Chapter 2

Review of Network Parameters and Transmission Line Theory

- Review of Network Parameters
- Transmission Line Theory

Review of Network Parameters

❖ Linear networks can be completely characterized by parameters measured at the network ports without knowing the content of the networks

❖ Networks can have any number of ports.

❖ Analysis of a **2-port network** is sufficient to explain the theory **of network parameters**

Fig. 2.1 Two port network

❖ The ports can be characterized with many **parameters** (**Z, Y, ABCD, S**)

Z- Parameters

- Consider the two-port network shown in Fig2.1.
- Since the network is linear, the superposition principle can be applied
- Assuming that it contains no independent sources, voltage V_1 and V_2 can be expressed in terms of I_1 and I_2 as follows

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$
$$V_2 = Z_{21} I_1 + Z_{22} I_2$$

- Using the matrix representation, we can write

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$ or $[V]=[Z][I]$

Continued

❖ Generally , for **N** port network we can express the network parameters using the **matrix** representation

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & & & \vdots \\ \vdots & & & \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

where $$Z_{ij} = \left. \frac{V_i}{I_j} \right|_{I_{k \neq j} = 0}$$ (**Open circuit impedance**)

Admittance Parameters

- Consider again the two-port network shown in Figure 2.1
- Since the network is linear, the superposition principle can be applied
- Assuming that it contains no independent sources, current I_1 **and** I_2 can be expressed in terms of two voltages:

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

Using the matrix representation, we can write

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$ or $$[I] = [Y][V]$$

Continued

- **Generally , for N** port network **we can express the network parameters using** the **matrix** representation

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & & & \vdots \\ \vdots & & & \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

$$[I] = [Y][V]$$

Where $$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_{k \neq j} = 0}$$ (**Short circuit admittance**)

Transmission (ABCD) Matrix

- The transmission matrix describes the network **given in Fig 2.1 above** in terms of both voltage and current waves

$$V_1 = AV_2 - BI_2$$
$$I_1 = CV_2 - DI_2$$

or

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V2 \\ -I2 \end{bmatrix}$$

- The **coefficients** can be calculated as follows:

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

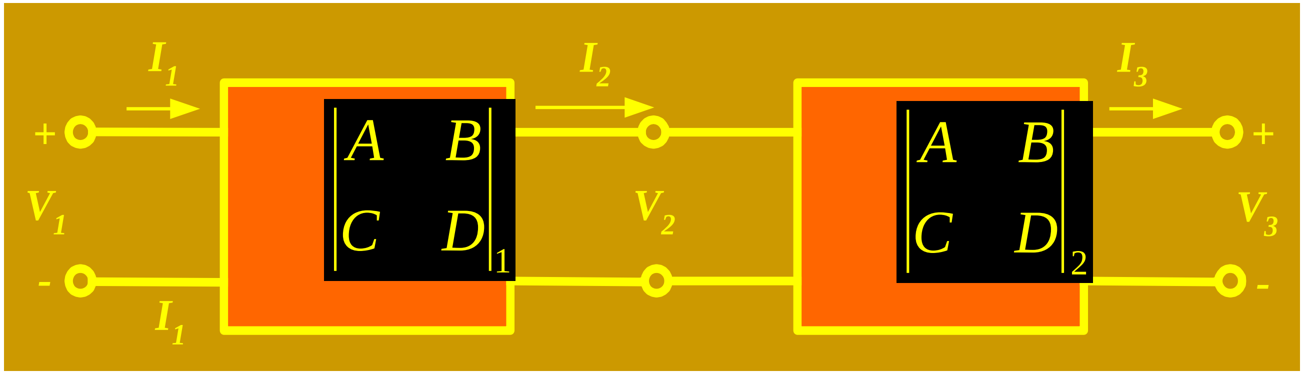
$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$

Continued

- Since the **ABCD** matrix represents the ports in terms of currents and voltages, it is well suited for cascading elements



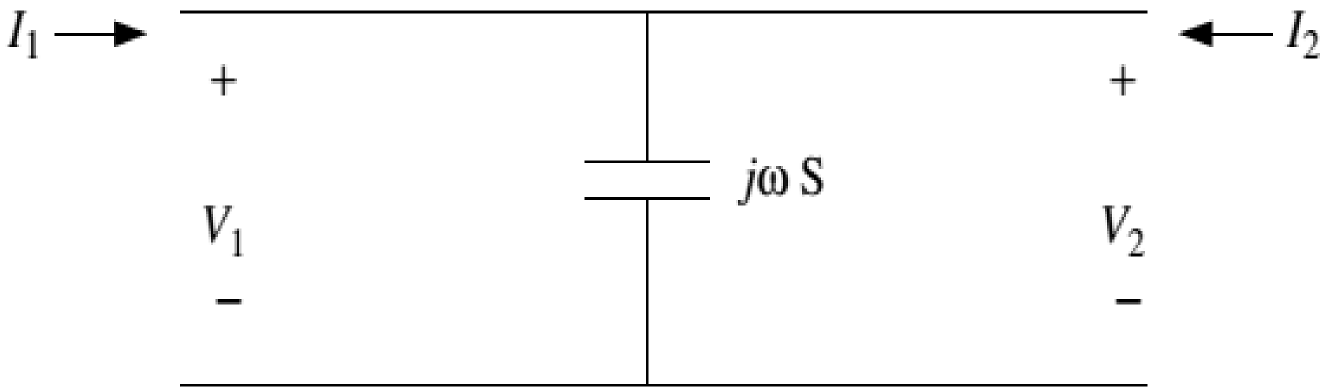
- The matrices can be mathematically cascaded by multiplication

$$\begin{vmatrix} V_1 \\ I_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}_1 \cdot \begin{vmatrix} V_2 \\ I_2 \end{vmatrix}$$

$$\begin{vmatrix} V_2 \\ I_2 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}_2 \cdot \begin{vmatrix} V_3 \\ I_3 \end{vmatrix}$$

$$\begin{vmatrix} V_1 \\ I_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}_1 \cdot \begin{vmatrix} A & B \\ C & D \end{vmatrix}_2 \cdot \begin{vmatrix} V_3 \\ I_3 \end{vmatrix}$$

Example. Determine the transmission parameters of the network shown **in fig** 2.3 **below**



Continued

SOLUTION With a source connected at port 1 while port 2 has a short circuit (so that V_2 is zero),

$$I_2 = -I_1 \quad \text{and} \quad V_1 = 0\text{V}$$

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = 0\,\Omega$$

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = 1$$

❖ Similarly, with a source connected at port 1 while port 2 is open (so that I_2 is zero), $V_2 = V_1$

$$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1$$

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = j\omega \qquad \text{S}$$

Hence, transmission matrix of this network is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ j\omega & 1 \end{bmatrix}$$

Hybrid Parameters

- Reconsider the two-port network of Figure 2.1
- Since the network is linear, **the superposition principle** can be applied
- Assuming that it contains no independent sources, voltage V_1 at port 1 can be expressed in terms of current I_1 at port 1 and voltage V_2 at port 2
- Similarly, we can write I_2 in terms of I_1 and V_2

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Continued

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

- **h**$_{11}$ = input impedance
- **h**$_{21}$ = forward current gain
- **h**$_{12}$ = reverse voltage gain
- **h**$_{22}$ = reverse output admittance circuit
- In transistor circuit analysis, these are generally denoted by hi ,hf ,hr ,and h0, respectively

Transmission Line Theory

- **Transmission line** is a distributed-parameter network, where **voltages** and **currents** can **vary** in **magnitude** and **phase** over the length of the line

Lumped Element Model for a Transmission Line

- **Transmission lines** usually consist of **2 parallel** conductors
- A short segment Δz of transmission line can be modeled as a lumped-element circuit.

Continued

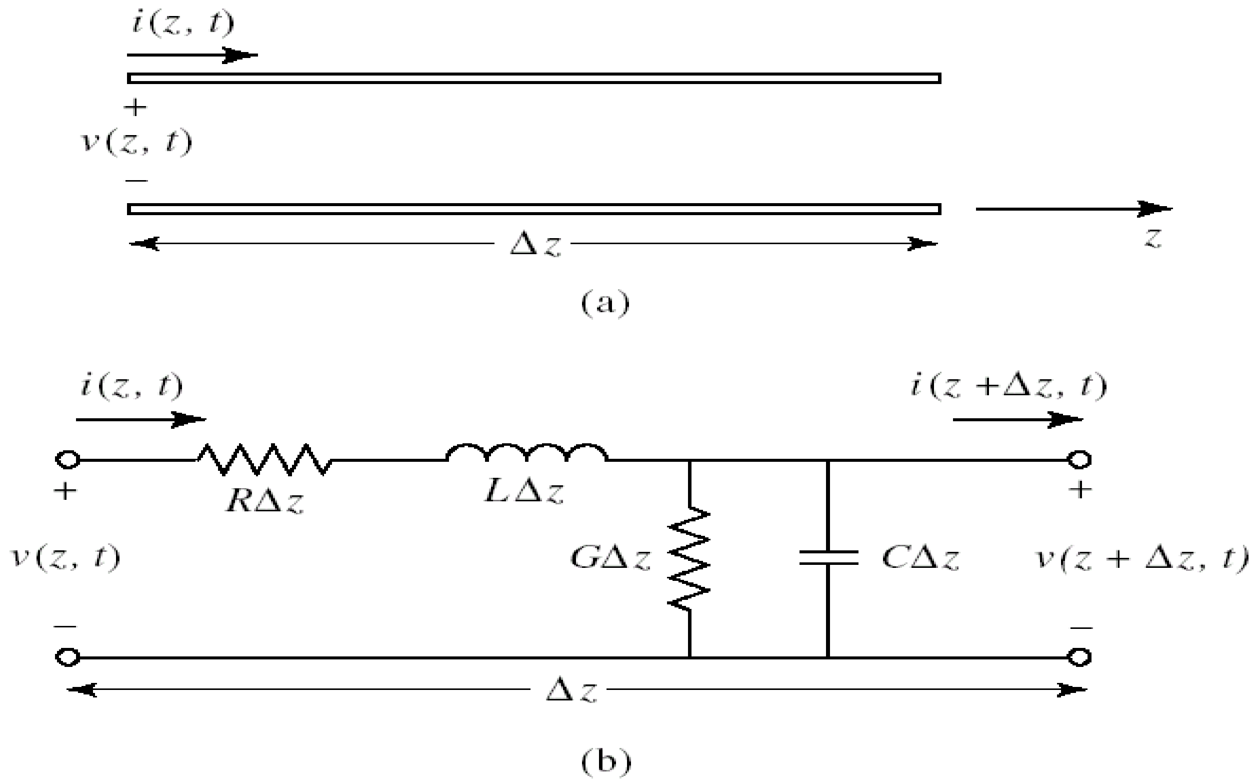


Figure 2.4 **Voltage and current definitions and equivalent circuit for an incremental length of transmission line. (*a*) Voltage and current definitions. (*b*) Lumped-element equivalent circuit.**

Transmission line parameters

- **R** = Series resistance per unit length for both conductors
- **L** = Series inductance per unit length for both conductors
- **G** = Shunt conductance per unit length
- **C** = Shunt capacitance per unit length
- Applying **KVL** and **KCL**,

$$v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \quad (2.1a)$$

$$i(z,t) - G\Delta z v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0 \quad (2.1b)$$

Continued

- Dividing (2.1) by Δz and $\Delta z \rightarrow 0$,

$$\frac{\partial v(z,t)}{\partial z} = -Ri(z,t) - L\frac{\partial i(z,t)}{\partial t} \quad (2.2a)$$

$$\frac{\partial i(z,t)}{\partial z} = -Gv(z,t) - C\frac{\partial v(z,t)}{\partial t} \quad (2.2b)$$

→Time-domain form of the transmission line, or telegrapher, equation

- For the sinusoidal steady-state condition with cosine-based phasors

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z) \quad (2.3b)$$

Wave Propagation on a Transmission Line

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \quad (2.4a)$$

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \quad (2.4b)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

γ Propagation Constant in per meter

α Attenuation Constant in nepers per meter

β Phase Constant in radians per meter

Continued

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}, \; I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \quad (2.6)$$

Wave propagation in +z directon

Wave propagation in -z directon

Applying (2.3a) to the voltage of (2.6),

$$I(z) = \frac{\gamma}{R + j\omega L}\left[V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}\right]$$

$$Z_0 = \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, (2.7)$$

$$\frac{V_0^+}{I_0^+} = Z_0 = -\frac{V_0^-}{I_0^-}$$

(2.6) can be rewritten

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z} \quad (2.8)$$

Converting the phasor voltage of (2.6) to the time domain:

$$v(z,t) = \left|V_0^+\right| \cos(\omega t - \beta z + \phi^+) e^{-\alpha z} + \left|V_0^-\right| \cos(\omega t + \beta z + \phi^-) e^{\alpha z} \quad (2.9)$$

The **wavelength** of the traveling waves:

$$\lambda = \frac{2\pi}{\beta} \qquad (2.10)$$

The **phase velocity** of the wave is defined as the **speed** at which a constant **phase** point travels down the line,

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \lambda f \qquad (2.11)$$

Lossless Transmission Lines

- R = G = 0 gives $\gamma = \alpha + j\beta = j\omega\sqrt{LC}$ or

$$\beta = \omega\sqrt{LC}, \quad \alpha = 0 \quad (2.12)$$

$$Z_0 = \sqrt{\frac{L}{C}} \quad (2.13)$$

- The general solutions for voltage and current on a lossless transmission line:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z},$$

$$I(z) = \frac{I_0^+}{Z_0} e^{-j\beta z} - I_0^- e^{j\beta z} \quad (2.14)$$

Continued

- The **wavelength** on the line:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}} \qquad (2.15)$$

- The **phase velocity** on the line:

$$v_p = \frac{\omega}{\beta} = \sqrt{\frac{1}{LC}} \qquad (2.16)$$

The Terminated Lossless Transmission Lines

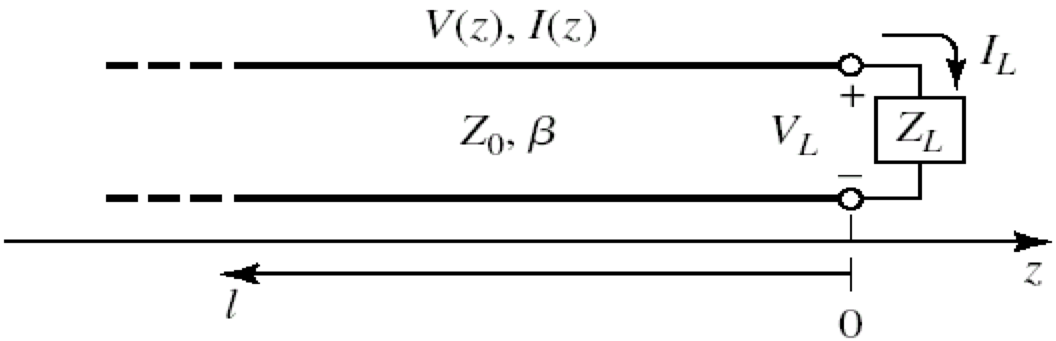


Fig 2.5 Terminated Lossless Transmission Lines

❖ The total voltage and current on the line:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z},$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{+j\beta z} \quad (2.34)$$

❖ The **total voltage** and **current** at the load are related by the load impedance, so at $z = 0$

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 \qquad V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

The **voltage reflection coefficient**: $$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (2.35)$$

The total voltage and current on the line:

$$V(z) = V_0^+ \left[e^{-j\beta z} + \Gamma e^{j\beta z} \right],$$

$$I(z) = \frac{V_0^+}{Z_0} \left[e^{-j\beta z} - \Gamma e^{j\beta z} \right] \qquad (2.36)$$

Continued

- It is seen that the voltage and current on the line consist of a superposition of an incident and reflected wave. → ***Standing waves***
- When Γ= 0 → matched.
- For the **time-average power** flow along the line at the point *z*:

$$P_{avg} = \frac{1}{2}\mathrm{Re}\left\{V(z)I^*(z)\right\} = \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}\mathrm{Re}\left\{1-\Gamma^* e^{-2j\beta z} + \Gamma e^{2j\beta z} - \left|\Gamma\right|^2\right\}$$

$$= \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}\left(1-\left|\Gamma\right|^2\right)$$

- When the load is mismatched, not all of the available power from the generator is delivered to the load.
- This loss is **return loss (RL):**

RL = -20 log|Γ| dB

- If the load is matched to the line, Γ= 0 and $|V(z)| = |V_0^+|$ (constant) → flat When the load is mismatched,

$$|V(z)| = |V_0^+||1+\Gamma e^{2j\beta z}| = |V_0^+||1+\Gamma e^{-2j\beta l}| = |V_0^+||1+|\Gamma|e^{j(\theta-2\beta l)}| \quad (2.39)$$

$$V_{max} = |V_0^+|(1+|\Gamma|), \quad V_{min} = |V_0^+|(1-|\Gamma|) \quad (2.40)$$

- A measure of the mismatch of a line, called the voltage standing wave ratio (**VSWR**)

$$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad (1 < VSWR < \infty)$$

- From (2.39), the distance between 2 successive voltage maxima (or minima) is $l = 2\pi/2\beta = \lambda/2$ $(2\beta l = 2\pi)$, while the distance between a maximum and a minimum is $l = \pi/2\beta = \lambda/4$.
- From (2.34) with $z = -l$,

$$\Gamma(l) = \frac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}} = \Gamma(0)\ e^{-2j\beta l} \quad (2.42)$$

At a distance $l = -z$,

$$Z_{in} = \frac{V(-l)}{I(-l)} = Z_0 \frac{V_0^+}{V_0^+}\left[\frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}}\right] = Z_0 \frac{1+\Gamma e^{-2j\beta l}}{1-\Gamma e^{-2j\beta l}} \qquad (2.43)$$

$$= Z_0 \frac{(Z_L + Z_0)e^{j\beta l} + (Z_L - Z_0)e^{-j\beta l}}{(Z_L + Z_0)e^{j\beta l} - (Z_L - Z_0)e^{-j\beta l}}$$

$$= Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l}$$

$$= Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \qquad (2.44)$$

→Transmission line impedance equation

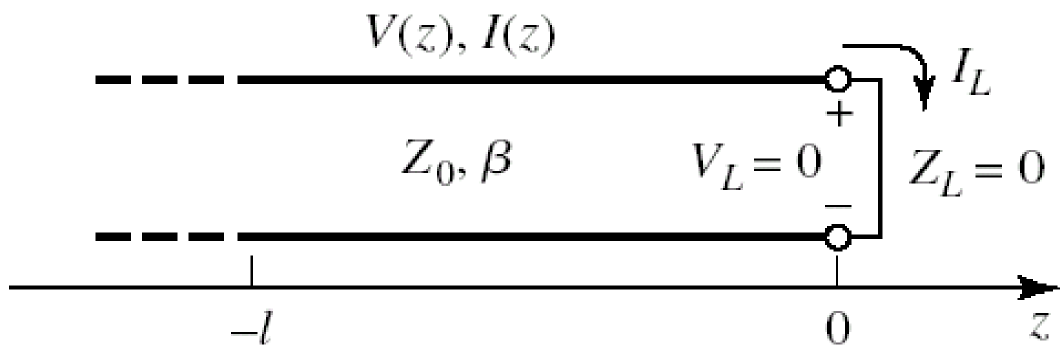
Special Cases of Terminated Transmission Lines

- **Short-circuited line**
- $Z_L = 0 \rightarrow \Gamma = -1$

$$V(z) = V_0^+\left[e^{-j\beta z} - e^{j\beta z}\right] = -2jV_0^+ \sin\beta z,$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} + e^{j\beta z}\right] = 2\frac{V_0^+}{Z_0}\cos\beta z$$

$$Z_{in} = jZ_0 \tan\beta l \qquad (2.45)$$



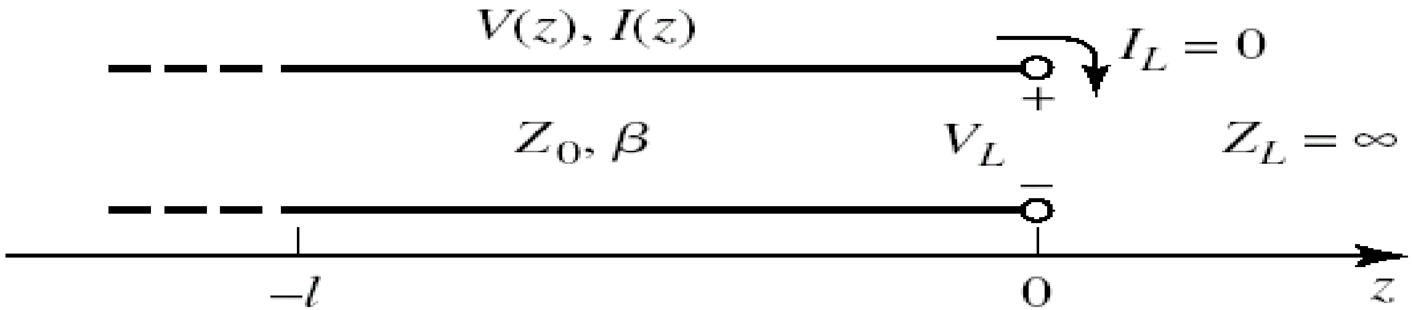
- **Open-circuited line**

$Z_L = \infty \rightarrow \Gamma = 1$

$$V(z) = V_0^+\left[e^{-j\beta z} + e^{j\beta z}\right] = 2V_0^+ \cos\beta z,$$

$$I(z) = \frac{V_0^+}{Z_0}\left[e^{-j\beta z} - e^{j\beta z}\right] = \frac{-2jV_0^+}{Z_0}\sin\beta z \qquad (2.46)$$

$$Z_{in} = -jZ_0 \cot\beta l$$



Generator and Load Mismatches

- Because both the **generator** and **load** are mismatched, multiple reflections can occur on the line.
- In the steady state, the net result is a single wave traveling toward the load, and a single reflected wave traveling toward the generator.

where $z = -l$,

$$Z_{in} = Z_0 \frac{1+\Gamma_l e^{-2j\beta l}}{1-\Gamma_l e^{-2j\beta l}} = Z_0 \frac{Z_l + jZ_0 \tan \beta l}{Z_0 + jZ_l \tan \beta l} \qquad (2.67)$$

$$\Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0} \qquad (2.68)$$

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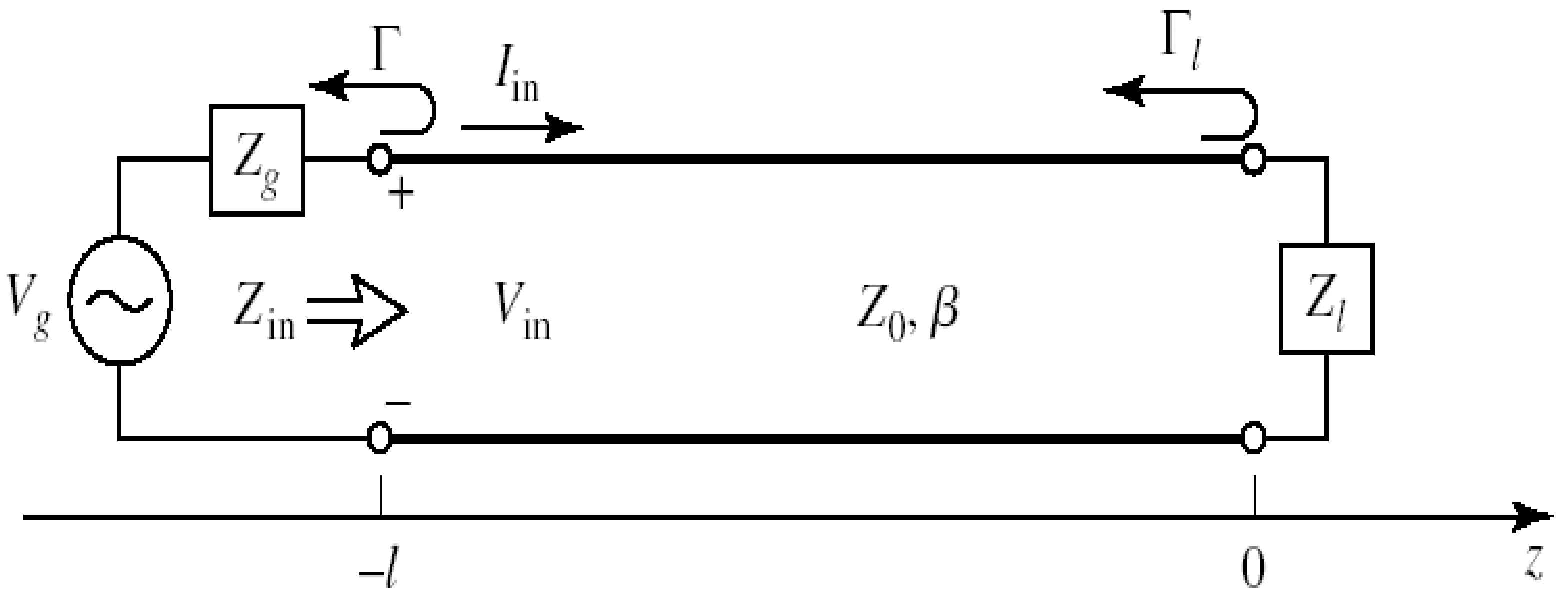

Figure 2.8 Transmission line circuit for mismatched load and generator

- The voltage on the line:

$$V(-l) = V_g \frac{Z_{in}}{Z_{in} + Z_g} = V_0^+ (e^{j\beta l} + \Gamma_l e^{-j\beta l})$$

$$V_0^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{1}{e^{j\beta l} + \Gamma_l e^{-j\beta l}} \qquad (2.70)$$

By (2.67) &

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0}$$

$$V_0^+ = V_g \frac{Z_0}{Z_0 + Z_g} \frac{e^{-j\beta l}}{\left(1 - \Gamma_l \Gamma_g e^{-2j\beta l}\right)} \qquad (2.71)$$

$$SWR = \frac{1 + |\Gamma_l|}{1 - |\Gamma_l|}$$

Power delivered to the load:

$$P_l = \frac{1}{2}\mathrm{Re}\left\{V_{in}I_{in}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{|V_{in}|^2 \frac{1}{Z_{in}^*}\right\} = \frac{1}{2}\mathrm{Re}\left\{|V_g|^2 \left|\frac{Z_{in}}{Z_{in}+Z_g}\right|^2 \frac{1}{Z_{in}^*}\right\}$$

$$= \frac{1}{2}|V_g|^2 \frac{R_{in}}{(R_{in}+R_g)^2+(X_{in}+X_g)^2} \quad (2.39)$$

Case 1: the load is matched to the line, $Z_l = Z_0$, $\Gamma_l = 0$, SWR $= 1$, $Z_{in} = Z_0$,

$$P_l = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0+R_g)^2+X_g^2} \quad (2.40)$$

- **Case 2**: the generator is matched to the input impedance of a **transmission** line, $Z_{in} = Z_g$

$$P_l = \frac{1}{2}|V_g|^2 \frac{R_g}{4\left(R_g^2 + X_g^2\right)} \qquad (2.41)$$

- If Z_g is fixed, to maximize P_l,

$$\frac{\partial P_l}{\partial R_{in}} = 0 \rightarrow \frac{1}{(R_{in}+R_g)^2+(X_{in}+X_g)^2} - \frac{2R_{in}(R_{in}+R_g)}{\left[(R_{in}+R_g)^2+(X_{in}+X_g)^2\right]^2} = 0$$

or $$R_g^2 - R_{in}^2 + (X_{in} + X_g)^2 = 0$$

$$\frac{\partial P_l}{\partial X_{in}} = 0 \rightarrow \frac{-2X_{in}(X_{in} + X_g)}{\left[(R_{in} + R_g)^2 + (X_{in} + X_g)^2\right]^2} = 0$$

or $$X_{in}(X_{in} + X_g) = 0$$

- Therefore, $R_{in} = R_g$ and $X_{in} = -X_g$, or $Z_{in} = Z_g^*$
- Under these conditions

$$P_l = \frac{1}{2}|V_g|^2 \frac{1}{4R_g} \qquad (2.44)$$

Example 1

An air line has characteristic impedance of 70 Ω and phase constant of 3 rad/m at 100 MHz. Calculate the inductance per meter and the capacitance per meter of the line.

Solution:

An air line can be regarded as a lossless line since $\sigma \simeq 0$. Hence

$$R = 0 = G \qquad \text{and} \qquad \alpha = 0$$

$$Z_o = R_o = \sqrt{\frac{L}{C}}$$

$$\beta = \omega \sqrt{LC}$$

$$\frac{R_o}{\beta} = \frac{1}{\omega C}$$

cont'd

$$C = \frac{\beta}{\omega R_o} = \frac{3}{2\pi \times 100 \times 10^6 (70)} = 68.2 \text{ pF/m}$$

$$L = R_o^2 C = (70)^2 (68.2 \times 10^{-12}) = 334.2 \text{ nH/m}$$

A transmission line operating at 500 MHz has $Z_o = 80\ \Omega$, $\alpha = 0.04$ Np/m, $\beta = 1.5$ rad/m. Find the line parameters R, L, G, and C.

Answer: 3.2 Ω/m, 38.2 nH/m, 5×10^{-4} S/m, 5.97 pF/m.

Example 2

A distortionless line has $Z_0 = 60\ \Omega$, $\alpha = 20$ mNp/m, $u = 0.6c$, where c is the speed of light in a vacuum. Find R, L, G, C, and λ at 100 MHz.

Solution:

For a distortionless line,

$$RC = GL \qquad \text{or} \qquad G = \frac{RC}{L}$$

and hence

$$Z_o = \sqrt{\frac{L}{C}}$$

$$\alpha = \sqrt{RG} = R\sqrt{\frac{C}{L}} = \frac{R}{Z_o}$$

or

$$R = \alpha Z_o$$

Cont'd

But

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

$$R = \alpha Z_o = (20 \times 10^{-3})(60) = 1.2\ \Omega/\text{m}$$

$$L = \frac{Z_o}{u} = \frac{60}{0.6\,(3 \times 10^8)} = 333\ \text{nH/m}$$

$$G = \frac{\alpha^2}{R} = \frac{400 \times 10^{-6}}{1.2} = 333\ \mu\text{S/m}$$

$$uZ_o = \frac{1}{C}$$

or

$$C = \frac{1}{uZ_o} = \frac{1}{0.6\,(3 \times 10^8)\,60} = 92.59\ \text{pF/m}$$

$$\lambda = \frac{u}{f} = \frac{0.6\,(3 \times 10^8)}{10^8} = 1.8\ \text{m}$$

Example 3

A certain transmission line operating at $\omega = 10^6$ rad/s has $\alpha = 8$ dB/m, $\beta = 1$ rad/m, and $Z_o = 60 + j40\ \Omega$, and is 2 m long. If the line is connected to a source of $10\underline{/0^\circ}$ V, $Z_g = 40\ \Omega$ and terminated by a load of $20 + j50\ \Omega$, determine

(a) The input impedance

(b) The sending-end current

(c) The current at the middle of the line

Solution:

(a) Since 1 Np = 8.686 dB,

$$\alpha = \frac{8}{8.686} = 0.921 \text{ Np/m}$$

$$\gamma = \alpha + j\beta = 0.921 + j1 \text{ /m}$$

$$\gamma \ell = 2(0.921 + j1) = 1.84 + j2$$

Using the formula for $\tanh(x + jy)$, we obtain

$$\tanh \gamma \ell = 1.033 - j0.03929$$

$$Z_{in} = Z_o \left(\frac{Z_L + Z_o \tanh \gamma \ell}{Z_o + Z_L \tanh \gamma \ell} \right)$$

$$= (60 + j40) \left[\frac{20 + j50 + (60 + j40)(1.033 - j0.03929)}{60 + j40 + (20 + j50)(1.033 - j0.03929)} \right]$$

$$Z_{in} = 60.25 + j38.79 \ \Omega$$

(b) The sending-end current is $I(z = 0) = I_o$.

$$I(z=0) = \frac{V_g}{Z_{in} + Z_g} = \frac{10}{60.25 + j38.79 + 40}$$
$$= 93.03\angle -21.15^\circ \text{ mA}$$

(c) To find the current at any point, we need V_o^+ and V_o^-. But

$$I_o = I(z=0) = 93.03\angle -21.15^\circ \text{ mA}$$

$$V_o = Z_{in} I_o = (71.66\angle 32.77^\circ)(0.09303\angle -21.15^\circ) = 6.667\angle 11.62^\circ \text{ V}$$

$$V_o^+ = \frac{1}{2}(V_o + Z_o I_o)$$
$$= \frac{1}{2}[6.667\angle 11.62^\circ + (60 + j40)(0.09303\angle -21.15^\circ)] = 6.687\angle 12.08^\circ$$

$$V_o^- = \frac{1}{2}(V_o - Z_o I_o) = 0.0518\angle 260^\circ$$

At the middle of the line, $z = \ell/2$, $\gamma z = 0.921 + j1$. Hence, the current at this point is

$$I_s(z = \ell/2) = \frac{V_o^+}{Z_o} e^{-\gamma z} - \frac{V_o^-}{Z_o} e^{\gamma z}$$
$$= \frac{(6.687e^{j12.08^\circ})e^{-0.921-j1}}{60 + j40} - \frac{(0.0518e^{j260^\circ})e^{0.921+j1}}{60 + j40}$$

Note that $j1$ is in radians and is equivalent to $j57.3^\circ$. Thus,

$$I_s(z = \ell/2) = \frac{6.687e^{j12.08^\circ}e^{-0.921}e^{-j57.3^\circ}}{72.1e^{j33.69^\circ}} - \frac{0.0518e^{j260^\circ}e^{0.921}e^{j57.3^\circ}}{72.1e^{j33.69^\circ}}$$
$$= 0.0369e^{-j78.91^\circ} - 0.001805e^{j283.61^\circ}$$
$$= 6.673 - j34.456 \text{ mA}$$
$$= 35.10\angle 281^\circ \text{ mA}$$

Example 4

A 30-m-long lossless transmission line with $Z_o = 50\ \Omega$ operating at 2 MHz is terminated with a load $Z_L = 60 + j40\ \Omega$. If $u = 0.6c$ on the line, find

(a) The reflection coefficient Γ

(b) The standing wave ratio s

(c) The input impedance

Solution:

This problem will be solved with and without using the Smith chart.

(a) $$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j40 - 50}{50 + j40 + 50} = \frac{10 + j40}{110 + j40} = 0.3523\underline{/56^\circ}$$

(b) $$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.3523}{1 - 0.3523} = 2.088$$

(c) Since $u = \omega/\beta$, or $\beta = \omega/u$,

$$\beta\ell = \frac{\omega\ell}{u} = \frac{2\pi\,(2\times10^6)(30)}{0.6\,(3\times10^8)} = \frac{2\pi}{3} = 120°$$

Note that $\beta\ell$ is the electrical length of the line.

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o\tan\beta\ell}{Z_o + jZ_L\tan\beta\ell}\right]$$

$$= \frac{50\,(60 + j40 + j50\tan 120°)}{[50 + j(60 + j40)\tan 120°]}$$

$$= \frac{50\,(6 + j4 - j5\sqrt{3})}{(5 + 4\sqrt{3} - j6\sqrt{3})} = 24.01\,\underline{/3.22°}$$

$$= 23.97 + j1.35\ \Omega$$