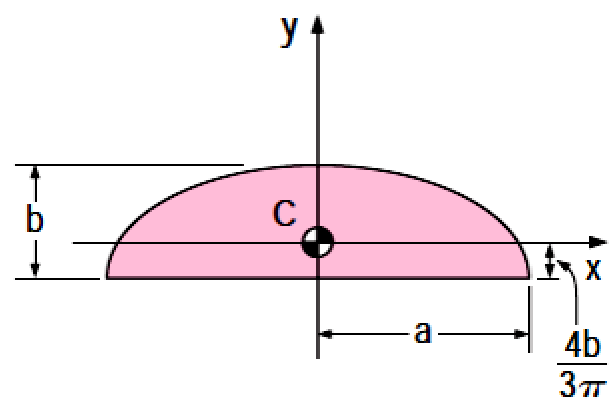
$A = ab/2,\ I_{xx,C} = ab^3/36$

(d) Triangle



$A = \pi R^2/2,\ I_{xx,C} = 0.109757R^4$

(e) Semicircle



$A = \pi ab/2,\ I_{xx,C} = 0.109757ab^3$

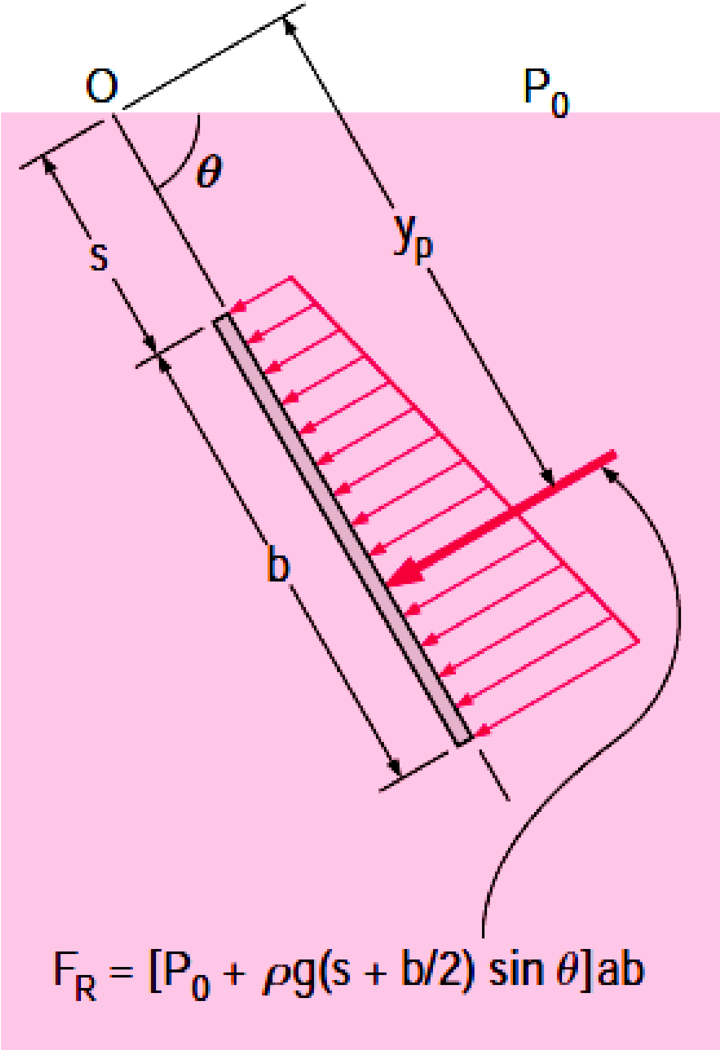
(f) Semiellipse

Hydrostatic Forces on Submerged Plane surfaces

Special Case:

Submerged Rectangular Plate

- Consider a completely submerged rectangular flat plate of height b and width a tilted at an angle θ from the horizontal and whose top edge is horizontal and is at a distance s from the free surface along the plane of the plate, as shown in the Fig.
- The resultant hydrostatic force on the upper surface is equal to the average pressure, which is the pressure at the midpoint of the surface, times the surface area A. That is,



(a) Tilted plate

Hydrostatic Forces on Submerged Plane surfaces

$$F_R = P_C A = [P_0 + \rho g(s + b/2)\sin\theta]ab$$

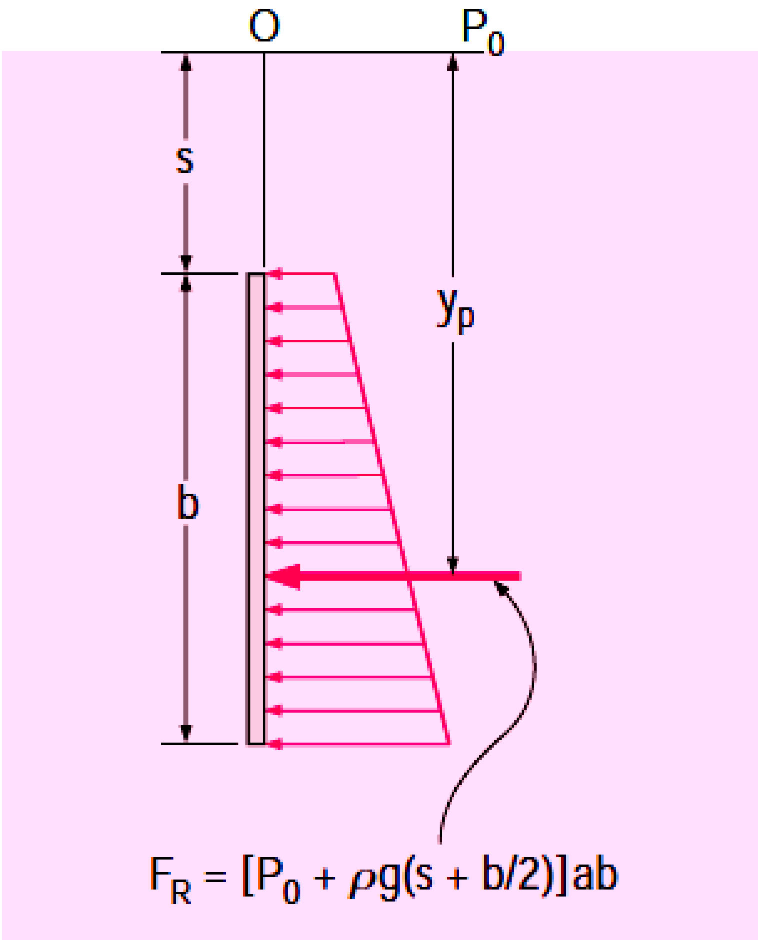
- The force acts at a vertical distance of $h_P = y_P \sin\theta$ *from the free surface* directly beneath the centroid of the plate where

$$y_P = s + \frac{b}{2} + \frac{ab^3/12}{[s + b/2 + P_0/(\rho g \sin\theta)]ab}$$
$$= s + \frac{b}{2} + \frac{b^2}{12[s + b/2 + P_0/(\rho g \sin\theta)]}$$

- When the upper edge of the plate is at the free surface and thus *s =0*

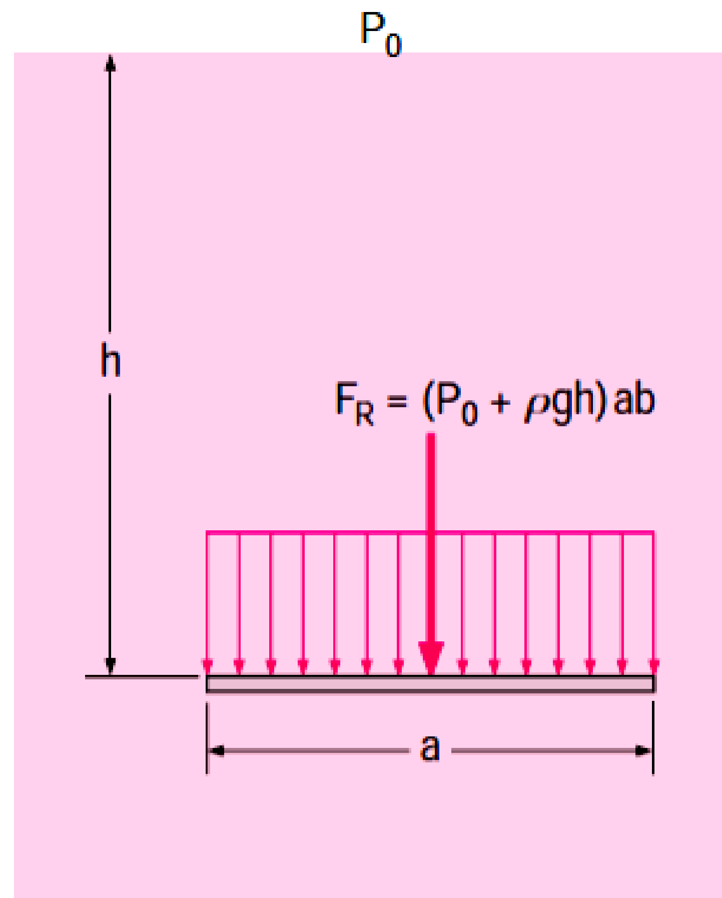
$$\text{Tilted rectangular plate } (s = 0): \qquad F_R = [P_0 + \rho g(b \sin\theta)/2]ab$$

Hydrostatic Forces on Submerged Plane surfaces



(b) Vertical plate

$$F_R = [P_0 + \rho g(s + b/2)]ab$$



(c) Horizontal plate

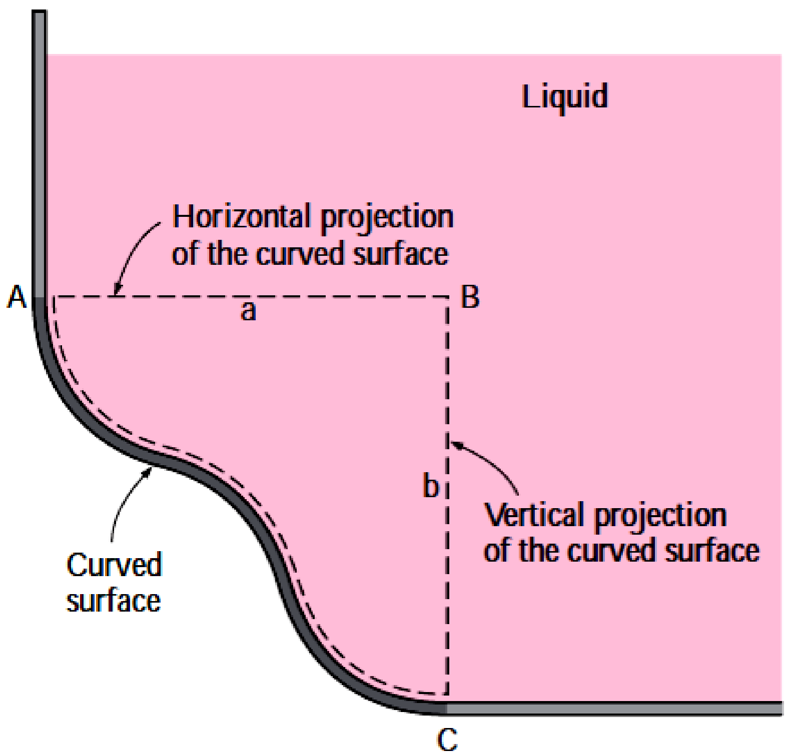
$$F_R = (P_0 + \rho gh)ab$$

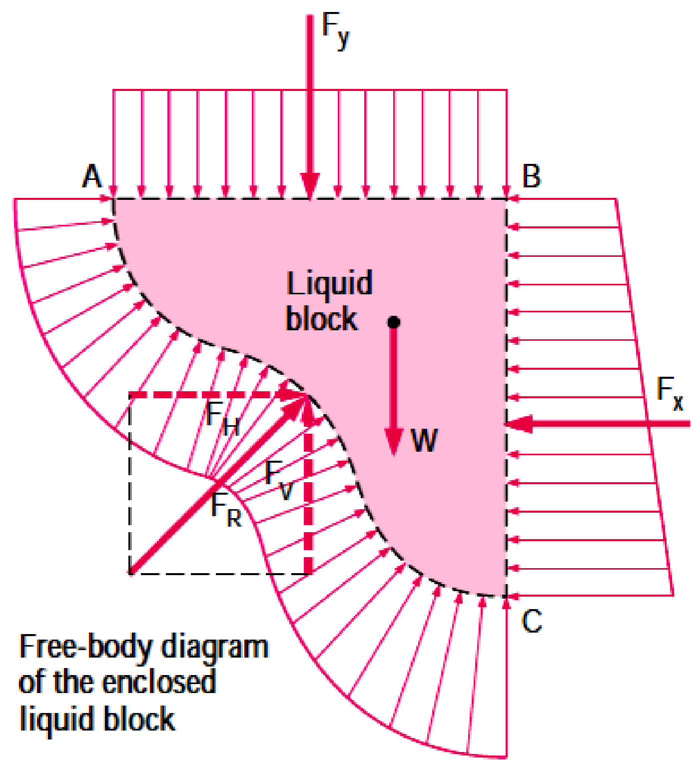
Hydrostatic Forces on submerged curved surfaces

- For a submerged curved surface, the determination of the resultant hydrostatic force is more involved since it typically requires the integration of the pressure forces that change direction along the curved surface.
- The easiest way to determine the resultant hydrostatic force F_R acting on a two-dimensional curved surface is to determine the horizontal and vertical components F_H and F_V separately.
- Consider the free-body diagram of the liquid block enclosed by the curved surface and the two plane surfaces (one horizontal and one vertical) passing through the two ends of the curved surface as shown in the fig. below.

Hydrostatic Forces on submerged curved surfaces

- The resultant force acting on the curved solid surface is then equal and opposite to the force acting on the curved liquid surface (Newton's third law).





Hydrostatic Forces on submerged curved surfaces

- The weight of the enclosed liquid block of volume V is simply W =ρgV, and it acts downward through the centroid of this volume.
- Noting that the fluid block is in static equilibrium, the force balances in the horizontal and vertical directions give

Horizontal force component on curved surface: $F_H = F_x$

Vertical force component on curved surface: $F_V = F_y + W$

- where the summation F_y +W is a vector addition (i.e., add magnitudes if both act in the same direction and subtract if they act in opposite directions).
- The magnitude of the resultant hydrostatic force acting on the curved surface is $F_R = \sqrt{F_H^2 + F_V^2}$, , and the tangent of the angle it makes with the horizontal is $\tan \alpha = F_V/F_H$

Hydrostatic Forces on submerged curved surfaces

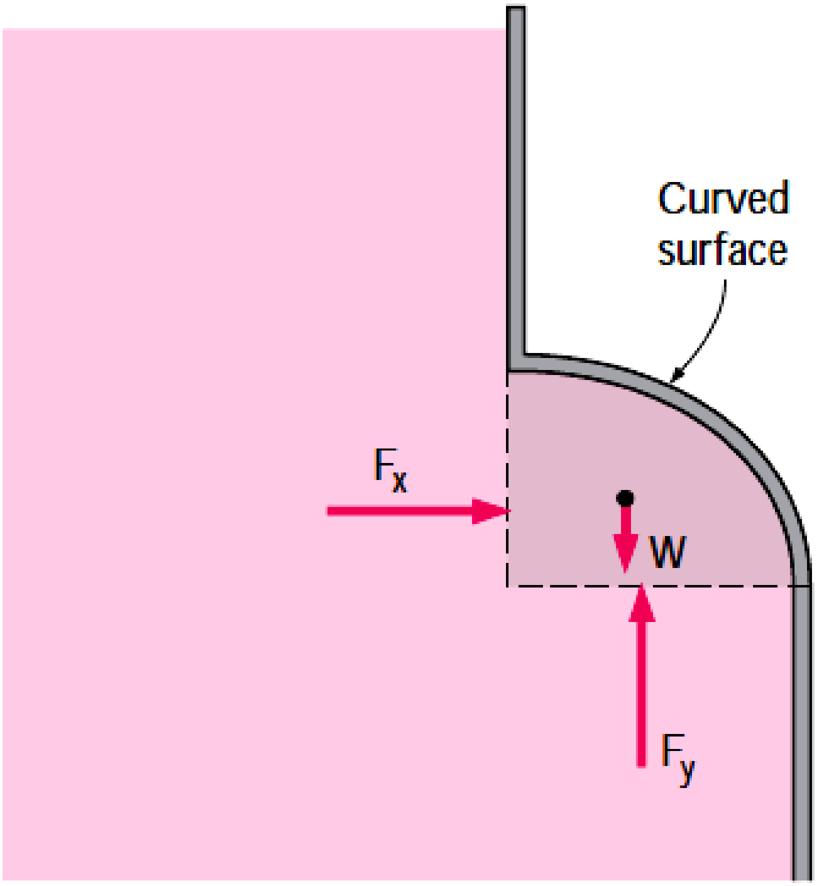
- Thus, we conclude that

1. The horizontal component of the hydrostatic force acting on a curved surface is equal (in both magnitude and the line of action) to the hydrostatic force acting on the vertical projection of the curved surface.
2. The vertical component of the hydrostatic force acting on a curved surface is equal to the hydrostatic force acting on the horizontal projection of the curved surface, plus (minus, if acting in the opposite direction) the weight of the fluid block.

- The location of the line of action of the resultant force (e.g., its distance from one of the end points of the curved surface) can be determined by taking a moment about an appropriate point.

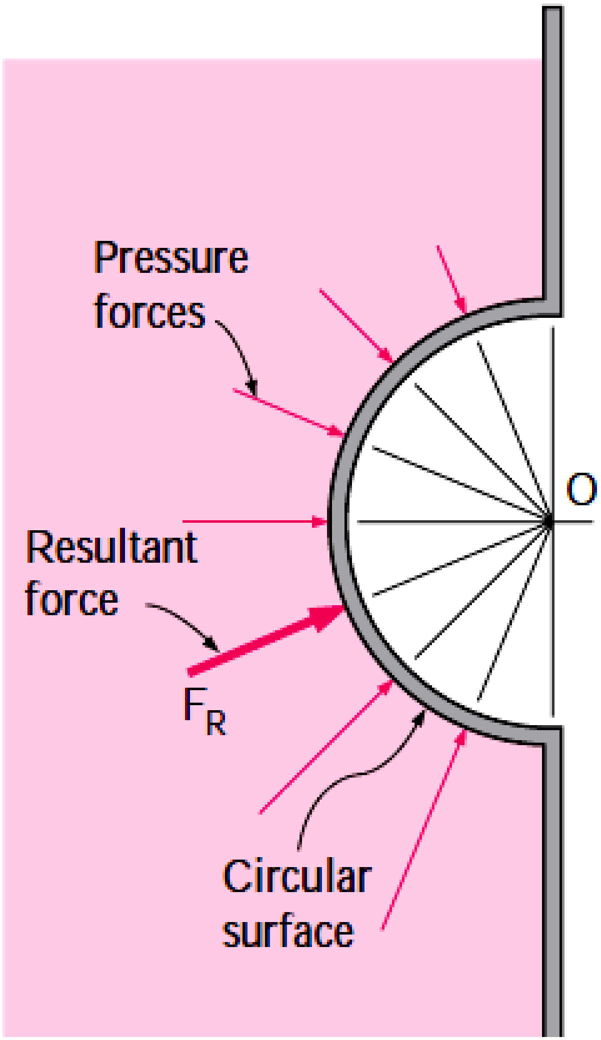
Hydrostatic Forces on submerged curved surfaces

- When a curved surface is above the liquid, the weight of the liquid and the vertical component of the hydrostatic force act in the opposite directions.



Hydrostatic Forces on submerged curved surfaces

- When the curved surface is a circular arc (full circle or any part of it), the resultant hydrostatic force acting on the surface always passes through the center of the circle.
- This is because the pressure forces are normal to the surface, and all lines normal to the surface of a circle pass through the center of the circle.
- Thus, the pressure forces form a concurrent force system at the center, which can be reduced to a single equivalent force at that point



Hydrostatic Forces on submerged curved surfaces

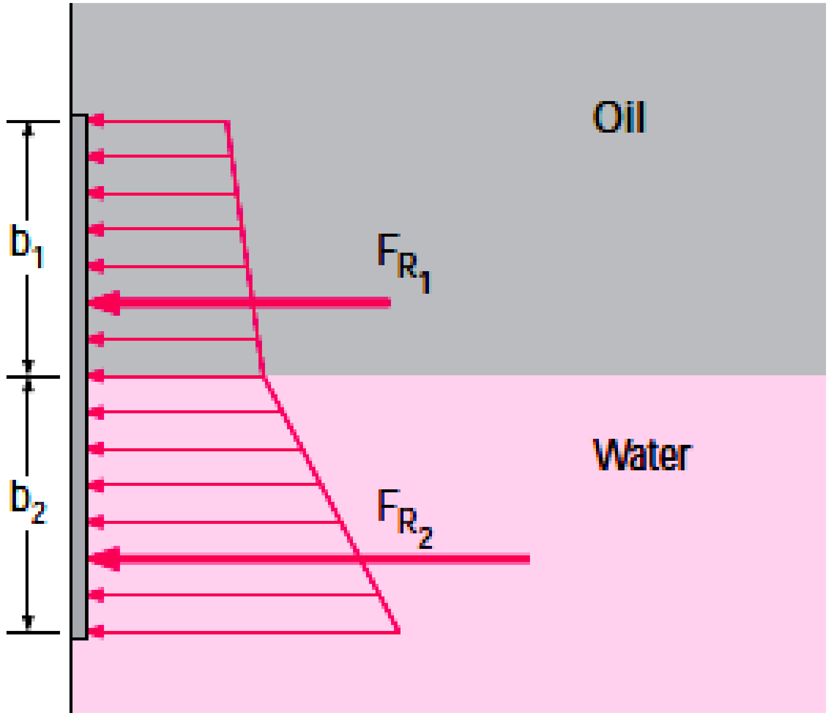
- Finally, hydrostatic forces acting on a plane or curved surface submerged in a **multilayered fluid** of different densities can be determined by considering different parts of surfaces in different fluids as different surfaces, finding the force on each part, and then adding them using vector addition.
- For a plane surface, it can be expressed as

Plane surface in a multilayered fluid: $$F_R = \sum F_{R,i} = \sum P_{C,i} A_i$$

- Where $P_{C,i} = P_0 + \rho_i g h_{C,i}$ *is the pressure at the centroid of the portion of* the surface in fluid *i and* A_i *is the area of the plate in that fluid.*
- *The line of* action of this equivalent force can be determined from the requirement that the moment of the equivalent force about any point is equal to the sum of the moments of the individual forces about the same point.

Hydrostatic Forces on submerged curved surfaces

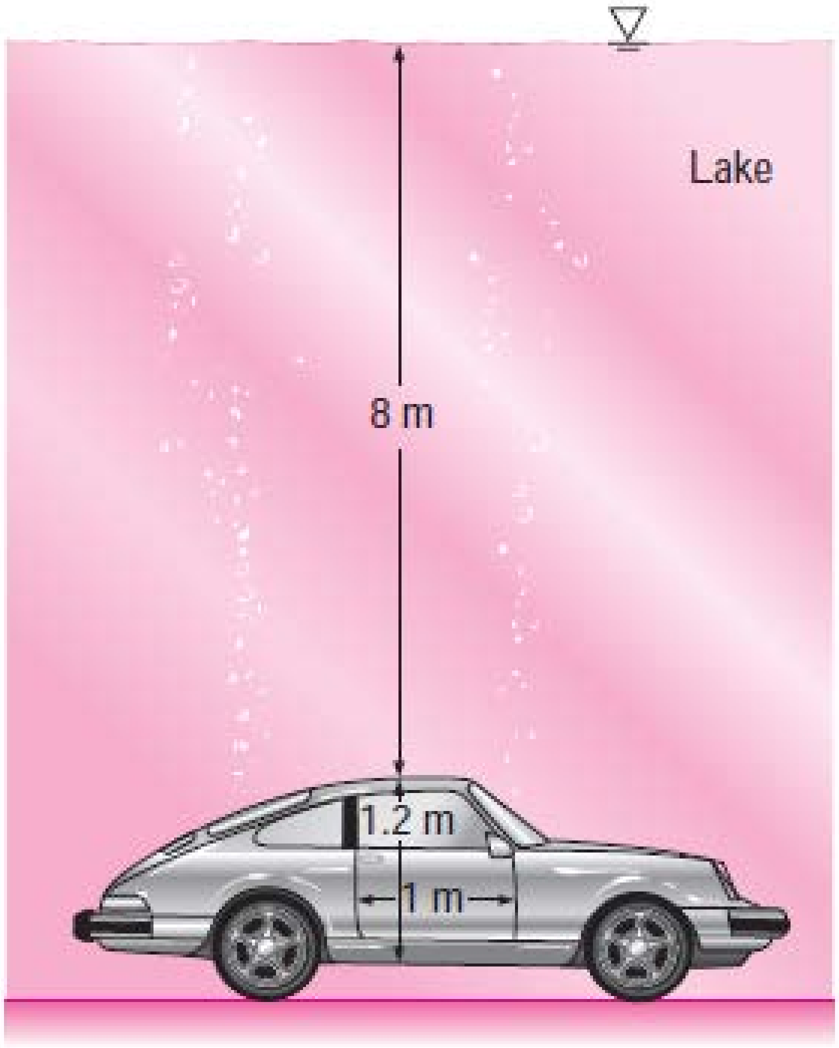
- The hydrostatic force on a surface submerged in a multilayered fluid can be determined by considering parts of the surface in different fluids as different surfaces.



Plane surface in a multilayered fluid: $F_R = \sum F_{R,i} = \sum P_{C,i} A_i$

Example 3. Hydrostatic Force Acting on the Door of a Submerged Car

- A heavy car plunges into a lake during an accident and lands at the bottom of the lake on its wheels. The door is 1.2 m high and 1 m wide, and the top edge of the door is 8 m below the free surface of the water.
- Determine the hydrostatic force on the door and the location of the pressure center.



Solution

Assumptions

1. The bottom surface of the lake is horizontal.
2. The passenger cabin is well-sealed so that no water leaks inside.
3. The door can be approximated as a vertical rectangular plate.
4. The pressure in the passenger cabin remains at atmospheric value since there is no water leaking in, and thus no compression of the air inside. Therefore, atmospheric pressure cancels out in the calculations since it acts on both sides of the door.
5. The weight of the car is larger than the buoyant force acting on it.

Solution

Analysis The average pressure on the door is the pressure value at the centroid (midpoint) of the door and is determined to be

$$P_{ave} = P_C = \rho g h_C = \rho g(s + b/2)$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(8 + 1.2/2\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= \mathbf{84.4\ kN/m^2}$$

Then the resultant hydrostatic force on the door becomes

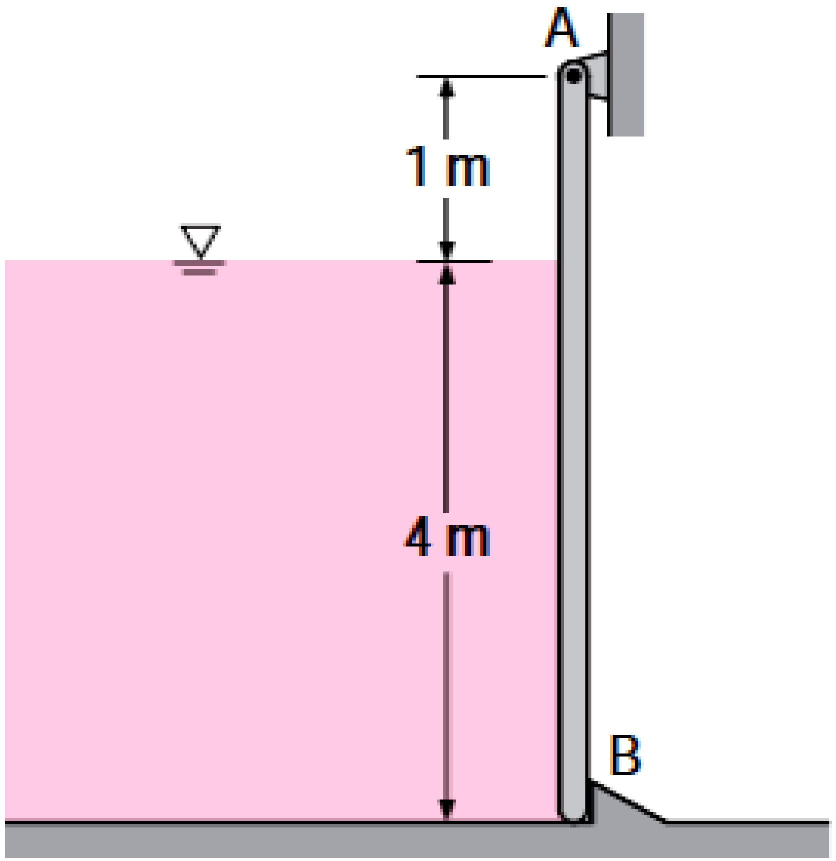
$$F_R = P_{ave}A = (84.4\ \text{kN/m}^2)(1\ \text{m} \times 1.2\ \text{m}) = \mathbf{101.3\ kN}$$

The pressure center is directly under the midpoint of the door, and its distance from the surface of the lake is determined from Eq. 3–24 by setting $P_0 = 0$ to be

$$y_P = s + \frac{b}{2} + \frac{b^2}{12(s + b/2)} = 8 + \frac{1.2}{2} + \frac{1.2^2}{12(8 + 1.2/2)} = \mathbf{8.61\ m}$$

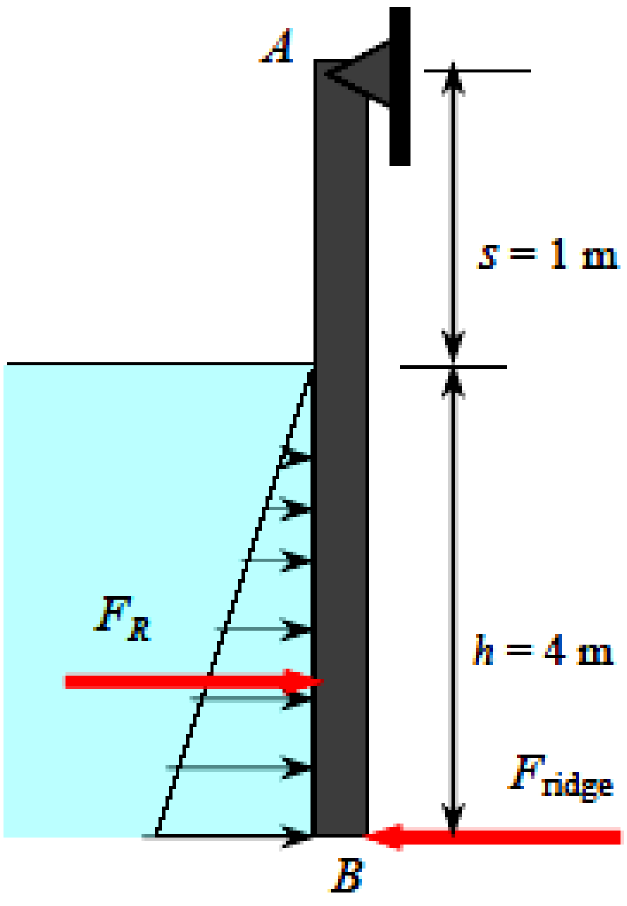
Example 4

- A 4-m-high, 5-m-wide rectangular plate blocks the end of a 4-m-deep freshwater channel, as shown in the Fig. The plate is hinged about a horizontal axis along its upper edge through a point A and is restrained from opening by a fixed ridge at point B.
- Determine the force exerted on the plate by the ridge.



Solution

- A rectangular plate hinged about a horizontal axis along its upper edge blocks a fresh water channel. The plate is restrained from opening by a fixed ridge at a point B. The force exerted to the plate by the ridge is to be determined.
- **Assumptions.** Atmospheric pressure acts on both sides of the plate, and thus it can be ignored in calculations for convenience.



Properties We take the density of water to be 1000 kg/m^3 throughout.

Analysis The average pressure on a surface is the pressure at the centroid (midpoint) of the surface, and is determined to be

$$P_{\text{avg}} = P_C = \rho g h_C = \rho g(h/2)$$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4/2 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg}\cdot\text{m/s}^2}\right) = 19.62 \text{ kN/m}^2$$

Then the resultant hydrostatic force on each wall becomes

$$F_R = P_{\text{avg}} A = (19.62 \text{ kN/m}^2)(4 \text{ m} \times 5 \text{ m}) = 392 \text{ kN}$$

The line of action of the force passes through the pressure center, which is $2h/3$ from the free surface,

$$y_P = \frac{2h}{3} = \frac{2 \times (4 \text{ m})}{3} = 2.667 \text{ m}$$

Taking the moment about point A and setting it equal to zero gives

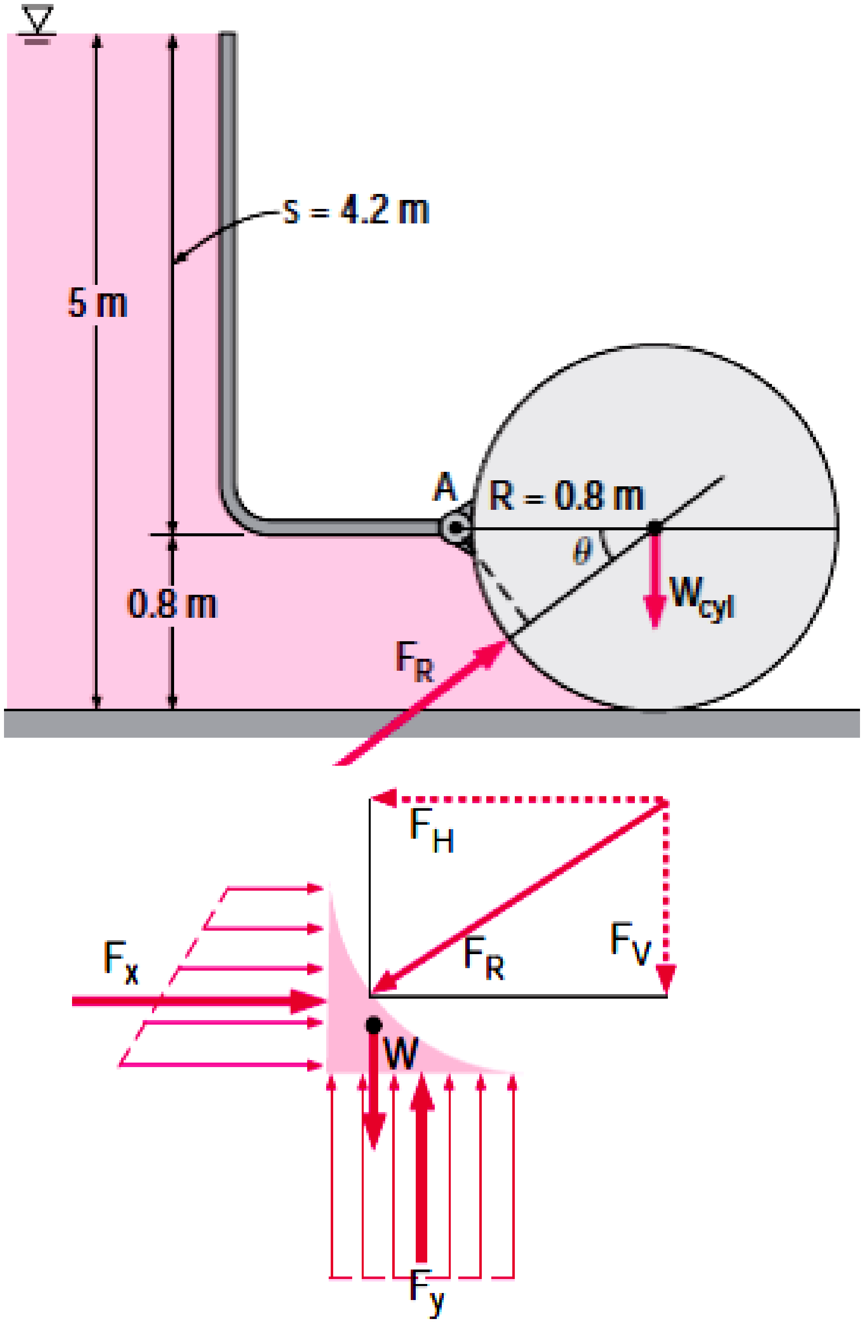
$$\sum M_A = 0 \quad \rightarrow \quad F_R(s + y_P) = F_{\text{ridge}}\overline{AB}$$

Solving for F_{ridge} and substituting, the reaction force is determined to be

$$F_{\text{ridge}} = \frac{s + y_P}{\overline{AB}} F_R = \frac{(1 + 2.667) \text{ m}}{5 \text{ m}}(392 \text{ kN}) = \mathbf{288 \text{ kN}}$$

Example 5

- A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate. When the water level reaches 5 m, the gate opens by turning about the hinge at point A.
- Determine the hydrostatic force acting on the cylinder and its line of action when the gate opens and



SOLUTION The height of a water reservoir is controlled by a cylindrical gate hinged to the reservoir. The hydrostatic force on the cylinder and the weight of the cylinder per m length are to be determined.

Assumptions **1** Friction at the hinge is negligible. **2** Atmospheric pressure acts on both sides of the gate, and thus it cancels out.

Properties We take the density of water to be 1000 kg/m^3 throughout.

Analysis (*a*) We consider the free-body diagram of the liquid block enclosed by the circular surface of the cylinder and its vertical and horizontal projections. The hydrostatic forces acting on the vertical and horizontal plane surfaces as well as the weight of the liquid block are determined as

Horizontal force on vertical surface:

$$F_H = F_x = P_{ave} A = \rho g h_C A = \rho g (s + R/2) A$$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(4.2 + 0.8/2 \text{ m})(0.8 \text{ m} \times 1 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$= \mathbf{36.1\ kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{ave} A = \rho g h_C A = \rho g h_{bottom} A$$

$$= (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(5 \text{ m})(0.8 \text{ m} \times 1 \text{ m})\left(\frac{1 \text{ kN}}{1000 \text{ kg} \cdot \text{m/s}^2}\right)$$

$$= 39.2 \text{ kN}$$

Weight of fluid block per m length (downward):

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\ \text{m})$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 1.3\ \text{kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

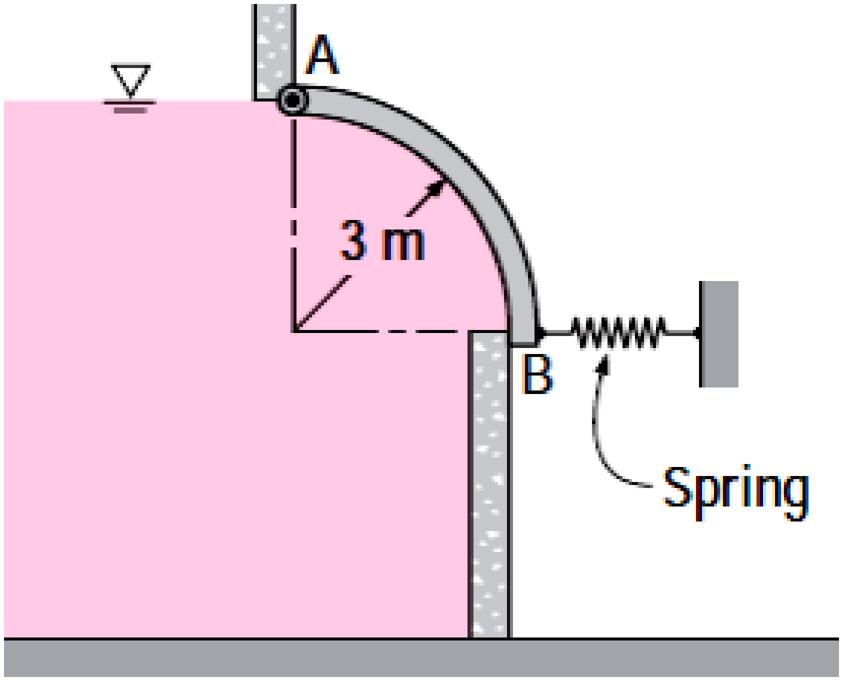
Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = \mathbf{52.3\ kN}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4^\circ$$

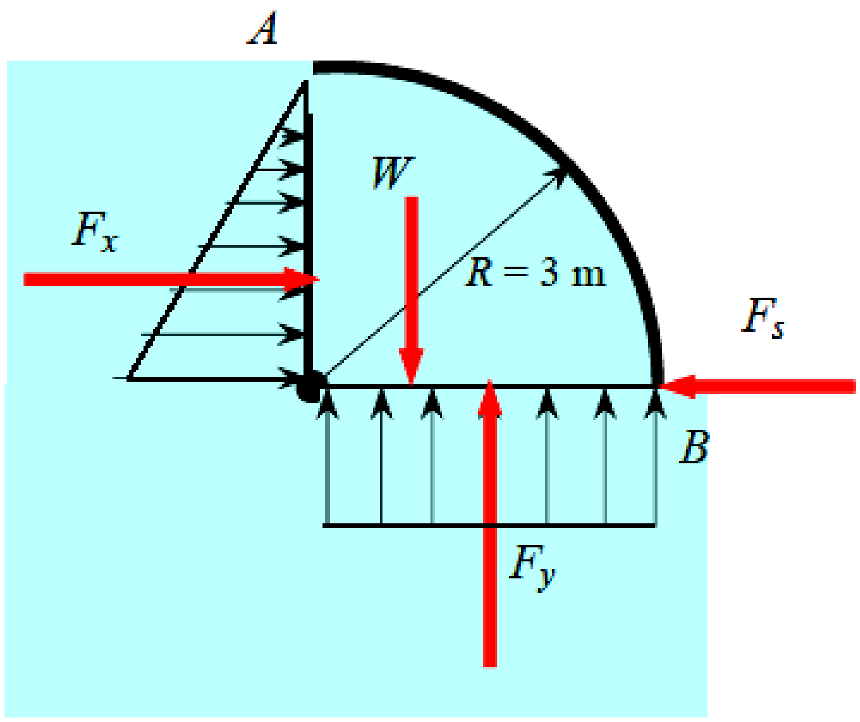
Example 6

- A 4-m-long quarter-circular gate of radius 3 m and of negligible weight is hinged about its upper edge A, as shown in the Fig. The gate controls the flow of water over the ledge at B, where the gate is pressed by a spring. Determine the minimum spring force required to keep the gate closed when the water level rises to A at the upper edge of the gate.



Solution

- **Assumptions** 1. The hinge is frictionless. 2. Atmospheric pressure acts on both sides of the gate, and thus it can be ignored in calculations for convenience. 3. The weight of the gate is negligible.
- We consider the free body diagram of the liquid block enclosed by the circular surface of the gate and its vertical and horizontal projections.
- The hydrostatic forces acting on the vertical and horizontal plane surfaces as well as the weight of the liquid block are determined as follows
- We take the density of water to be 1000 kg/m3 throughout.



Horizontal force on vertical surface:

$$F_H = F_x = P_{ave} A = \rho g h_C A = \rho g (R/2) A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(3/2\ \text{m})(4\ \text{m} \times 3\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 176.6\ \text{kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{\text{avg}} A = \rho g h_C A = \rho g h_{\text{bottom}} A$$

$$= \left(1000\ \text{kg/m}^3\right)\left(9.81\ \text{m/s}^2\right)\left(3\ \text{m}\right)\left(4\ \text{m} \times 3\ \text{m}\right)\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right) = 353.2\ \text{kN}$$

The weight of fluid block per 4-m length (downwards):

$$W = \rho g V = \rho g\left[w \times \pi R^2/4\right]$$

$$= \left(1000\ \text{kg/m}^3\right)\left(9.81\ \text{m/s}^2\right)\left[\left(4\ \text{m}\right)\pi(3\ \text{m})^2/4\right]\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right) = 277.4\ \text{kN}$$

Therefore, the net upward vertical force is

$$F_V = F_y - W = 353.2 - 277.4 = 75.8\ \text{kN}$$

- Then the magnitude and direction of the hydrostatic force acting on the surface of the 4-m long quarter-circular section of the gate become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{(176.6\,\text{kN})^2 + (75.8\,\text{kN})^2} = 192.2\ \text{kN}$$

$$\tan\theta = \frac{F_V}{F_H} = \frac{75.8\,\text{kN}}{176.6\,\text{kN}} = 0.429 \quad \rightarrow \quad \theta = 23.2°$$

- The minimum spring force needed is determined by taking a moment about the point A where the hinge is, and setting it equal to zero,

$$\sum M_A = 0 \quad \rightarrow \quad F_R R \sin(90-\theta) - F_{\text{spring}} R = 0$$

Solving for F_{spring} and substituting, the spring force is determined to be

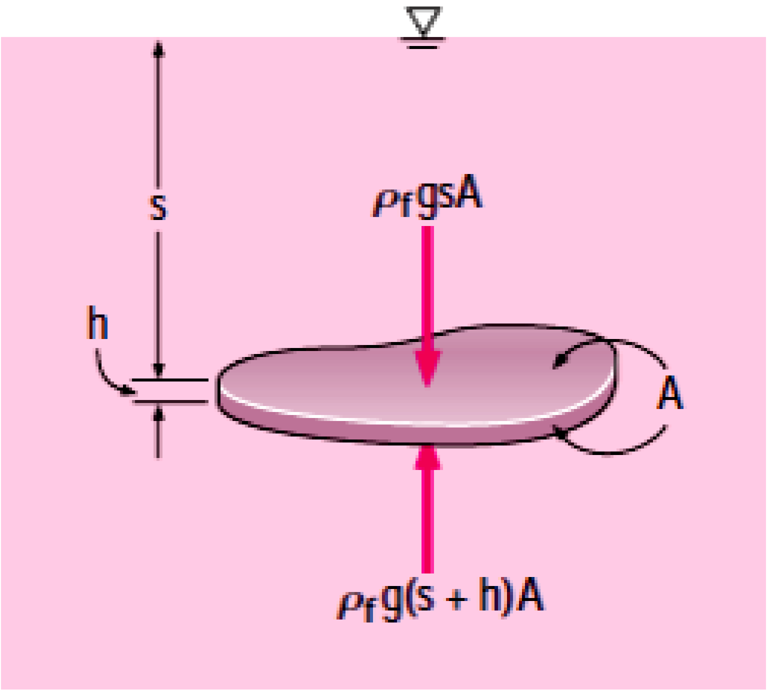
$$F_{\text{spring}} = F_R \sin(90 - \theta) = (192.2\,\text{kN})\sin(90° - 23.2°) = \mathbf{177\ kN}$$

Buoyancy, Floatation and stability

- It is a common experience that an object feels lighter and weighs less in a liquid than it does in air. This can be demonstrated easily by weighing a heavy object in water by a waterproof spring scale. Also, objects made of wood or other light materials float on water.
- These and other observations suggest that a fluid exerts an upward force on a body immersed in it. This force that tends to lift the body is called the **buoyant force** and is denoted by F_B.
- The buoyant force is caused by the increase of pressure in a fluid with depth.

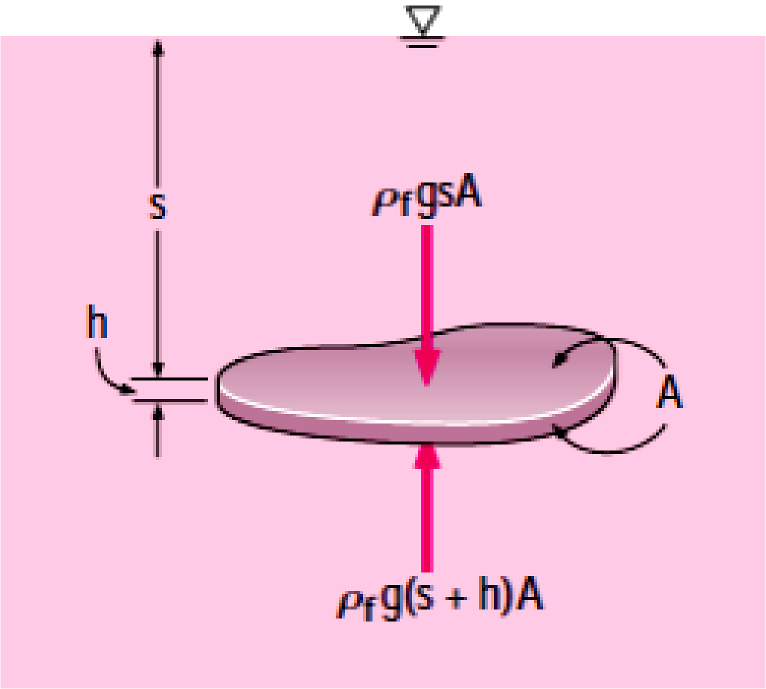
Buoyancy, Floatation and stability

- Consider, for example, a flat plate of thickness h submerged in a liquid of density ρ_f parallel to the free surface, as shown in the Fig.
- The area of the top (and also bottom) surface of the plate is A, and its distance to the free surface is s.
- The pressures at the top and bottom surfaces of the plate are $\boldsymbol{\rho_f\, gs}$ and $\boldsymbol{\rho_f\, g(s + h)}$, respectively.



Buoyancy, Floatation and stability

- Then the hydrostatic force $\mathbf{F_{top} = \rho_f\, gsA}$ acts downward on the top surface, and the larger force $\mathbf{F_{bottom} = \rho_f\, g(s + h)A}$ acts upward on the bottom surface of the plate.
- The difference between these two forces is a net upward force, which is the buoyant force,



$$F_B = F_{bottom} - F_{top} = \rho_f g(s + h)A - \rho_f gsA = \rho_f ghA = \rho_f gV$$

Buoyancy, Floatation and stability

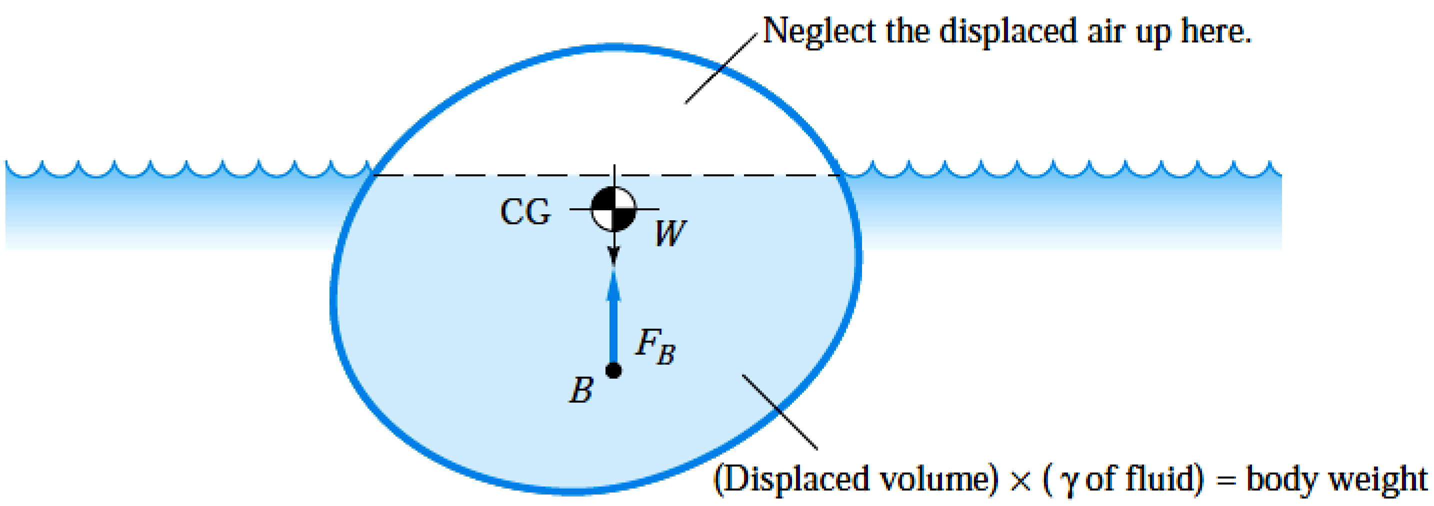
- where $\mathcal{V} = hA$ is the volume of the plate. But the relation $\boldsymbol{\rho_f\, g\mathcal{V}}$ is simply the weight of the liquid whose volume is equal to the volume of the plate.
- Thus, we conclude that the buoyant force acting on the plate is equal to the weight of the liquid displaced by the plate.
- Note that the buoyant force is independent of the distance of the body from the free surface. It is also independent of the density of the solid body.
- This is known as **Archimedes' principle,** after the Greek mathematician Archimedes (287–212 BC), and is expressed as

> The buoyant force acting on a body immersed in a fluid is equal to the weight of the fluid displaced by the body, and it acts upward through the centroid of the displaced volume.

Buoyancy, Floatation and stability

- Floating bodies are a special case; only a portion of the body is submerged, with the remainder poking up out of the free surface.

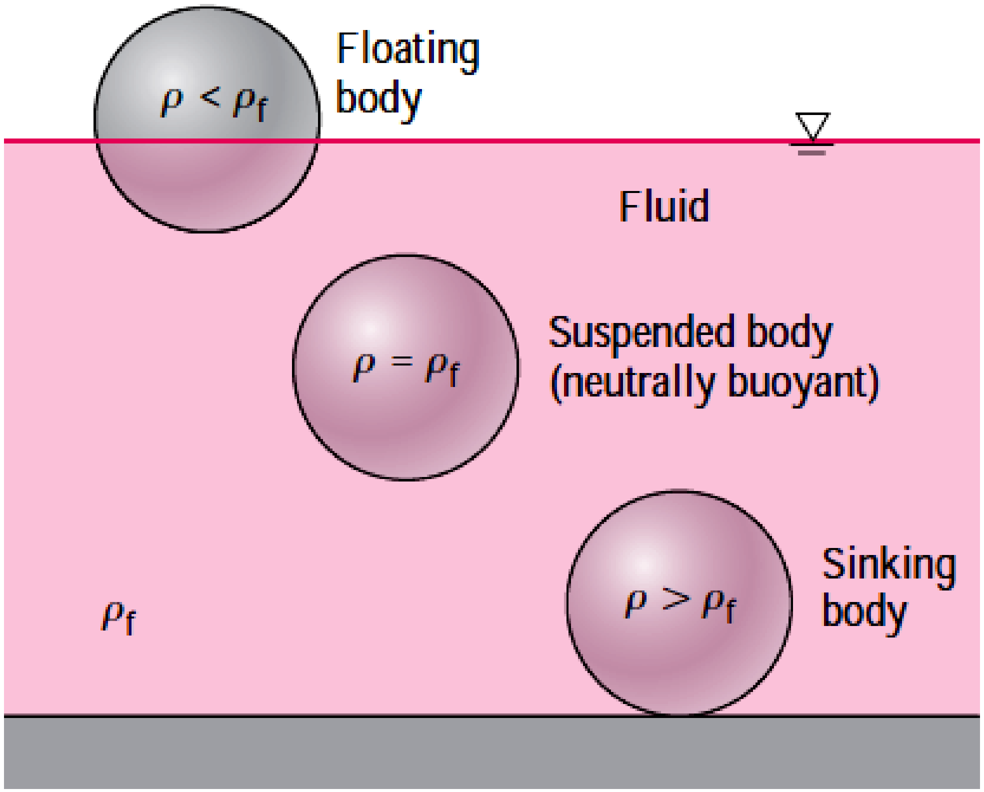
$$F_B = (\gamma)(\text{displaced volume}) = \text{floating-body weight}$$



Buoyancy, Floatation and stability

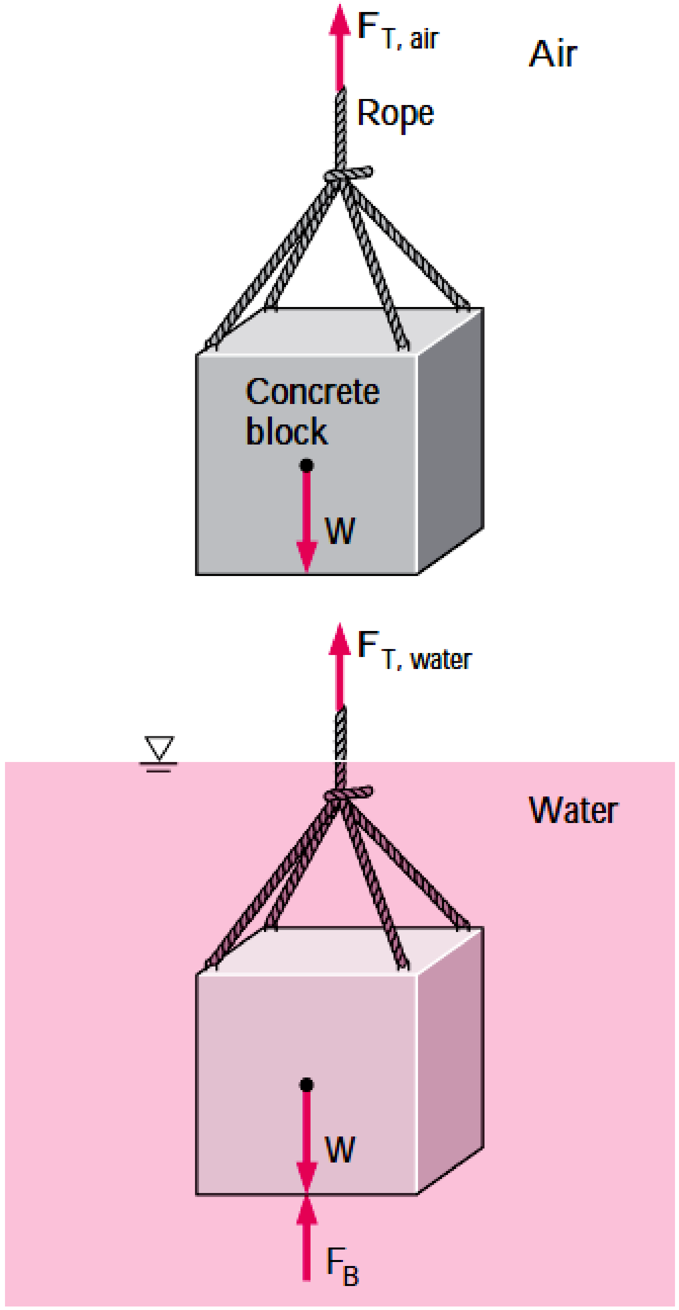
- A body immersed in a fluid

1) Remains at rest at any point in the fluid when its density is equal to the density of the fluid,
2) Sinks to the bottom when its density is greater than the density of the fluid, and
3) Rises to the surface of the fluid and floats when the density of the body is less than the density of the fluid



Example 1

- A crane is used to lower weights into the sea (density =1025 kg/m3) for an underwater construction project.
- Determine the tension in the rope of the crane due to a rectangular 0.4-m x 0.4-m x 3-m concrete block (density = 2300 kg/m3) when it is (a) suspended in the air and (b) completely immersed in water.



Analysis (*a*) Consider the free-body diagram of the concrete block. The forces acting on the concrete block in air are its weight and the upward pull action (tension) by the rope. These two forces must balance each other, and thus the tension in the rope must be equal to the weight of the block:

$$V = (0.4\ \text{m})(0.4\ \text{m})(3\ \text{m}) = 0.48\ \text{m}^3$$

$$F_{T,\,\text{air}} = W = \rho_{\text{concrete}}\, gV$$

$$= (2300\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.48\ \text{m}^3)\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{10.8\ kN}$$

(*b*) When the block is immersed in water, there is the additional force of buoyancy acting upward. The force balance in this case gives

$$F_B = \rho_f\, gV = (1025\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.48\ \text{m}^3)\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right) = \mathbf{4.8\ kN}$$

$$F_{T,\,\text{water}} = W - F_B = 10.8 - 4.8 = \mathbf{6.0\ kN}$$

Discussion Note that the weight of the concrete block, and thus the tension of the rope, decreases by $(10.8 - 6.0)/10.8 = 55$ percent in water.

Example 2

- A 170-kg granite rock (ρ = 2700 kg/m3) is dropped into a lake. A man dives in and tries to lift the rock. Determine how much force the man needs to apply to lift it from the bottom of the lake. Do you think he can do it?

Analysis The weight and volume of the rock are

$$W = mg = (170\,\text{kg})(9.81\,\text{m/s}^2)\left(\frac{1\,\text{N}}{1\,\text{kg}\cdot\text{m/s}^2}\right) = 1668\,\text{N}$$

$$V = \frac{m}{\rho} = \frac{170\,\text{kg}}{2700\,\text{kg/m}^3} = 0.06296\,\text{m}^3$$

The buoyancy force acting on the rock is

$$F_B = \rho_{\text{water}} g V = \left(1000\,\text{kg/m}^3\right)\left(9.81\,\text{m/s}^2\right)\left(0.06296\,\text{m}^3\right)\left(\frac{1\,\text{N}}{1\,\text{kg}\cdot\text{m/s}^2}\right) = 618\,\text{N}$$

The weight of a body submerged in water is equal to the weigh of the body in air minus the buoyancy force,

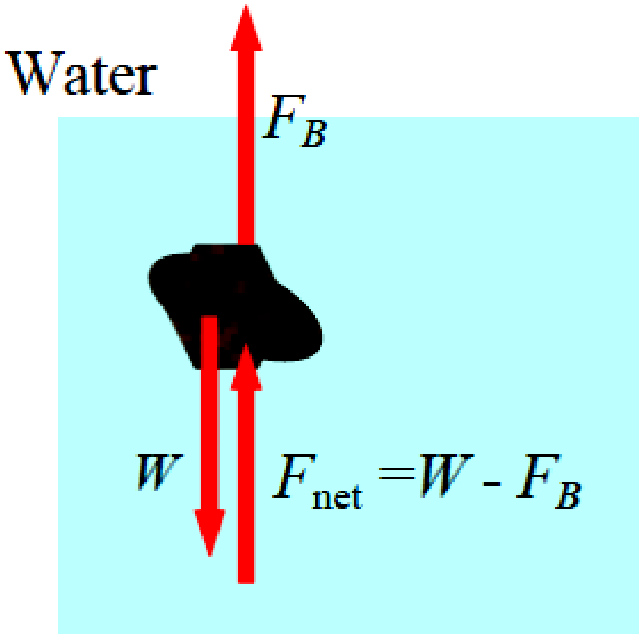
$$W_{\text{in water}} = W_{\text{in air}} - F_B = 1668 - 618 = \mathbf{1050\ N}$$

Example 2

- This force corresponds to a mass of

$$m = \frac{W_{\text{in water}}}{g} = \frac{1050\ \text{N}}{9.81\ \text{m/s}^2}\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right) = 107\ \text{kg}$$

- Therefore, a person who can lift 107 kg on earth can lift this rock in water.

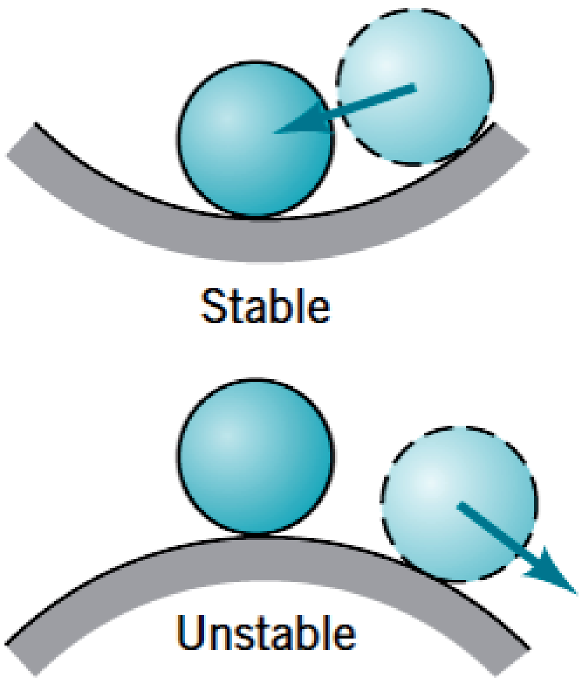


Stability of Immersed and Floating Bodies

- An important application of the buoyancy concept is the assessment of the stability of immersed and floating bodies with no external attachments.
- This topic is of great importance in the design of ships and submarines

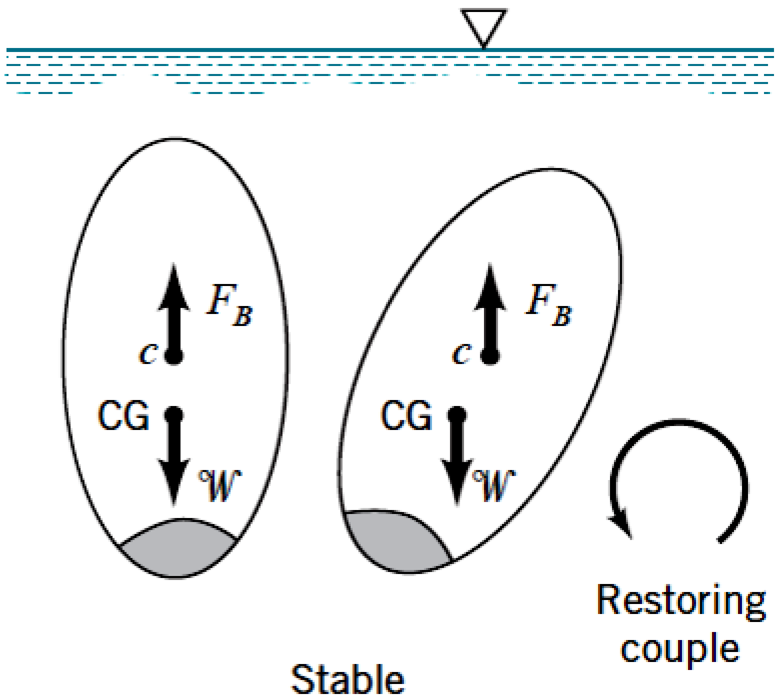
Stability of Immersed and Floating Bodies

- A body is said to be in a stable equilibrium position if, when displaced, it returns to its equilibrium position.
- Conversely, it is in an unstable equilibrium position if, when displaced (even slightly), it moves to a new equilibrium position.
- Stability considerations are particularly important for submerged or floating bodies since the centers of buoyancy and gravity do not necessarily coincide.
- A small rotation can result in either a restoring or overturning couple.



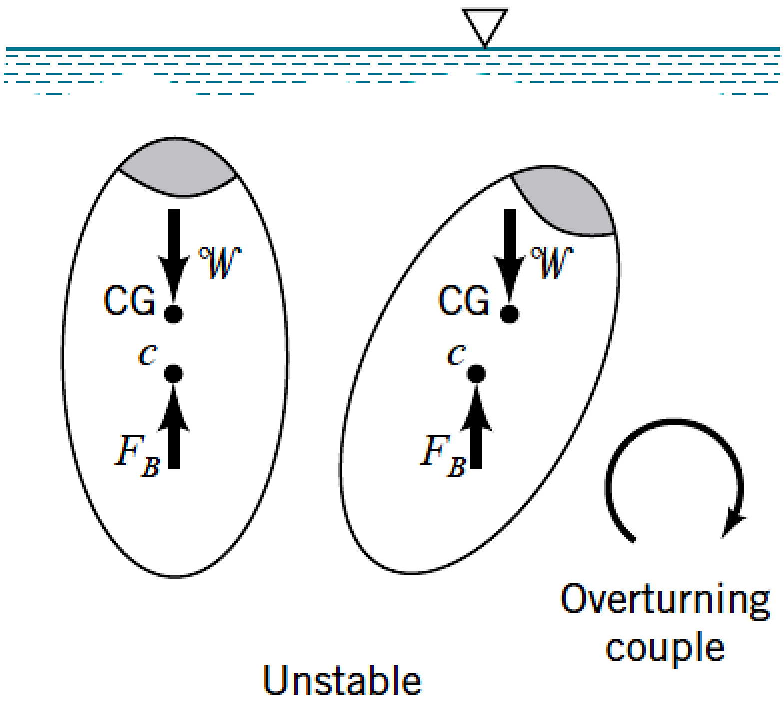
Stability of Immersed and Floating Bodies

- For example, for the completely submerged body shown in the Fig. , which has **a center of gravity below the center of buoyancy**, a rotation from its equilibrium position will create a **restoring couple** formed by the weight , and the buoyant force, which causes the body to **rotate back to its original position.**
- Thus, for this configuration the body is stable. It is to be noted that as long as the center of gravity falls below the center of buoyancy, this will always be true; that is, the body is in a stable equilibrium position with respect to small rotations.



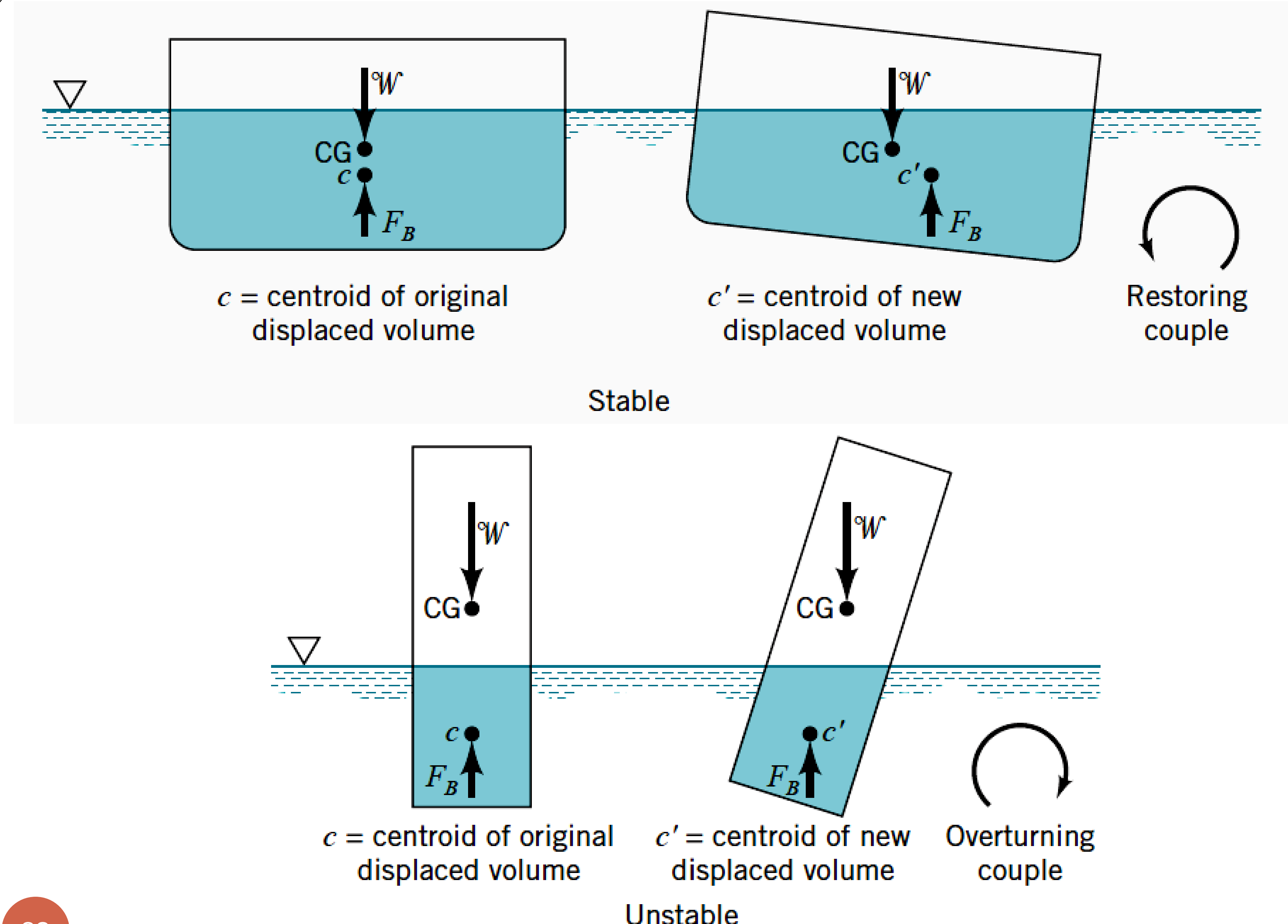
Stability of Immersed and Floating Bodies

- If the **center of gravity of the completely submerged body is above the center of buoyancy**, the resulting couple formed by the weight and the buoyant force will cause the body to overturn and move to a new equilibrium position.
- Thus, a completely submerged body with its center of gravity above its center of buoyancy is in an unstable equilibrium position.



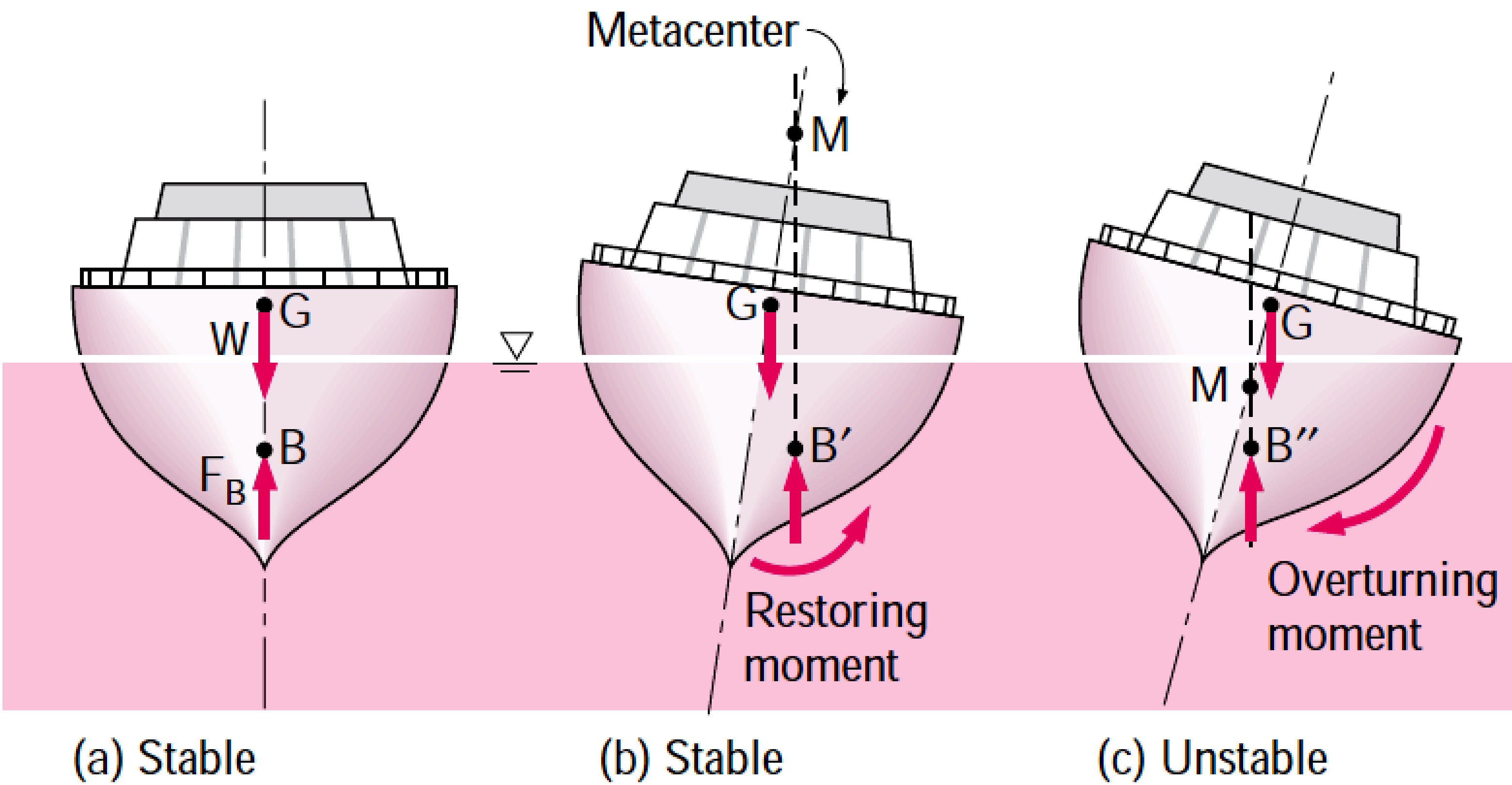
Stability of Immersed and Floating Bodies

- For floating bodies the stability problem is more complicated, since as the body rotates the location of the center of buoyancy (which passes through the centroid of the displaced volume) may change.
- A floating body can be stable even though the center of gravity lies above the center of buoyancy. This is true since as the body rotates the buoyant force, shifts to pass through the centroid of the newly formed displaced volume and, as illustrated, combines with the weight, to form a couple which will cause the body to return to its original equilibrium position.
- However, for the relatively tall, slender body shown in Fig. below, a small rotational displacement can cause the buoyant force and the weight to form an overturning couple as illustrated.



$\mathcal{W}$
CG
c
F_B
c = centroid of original displaced volume
$\mathcal{W}$
CG
c'
F_B
c' = centroid of new displaced volume
Restoring couple
Stable
$\mathcal{W}$
CG
c
F_B
c = centroid of original displaced volume
$\mathcal{W}$
CG
c'
F_B
c' = centroid of new displaced volume
Overturning couple
Unstable

Stability of Immersed and Floating Bodies



- A floating body is stable if the body is bottom-heavy and thus the center of gravity G is below the centroid B of the body, or if the metacenter M is above point G. However, the body is unstable if point M is below point G.

Part II

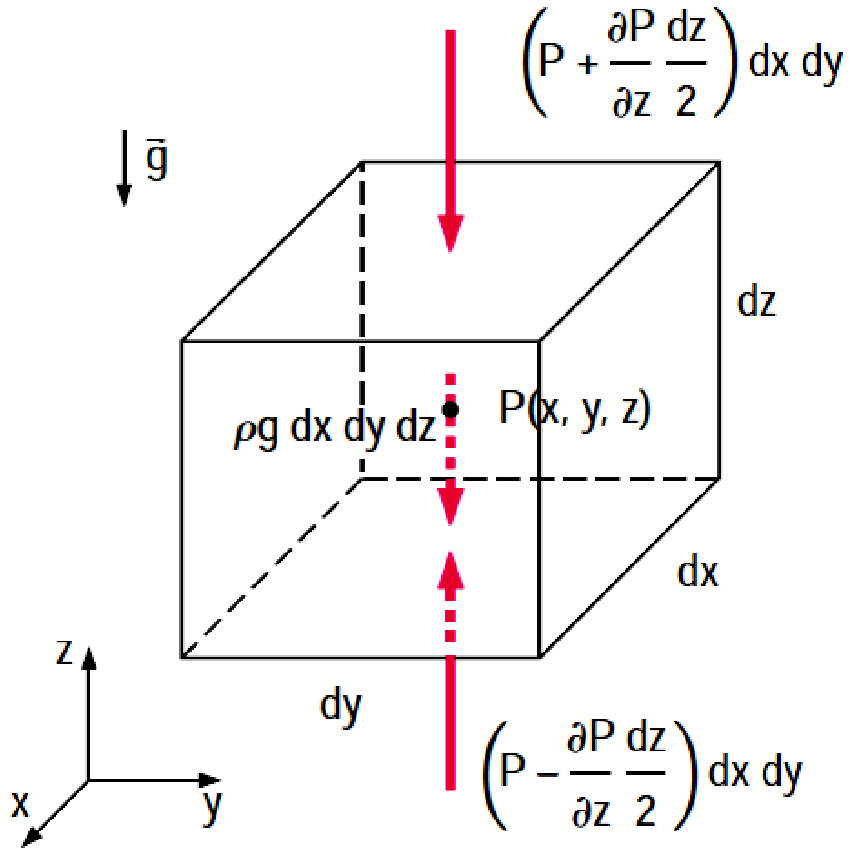
Fluids in Rigid-Body Motion

Fluids in Rigid-body Motion

- In this section we obtain relations for the variation of pressure in fluids moving like a solid body with or without acceleration in the absence of any shear stresses (i.e., no motion between fluid layers relative to each other).
- Many fluids such as milk and gasoline are transported in tankers. In an accelerating tanker, the fluid rushes to the back, and some initial splashing occurs. But then a new free surface (usually non-horizontal) is formed, each fluid particle assumes the same acceleration, and the entire fluid moves like a rigid body.
- No shear stresses develop within the fluid body since there is no deformation and thus no change in shape. Rigid-body motion of a fluid also occurs when the fluid is contained in a tank that rotates about an axis.

Fluids in Rigid-body Motion

- Consider a differential rectangular fluid element of side lengths *dx*, *dy*, and *dz* in the *x*-, *y*-, *and* z-directions, respectively, with the z-axis being upward in the vertical direction .
- Noting that the differential fluid element behaves like a rigid body, *Newton's second law of motion* for this element can be expressed as



$$\delta\vec{F} = \delta m \cdot \vec{a}$$

Where

$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$ is the mass of the fluid element

Fluids in Rigid-body Motion

- $\vec{a}$ is the acceleration, and $\delta\vec{F}$ is the net force acting on the element.
- Two types forces act on the fluid element
  i. *Body forces* such as gravity that act throughout the entire body of the element and are proportional to the volume of the body (and also electrical and magnetic forces, which will not be considered in this course), and
  ii. *Surface forces* such as the pressure forces that act on the surface of the element and are proportional to the surface area (shear stresses are also surface forces, but they do not apply in this case since the relative positions of fluid elements remain unchanged).
- Note that pressure represents the compressive force applied on the fluid element by the surrounding fluid and is always directed to the surface.

Fluids in Rigid-body Motion

- Taking the pressure at the center of the element to be *P*, the pressures at the top and bottom surfaces of the element can be expressed as $\dot{P} + (\partial P/\partial z)\, dz/2$ and $P - (\partial P/\partial z)\, dz/2$, respectively.
- Noting that the pressure force acting on a surface is equal to the average pressure multiplied by the surface area, the net surface force acting on the element in the *z*-direction is the difference between the pressure forces acting on the bottom and top faces,

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz$$

Similarly, the net surface forces in the x- and y-directions are

$$\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz \qquad \text{and} \qquad \delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz$$

Fluids in Rigid-body Motion

- Then the surface force (which is simply the pressure force) acting on the entire element can be expressed in vector form as

$$\delta\vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$
$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\,dy\,dz = -\vec{\nabla}P\,dx\,dy\,dz$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are the unit vectors in the x-, y-, and z-directions, respectively, and

$$\vec{\nabla}P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$ is the *pressure gradient.*

- Note that the $\vec{\nabla}$ or "del" is a vector operator that is used to express the gradients of a scalar function compactly in vector form.

Fluids in Rigid-body Motion

- The only body force acting on the fluid element is the weight of the element acting in the negative z-direction, and it is expressed as

$$\delta F_{B,z} = -g\delta m = -\rho g \, dx \, dy \, dz$$

or in vector form as

$$\delta \vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g \, dx \, dy \, dz\vec{k}$$

- Then the total force acting on the element becomes

$$\delta \vec{F} = \delta \vec{F}_S + \delta \vec{F}_B = -(\vec{\nabla} P + \rho g\vec{k}) \, dx \, dy \, dz$$

- Substituting into Newton's second law of motion

$$\delta \vec{F} = \delta m \cdot \vec{a} = \rho \, dx \, dy \, dz \cdot \vec{a}$$

and canceling *dx dy dz, the general* ***equation of motion*** for a fluid that acts as a rigid body (no shear stresses) is determined to be

Fluids in Rigid-body Motion

Rigid-body motion of fluids: $$\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}$$

- Resolving the vectors into their components, this relation can be expressed more explicitly as

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

or, in scalar form in the three orthogonal directions, as

Accelerating fluids: $$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = -\rho a_y, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

- where a_x, a_y, *and* a_z are accelerations in the *x*-, *y*-, *and* z-directions, respectively.

Fluids in Rigid-body Motion

Special Case 1: Fluids at Rest

- For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero, and the relations reduce to

Fluids at rest: $$\frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

- which confirm that, in fluids at rest, the pressure remains constant in any horizontal direction (*P is independent of x and y) and varies only in the vertical* direction as a result of gravity [and thus *P* = *P(z)]*.
- *These relations are* applicable for both compressible and incompressible fluids.

Fluids in Rigid-body Motion

Special Case 2: Free Fall of a Fluid Body

- A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero.
- Therefore, $a_x = a_y = 0$ *and* $a_z = -g$. Then *the equations of motion for* accelerating fluids reduce to

Free-falling fluids: $$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$$

- Therefore, in a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout.

Fluids in Rigid-body Motion

Special Case 2: Free Fall of a Fluid Body

- When the direction of motion is reversed and the fluid is forced to accelerate vertically with $a_z = +g$ by placing the fluid container in an elevator or a space vehicle propelled upward by a rocket engine, the pressure gradient in the z-direction is , $\partial P/\partial z = -2\rho g$.
- Therefore, the pressure difference across a fluid layer now doubles relative to the stationary fluid case

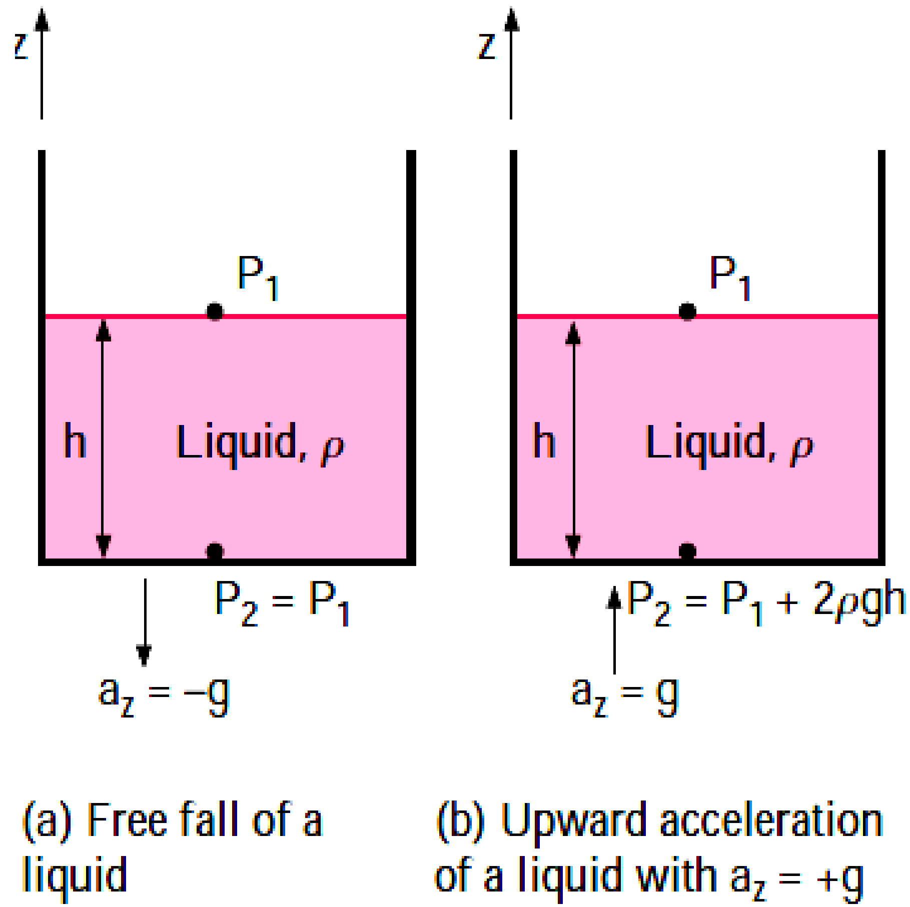
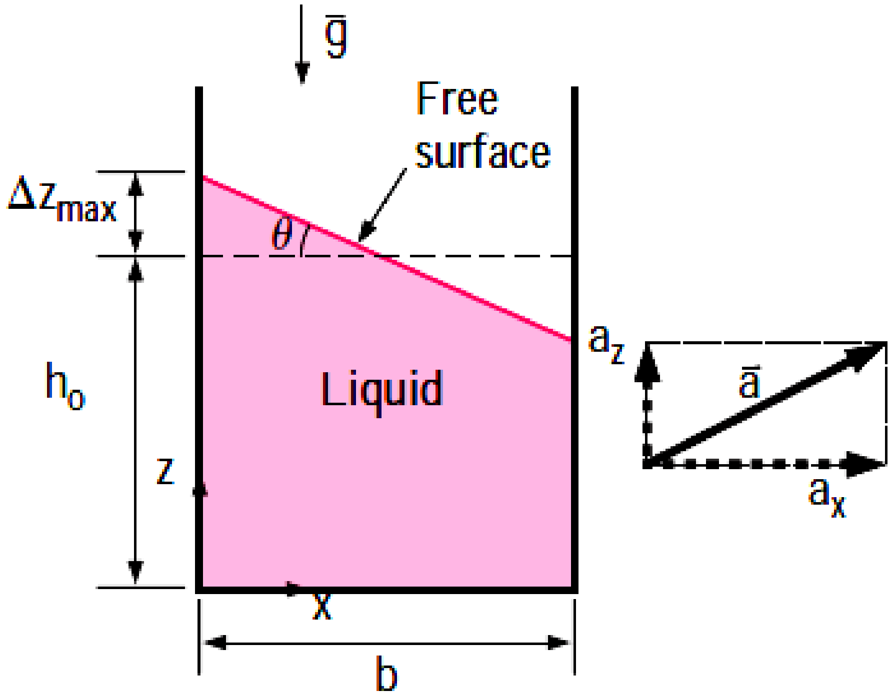


Fig. The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

Acceleration on a Straight Path

- Consider a container partially filled with a liquid. The container is moving on a straight path with a constant acceleration.
- We take the projection of the path of motion on the horizontal plane to be the *x-axis*, and the projection on the vertical plane to be the *z-axis*



- The *x- and z- components* of acceleration are a_x *and* a_z. There is no movement in the y-direction, and thus the acceleration in that direction is zero, $a_y = 0$.

Acceleration on a Straight Path

- Then the equations of motion for accelerating fluids reduce to $$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$
- Therefore, pressure is independent of *y*.
- *Then the total differential of P*

  $= P(x, z)$, which is $(\partial P/\partial x)\,dx + (\partial P/\partial z)\,dz$, becomes

$$dP = -\rho a_x\,dx - \rho(g + a_z)\,dz$$

- For ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integration to be

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

- Taking point 1 to be the origin (*x = 0, z = 0)* where the pressure is P_0 *and* point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as

Acceleration on a Straight Path

Pressure variation: $P = P_0 - \rho a_x x - \rho(g + a_z)z$

- The vertical rise (or drop) of the free surface at point 2 relative to point 1 can be determined by choosing both 1 and 2 on the free surface (so that $P_1 = P_2$), and solving for $z_2 - z_1$,

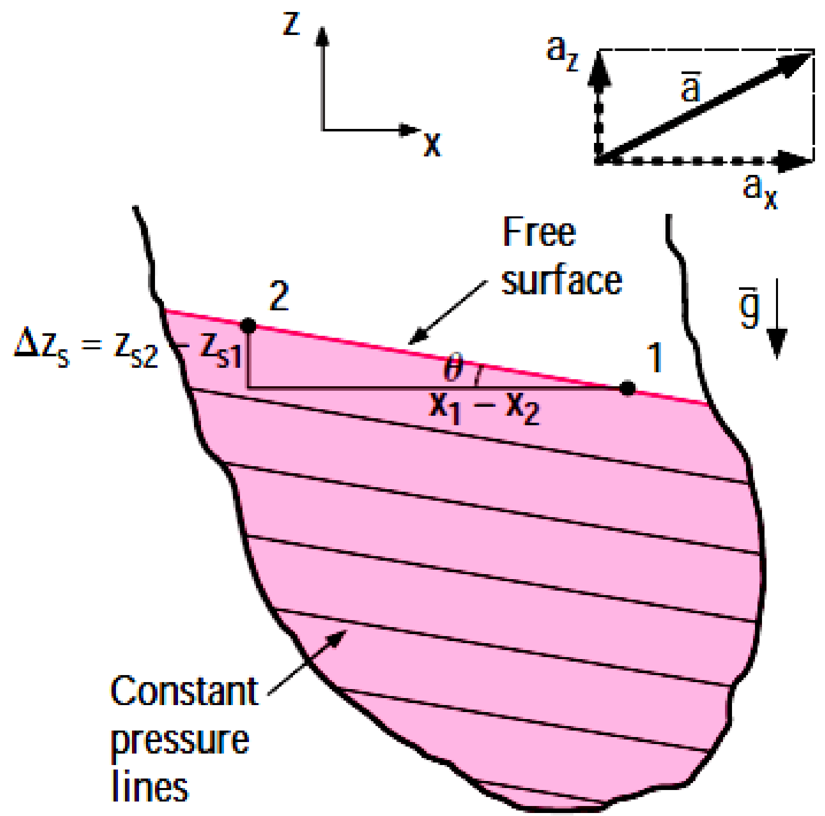
Vertical rise of surface: $\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$

- where z_s is the z-coordinate of the liquid's free surface
- The equation for surfaces of constant pressure, called **isobars,** is obtained from $dP = -\rho a_x\,dx - \rho(g + a_z)\,dz$ by setting $d_P = 0$ and replacing z by z_{isobar}, which is the z-coordinate (the vertical distance) of the surface as a function of *x*. It gives

Acceleration on a Straight Path

Surfaces of constant pressure: $$\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$$

- Thus we conclude that the isobars (including the free surface) in an incompressible fluid with constant acceleration in linear motion are parallel surfaces
- whose slope in the *xz-plane is*

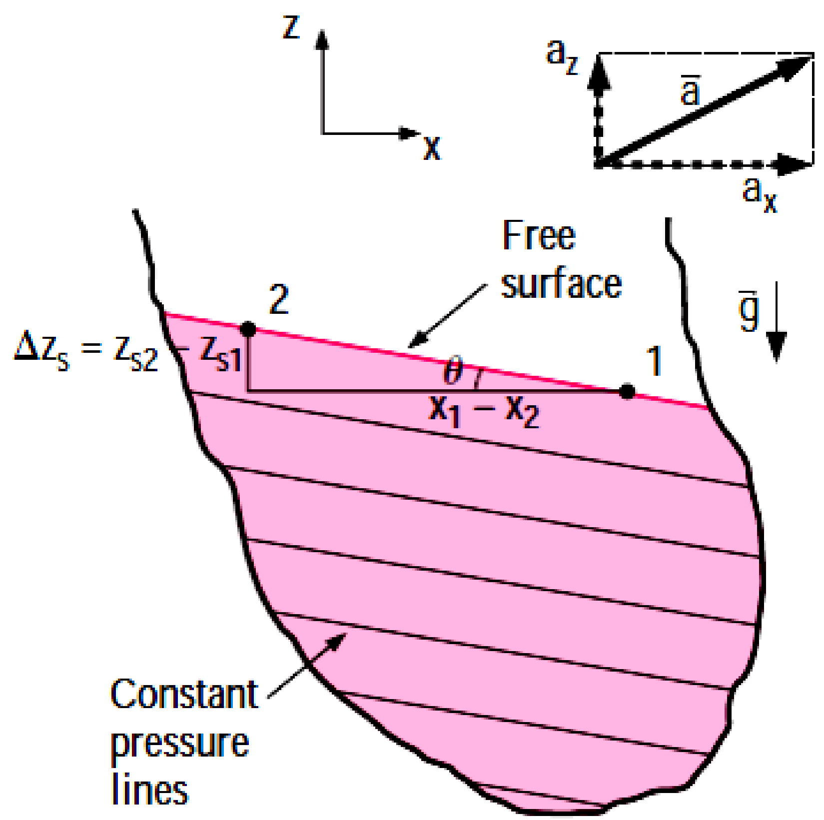


Slope of isobars:

$$\text{Slope} = \frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = -\tan\theta$$

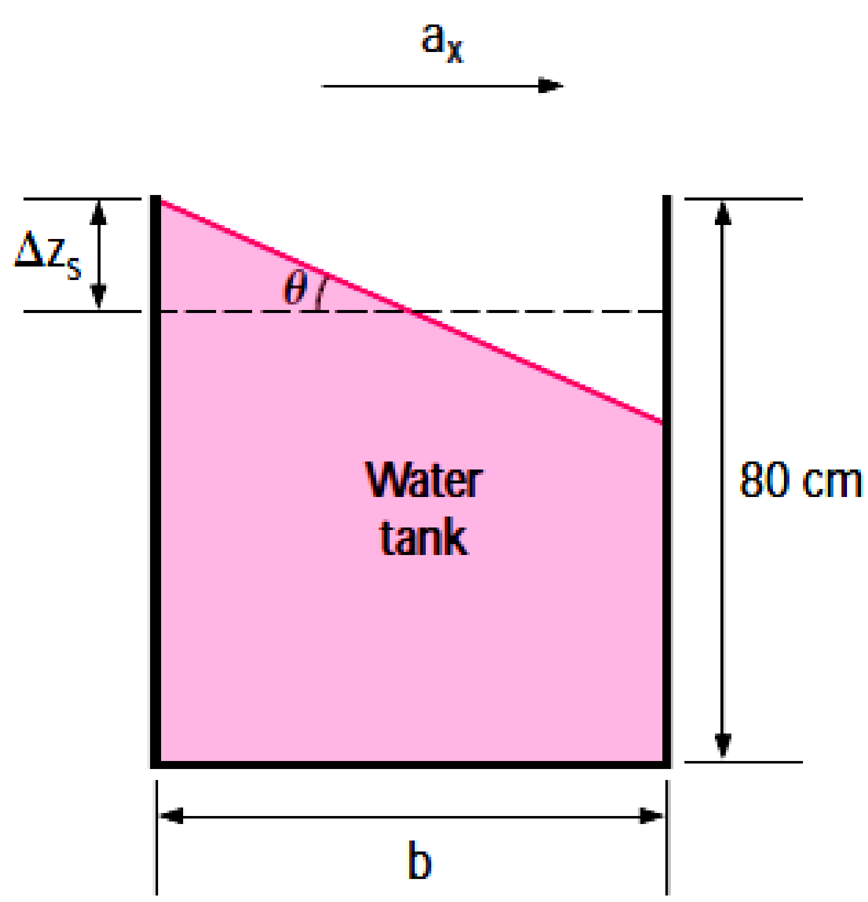
Acceleration on a Straight Path

- Obviously, the free surface of such a fluid is a plane surface, and it is inclined unless $a_x = 0$ (the acceleration is in the vertical direction only).
- Also, the conservation of mass together with the assumption of incompressibility ($\rho = constant$) requires that the volume of the fluid remain constant before and during acceleration.
- Therefore, the rise of fluid level on one side must be balanced by a drop of fluid level on the other side



Example. Overflow from a Water Tank During Acceleration

- An 80-cm-high fish tank of cross section 2 m x 0.6 m that is initially filled with water is to be transported on the back of a truck . The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?



Example. Overflow from a Water Tank During Acceleration

SOLUTION A fish tank is to be transported on a truck. The allowable water height to avoid spill of water during acceleration and the proper orientation are to be determined.

Assumptions **1** The road is horizontal during acceleration so that acceleration has no vertical component ($a_z = 0$). **2** Effects of splashing, braking, driving over bumps, and climbing hills are assumed to be secondary and are not considered. **3** The acceleration remains constant.

Analysis We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90 - 0)\text{ km/h}}{10\text{ s}}\left(\frac{1\text{ m/s}}{3.6\text{ km/h}}\right) = 2.5\text{ m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \qquad (\text{and thus } \theta = 14.3^\circ)$$

The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical midplane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the midplane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = \mathbf{25.5\text{ cm}}$$

Case 2: The short side is parallel to the direction of motion:

$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = \mathbf{7.6\text{ cm}}$$

Therefore, assuming tipping is not a problem, the tank should definitely be oriented such that its short side is parallel to the direction of motion. Emptying the tank such that its free surface level drops just 7.6 cm in this case will be adequate to avoid spilling during acceleration.

Discussion Note that the orientation of the tank is important in controlling the vertical rise. Also, the analysis is valid for any fluid with constant density, not just water, since we used no information that pertains to water in the solution.

End of Chapter 2

Next Lecture

Chapter 3: Integral Relations For A Control Volume