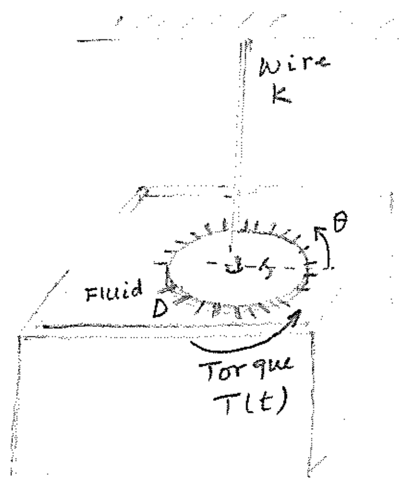
$$\left[\frac{(s^2M_2 + sD_2 + sD_1)(s^2M_1 + sD_1 + k) - s^2D_1^2}{s^2M_1 + sD_1 + K} \right] X_2(s) = F(s)$$

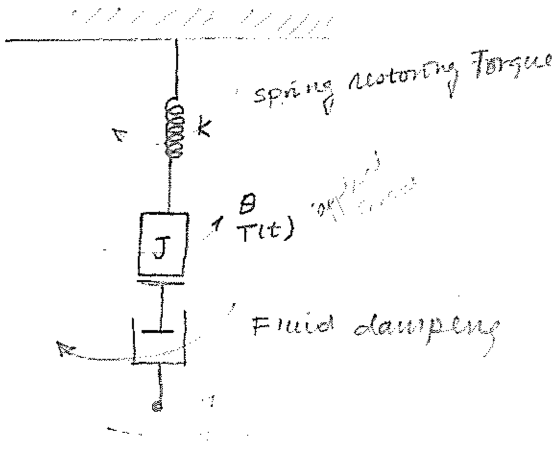
The transfer function $\frac{X_2(s)}{F(s)}$ becomes

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2M_1 + sD_1 + k}{(s^2M_2 + sD_2 + sD_1)(s^2M_1 + sD_1 + K) - s^2D_1^2}$$

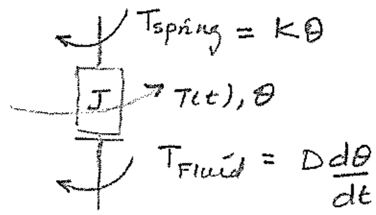
Example.

A system shown below has a mass with a moment of enertia J immersed in a fluid. A torque T is applied to the mass. The wire produces a reactive torque proportional to the stiffness K and the angle of twist. The fins moving through the fluid have a damping D which requires a torque proportional to the rate at which they are moving.





Free body diagram



$$T_{ap}(t) - T_{Fluid}(t) - T_{spring}(t) = J\frac{d^2\theta}{dt^2}$$

$$T(t) - D\frac{d\theta}{dt} - k\theta = J\frac{d^2\theta}{dt^2}$$

$$T(t) = J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + k\theta$$

Transfering using Laplace transform

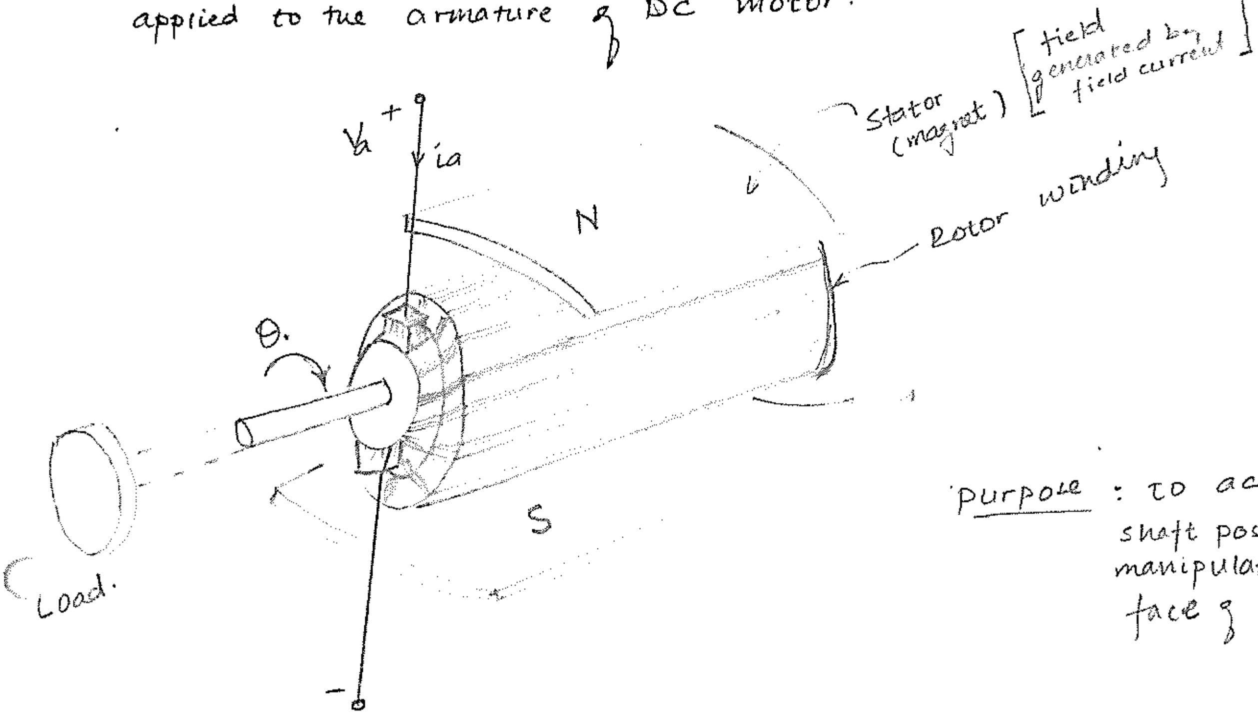
$$T(s) = [s^2J + sD + K]\,\theta(s)$$

The transfer function

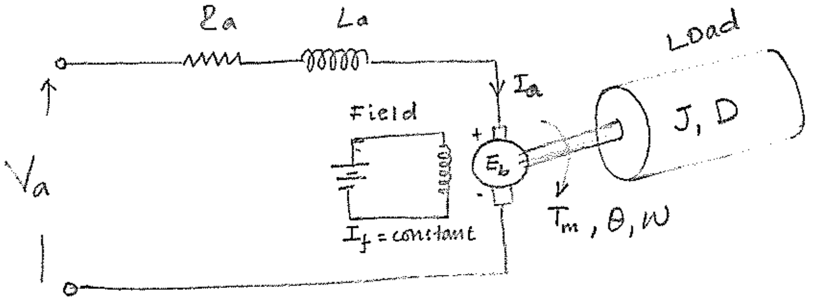
$$G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{s^2J + sD + K}$$

Transfer function of DC Motor.

Determine the transfer function of an armature controlled DC motor. The transfer function relating the angular position θ and the Voltage applied to the armature of DC motor.



Purpose : to achive a desired shaft position θ, by manipulating V_a, in the face of load.



Inertia = J
Friction = D or damping constant

The armature - controlled DC-motor utilizes a constant field-current.

1) The armature current is related to the input Voltage (V_a) applied to the armature

$$V_a = R_a i_a + L_a \frac{d i_a}{dt} + E_b \quad --- (1)$$

using Laplace transform

$$V_a(s) = R_a I_a(s) + s L_a I_a(s) + E_b(s) \quad -- (2)$$

The back emf ~~is~~ E_b is ~~expressed~~ proportional to speed or expressed as

$$E_b = k_e \phi N = \frac{60}{2\pi} k_e \phi W \Rightarrow E_b = K_b W$$

$$E_b(s) = K_b W(s) \quad ---- (3)$$

The armature current becomes

$$I_a(s) = \frac{V_a(s) - E_b(s)}{SL_a + R_a}$$

$w = \frac{d\theta}{dt}$

$$= \frac{V_a(s) - K_b W(s)}{SL_a + R_a} \quad \text{- - - - (4)}$$

i_a.

2) The motor torque is related to air-gap flux (ϕ) and the armature current

$$T_m = k_m \phi I_a$$

where air-gap flux $\phi = k_f I_f$ (constant)

b/c field current is constant

$$= (k_m k_f I_f) I_a$$

let $K_m = k_m k_f$

$$T_m = K_m I_a \quad \text{- - -(5)}$$

3) The dynamic Torque equation

The motor torque T_m is equal to the torque delivered to the load

$$T_m = T_L = J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt}$$

$T_m - T_D = J\frac{d^2\theta}{dt^2} = J\alpha$

$$T_m(s) = s^2 J\theta(s) + SD\,\theta(s)$$

$$= (s^2 J + SD)\,\theta(s) \quad \text{- - - -(6)}$$

4) Equating eq (5) and eq (6)

$$K_m I_a(s) = (s^2 J + SD)\,\theta(s)$$

$$\theta(s) = \frac{K_m I_a(s)}{s^2 J + SD} \quad \text{- - - -(7)}$$

5) Substituting eq (4) $I_a(s)$ into eqn (7)

$$K_m\left[\frac{V_a(s) - K_b W(s)}{SL_a + R_a}\right] = (s^2 J + SD)\,\theta(s)$$

$$\frac{K_m V_a(s)}{SL_a + R_a} - \frac{K_m k_b W(s)}{SL_a + R_a} = (s^2 J + SD)\,\theta(s)$$

$w = \frac{d\theta}{dt}$

$w(s) = s\theta(s)$

$$\frac{K_m V_a(s)}{sL_a + R_a} = (s^2 J + sD)\,\theta(s) + \frac{K_m K_b}{sL_a + R_a}\left[s\,\theta(s) \right]$$

$$= \left[\frac{(s^2 J + sD)(sL_a + R_a) + sK_m K_b}{sL_a + R_a} \right] \theta(s)$$

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{(s^2 J + sD)(sL_a + R_a) + sK_m K_b}$$

Also $\frac{\omega(s)}{V_a(s)} =$

$$= \frac{K_m}{s\left[(sJ + D)(L_a + R_a) + K_m K_b \right]}$$

$$G(s) = \frac{K_m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

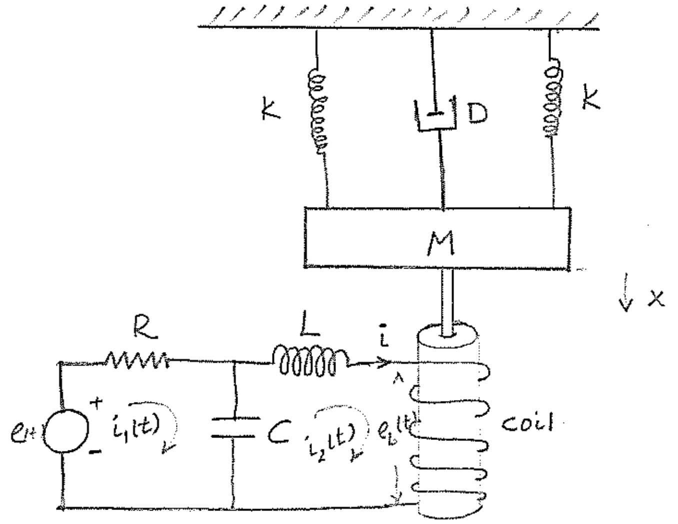
However for many DC motors, the time constant of the armature, $\tau_a = \frac{L_a}{R_a}$ is negligible and the transfer function reduced to

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s\left[R_a(sJ + D) + K_m K_b \right]}$$

Electro - Mechanical systems.

Eg. Consider the electomechanical System shown below. Determine the transfer function $\cancel{\frac{E(s)}{X(s)}}$ $\frac{X(t)}{E(t)}$ (as $e(t)$ as input and $x(t)$ as output)

[Hint
For a simplified analysis, assume that the coil has a back emf $e_b = k_b \frac{dx}{dt}$ and the coil current i produces a force $F_c = k_f i$ on the mass M)



* neglect the coil inductance and resistance

1 Consider the electrical ckt for loop ① and ② applying KVL

Loop 1

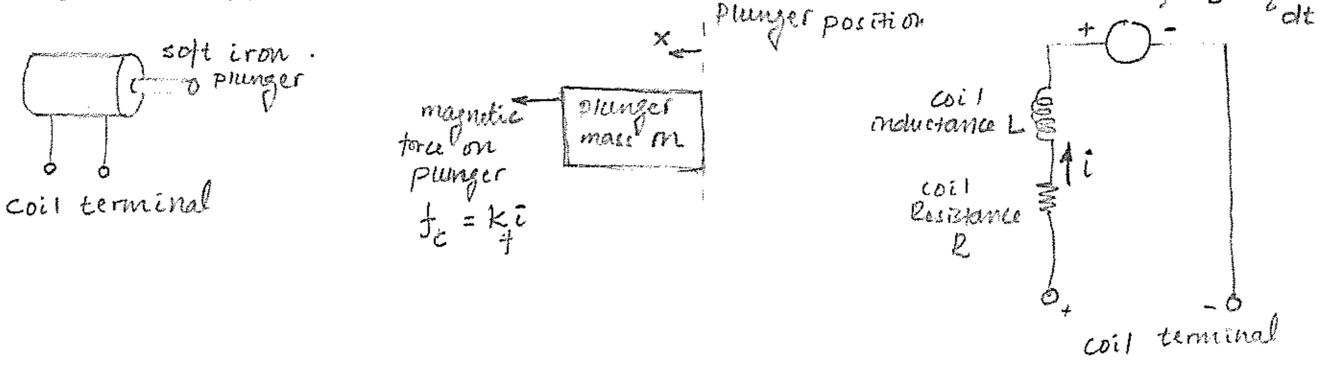
$$e(t) = R\, i_1(t) + \frac{1}{C}\int \left[i_1(t) - i_2(t) \right] dt$$

Laplace transform

$$E(s) = R I_1(s) + \frac{1}{sC} I_1(s) - \frac{1}{sC} I_2(s) \quad \text{---(1)}$$

Hint:

For linear actuator



Loop 2

$$e_b(t) = \frac{L\, di_2(t)}{dt} + \frac{1}{C}\int \left(i_2(t) - i_1(t) \right) dt$$

Laplace transform

$$E_b(s) = sL\, I_2(s) + \frac{1}{sC} I_2(s) - \frac{1}{sC} I_1(s) \quad \text{---(2)}$$

2. Consider the mechanical system

2-1 When the plunger moves because of the electromagnetic force generated when a current $i_2(t)$ passes through the coil a back emf is generated in the coil (Lenz law) given by

$$e_b(t) = K_b \frac{dx}{dt}$$

$$E_b(s) = S K_b X(s) \qquad --- (3)$$

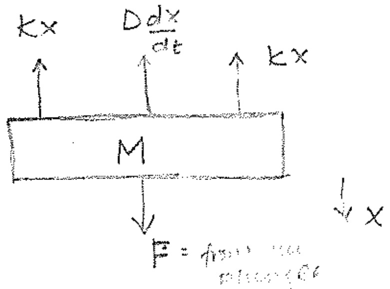
2-2. When a current $i_2(t)$ flows through the coil a mechanical force is developed which is given by

$$F(t) = K_f \, i_2(t)$$

$$F(s) = K_f I_2(s) \qquad --- (4)$$

where K_f is the force constant in N/A.

The free body diagram



$$F - 2(kx) - D\frac{dx}{dt} = M\frac{d^2x}{dt^2}$$

$$F(t) = M\frac{d^2x}{dt} + D\frac{dx}{dt} + 2kx$$

$$F(s) = s^2 M X(s) + SD X(s) + 2kx$$

$$F(s) = \left[s^2 M + SD + 2k\right] X(s) \qquad - \qquad (5)$$

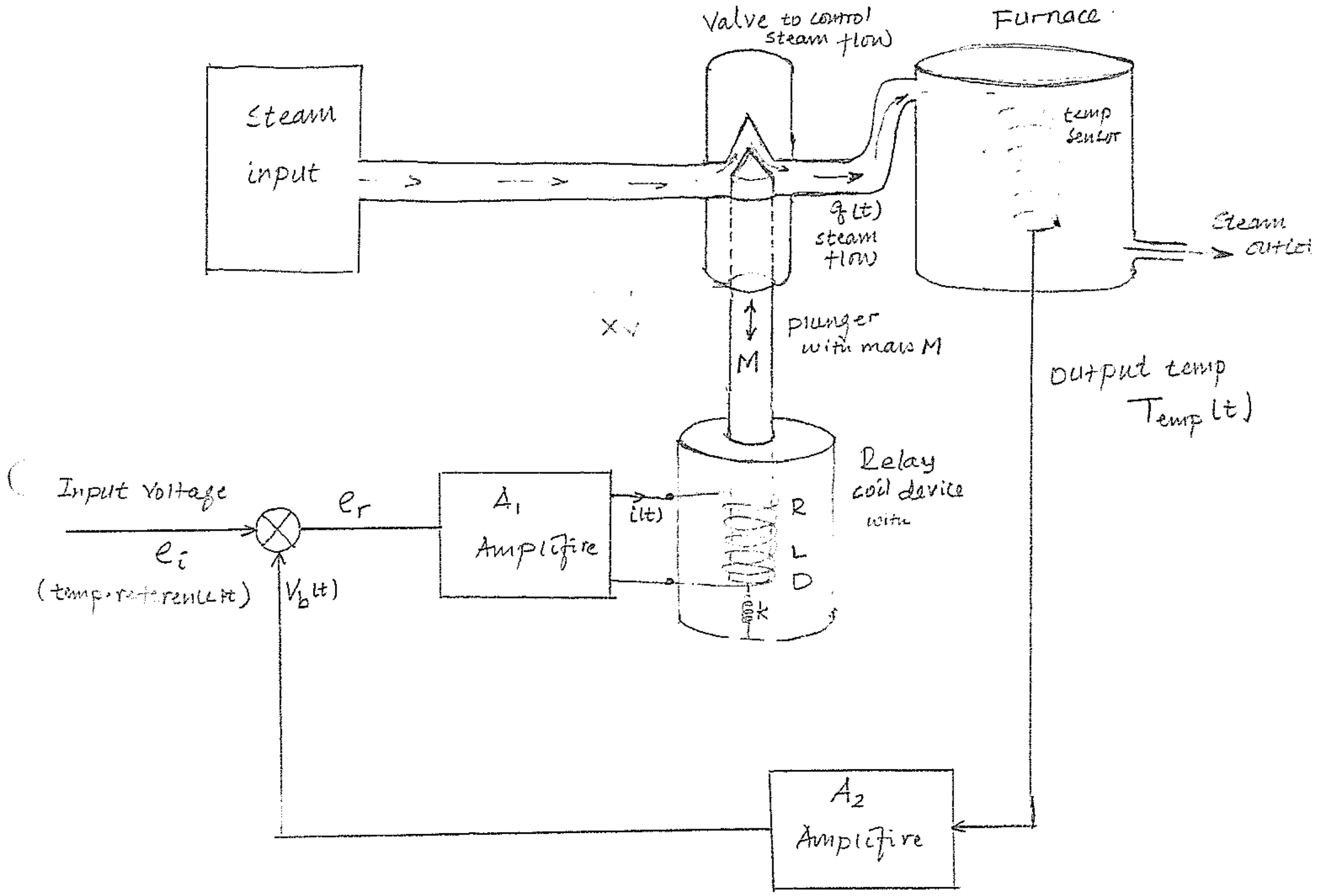
solving eq 1→2, 3→2, 2→4, →5

the transfer function

check!

$$T(s) = \frac{X(s)}{E(s)} = \frac{K_f}{\left\{RLCMs^4 + L(M+RCD)s^3 + \left[RM + LD + RC(2Lk + K_f K_b)\right]s^2 + (RD + 2Lk + K_f K_b)s + 2Rk\right\}}$$

Consider a process control of a Cement Factory.
The process controller controls the temperature along with terminal variable as shown below. If the input is Voltage V_i and output is temperature Temp. Find the transfer function



1. The Error Voltage $e_r = e_i - V_b$

$$E_r(s) = E_i(s) - V_B(s)$$

2. The feedback Voltage $V_b(t) = A_2 \, Temp(t)$

$$V_B(s) = A_2 \, Temp(s)$$

3. The amplifire output Voltage

$$V_{OA_1} = A_1 e_r(t) = R\,i(t) + L\frac{di(t)}{dt}$$

$$V_{OA_1}(s) = A_1 E_r(s) = (R + sL) I(s)$$

$$A_1 \left[E_i(s) - V_B(s) \right] = (R + sL) I(s)$$

$$A_1 \left[E_i - A_2 \, Temp \right] = (R + sL) I(s)$$

4. The mechanical Force developed

$$F(t) = k_m i(t)$$

$$F(t) - D\frac{dx}{dt} - kx = M\frac{d^2x}{dt^2}$$

$$F(t) = k_m i(t) = M\frac{d^2x}{dt} + D\frac{dx}{dt} + kx$$

$$F(s) = k_m I(s) = (s^2M + sD + k)X(s) \quad \text{---(2)}$$

5. The steam flow

$$q(t) = k_1 x(t)$$

$$Q(s) = k_1 X(s)$$

6. Output temperature of the furnace is given by

$$Temp(t) = k_2 q(t)$$

$$Temp(s) = k_2 Q(s)$$

$$= k_2 k_1 X(s) \quad - \quad (3)$$

7. From eqn 1, 2, 3 the transfer function

$$T(s) = \frac{Temp(s)}{E_i(s)} = \frac{G(s)}{1 + G(s)A_2}$$

check!

$$= \frac{(R+sL)(s^2M+sD+k)}{(R+sL)(s^2M+sD+K) + A_2A_1k_1k_f k_m}$$

where $G(s)$ = product gain of Forward path

$$= \frac{A_1 k_1 k_f k_m}{(R+sL)(s^2M+sD+k)}$$

Block diagram Models.

A control system may consists of a number of components. To show the functions performed by each component, in control Engineering, we commonly use a diagram called the block diagram.

A block diagram of a system is a pictorial representation of the function performed by each components and of the flow of signals. Such a diagram depicts the interrelationship that exists among the various components. For each block the transfer function provides the dynamical mathematical relationship between the input and output quantities.

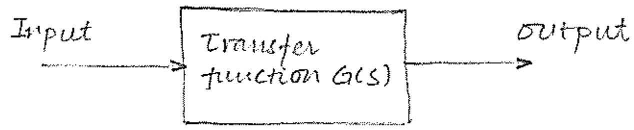
" The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output".

A block diagrams are used to describe the component parts of systems. They offer alternative to dealing directly with eqns

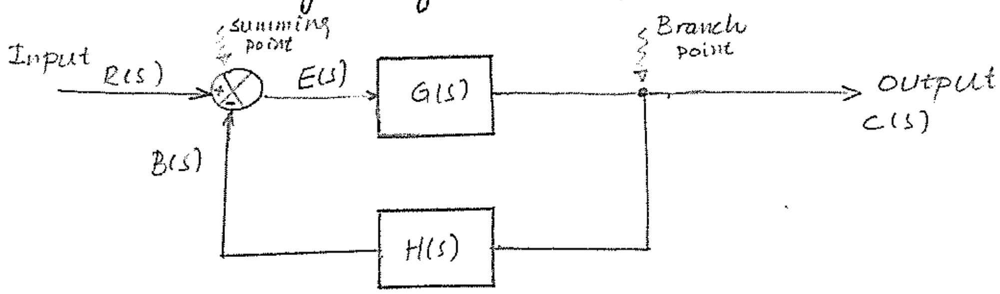
A block is used to indicate a proportional relationship b/w two Laplace-transformed signals

Example.

Block diagram representation of open-loop system.



Block diagram of closed loop system.



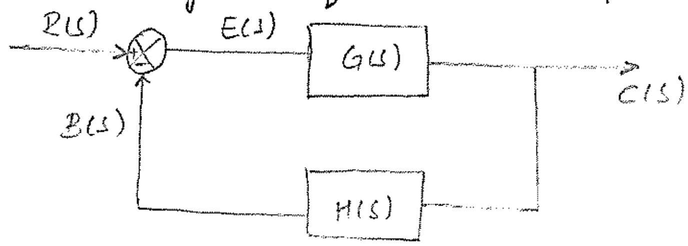
Any linear Control system may be represented by a block diagram consists of
- Blocks
- summing points
- Branch-point (also called pick-off point)

1- summing point ($\rightarrow\otimes$)

A circle with a cross is the symbol that indicate a summing operation. The plus or minus sign at each arrowhead indicates whether that signal is to be added or subtracted.

2- Branch / pick-off point - is a point from which the signal from a block goes concurrently to other blocks or summing points.

Block diagram of a closed-loop system.



C(s) - output signal

E(s) - actuating error signal

B(s) - feedback signal

Definitions.

The ratio of the feedback signal B(s) to the actuating error signal E(s) is called open-loop transfer function

$$\text{Open-loop transfer function} = \frac{B(s)}{E(s)} = G(s)H(s)$$

The ratio of the output C(s) to the actuating error signal E(s) is called feed-forward transfer function.

$$\text{Feed-forward transfer function} = \frac{C(s)}{E(s)} = G(s)$$

Closed-loop transfer function

The output C(s) and the input R(s) are related as follows:

$$C(s) = G(s)E(s) \quad \text{--- (1)}$$

$$E(s) = R(s) - B(s) \quad \text{--- (2)}$$

where $B(s) = H(s)C(s)$

$$= R(s) - H(s)C(s) \quad \text{--- (3)}$$

Eliminating E(s) from these eqn ((3) into (1))

$$C(s) = G(s)E(s)$$

$$= G(s)\left[R(s) - H(s)C(s)\right]$$

$$C(s) + G(s)H(s)C(s) = G(s)R(s)$$

$$C(s)\left[1 + G(s)H(s)\right] = G(s)R(s)$$

forward path

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \text{The closed-loop transfer function.}$$

open loop transf

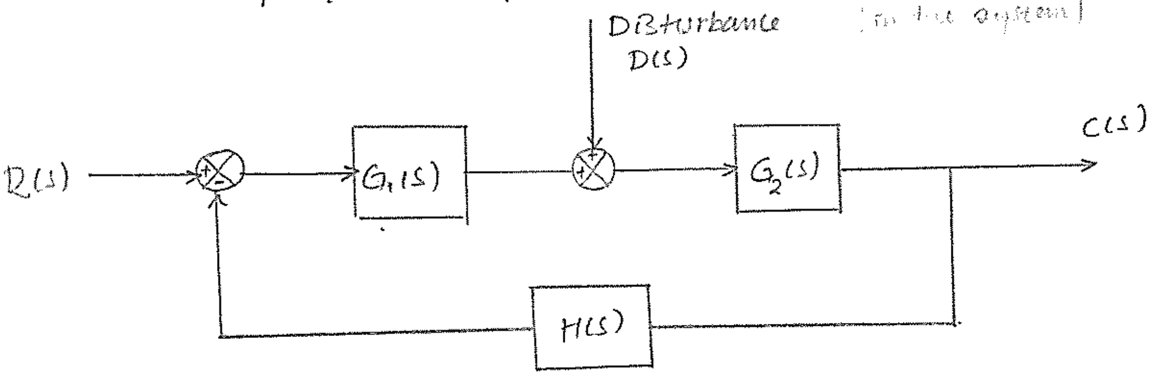
This transfer function relates the closed-loop system dynamics to the dynamics of the feedforward elements and feedback elements.

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

the output of the closed-loop system clearly depends on both the closed-loop transfer function and the nature of the input.

Closed-loop system subjected to a disturbance. (D(s))

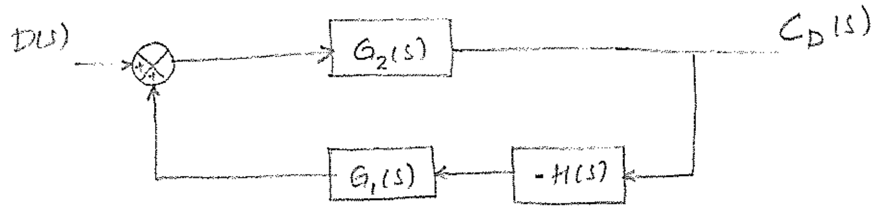
A closed-loop system subjected to a disturbance is shown below,



Closed-loop system subjected to disturbance.

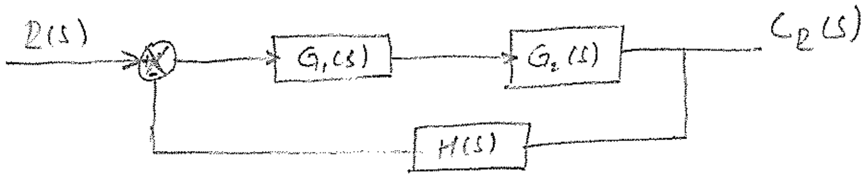
When two inputs (the reference input R(s) and disturbance D(s)) are present in a linear system, each input can be treated independently of other; and the outputs corresponding to each input alone can be added to give the complete output.

I - input only D(s), corresponding output $C_D(s)$, when $R(s) = 0$



$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}$$

II, when $D(s) = 0$, input only $R(s)$, corresponding output $C_R(s)$



$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}$$

The response to the simultaneous application of the reference input and disturbance can be obtained by adding the two individual response. (linearity property)

$$C(s) = C_R(s) + C_D(s)$$

$$= \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} \left[G_1(s)R(s) + D(s) \right]$$

Procedures for Drawing a block diagram (from its Transfer fn mathematical model)

To draw a block diagram for a system

1st - write the equations that describes the dynamic behavior of each component

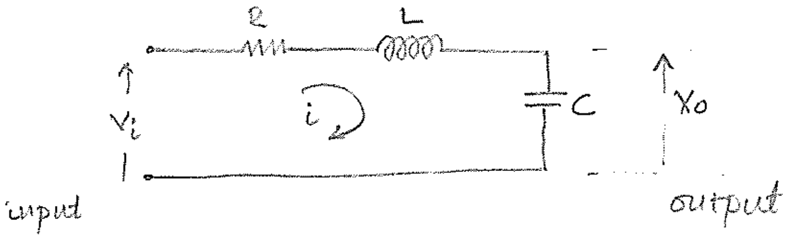
2nd - take the Laplace transform of the above equations, assuming zero initial conditions.

3rd - Represent the above each Laplace-transformed equation individually in block form

Finally, assemble the elements into a compact block diagram

Example.

consider the ~~ci~~ series RLC ckt, draw the block diagram representation



From input section.

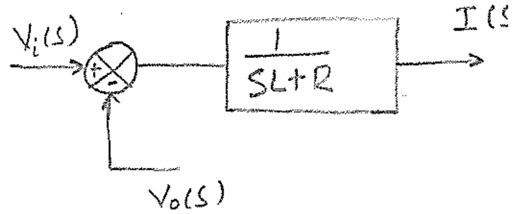
$$Ri + L\frac{di}{dt} + V_o = V_i$$

$$Ri + L\frac{di}{dt} = V_i - V_o$$

Using Laplace transform

$$[R + sL]\, I(s) = V_i(s) - V_o(s)$$

$$I(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$

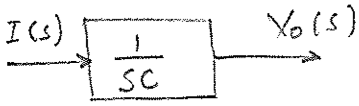


From output section

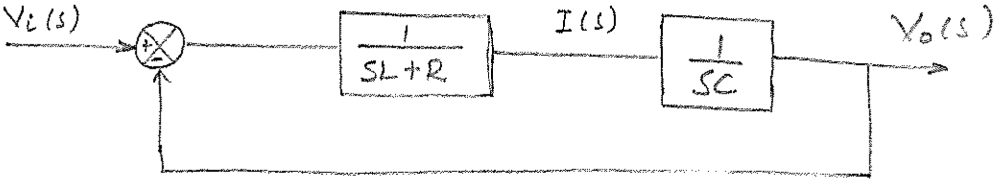
$$V_o = \frac{1}{C}\int i\,dt$$

using Laplace transform

$$V_o(s) = \frac{1}{sC} I(s)$$

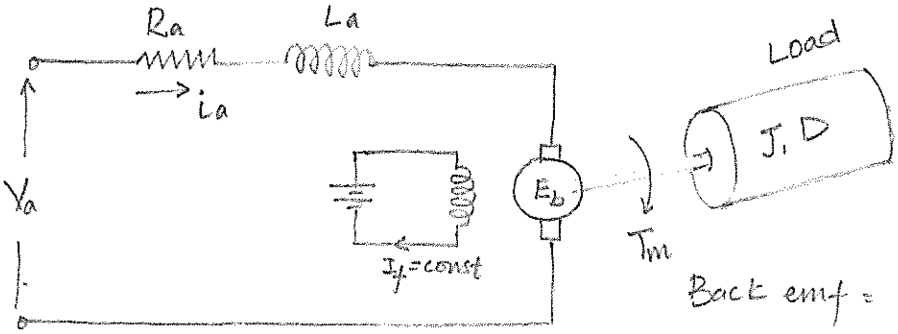


The over-all block-diagram representation.



Example.

Draw the block diagram representation of an armature-controlled DC-motor.



Back emf = E_b

$$E_b = k_e \phi N = \frac{60}{2\pi} k_e \phi \omega$$

$$E_b = k_b \omega$$

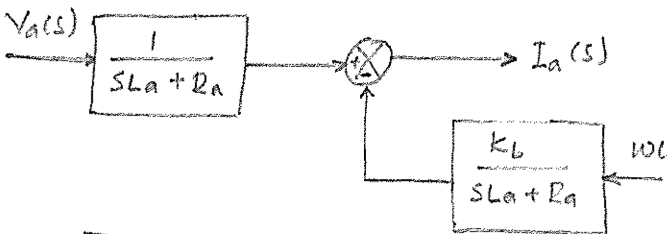
From input section.

$$V_a = R_a i_a + L_a \frac{di_a}{dt} + E_b$$

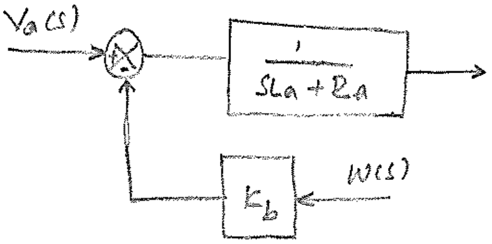
$$V_a(s) = R_a I_a(s) + sL_a I_a(s) + E_b(s)$$

$$= \left[R_a + sL_a\right] I_a(s) + k_b \omega(s)$$

$$I_a(s) = \frac{V_a(s)}{sL_a + R_a} - \frac{k_b}{sL_a + R_a}\omega(s)$$



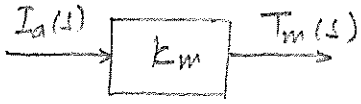
$$= \frac{V_a(s) - k_b \omega(s)}{sL_a + R_a}$$



From torque eqn

i) $T_m = k_m \phi I_a = (k_m k_f I_f) I_a = K_m I_a$, $I_f = \text{const.}$

$T_m(s) = K_m I_a(s)$

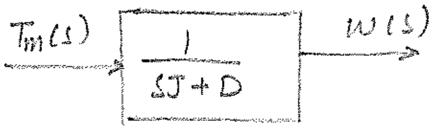


ii) $T_m = J\dfrac{d^2\theta}{dt^2} + D\dfrac{d\theta}{dt}$ and $\omega = \dfrac{d\theta}{dt}$

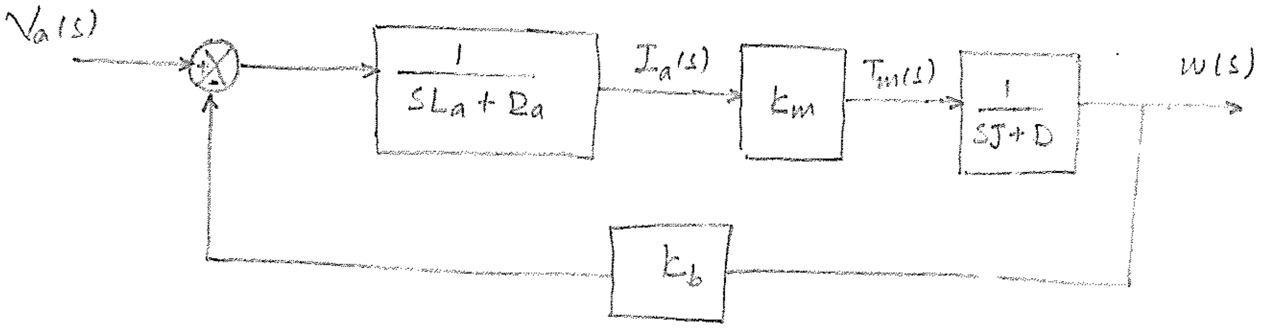
$$= J\frac{d\omega}{dt} + D\omega$$

$$T_m(s) = SJ\omega(s) + D\omega(s)$$
$$= (SJ + D)\,\omega(s)$$

$$\omega(s) = \frac{T_m(s)}{SJ+D}$$



The overall block diagram representation of an Armature-controlled DC-motor.

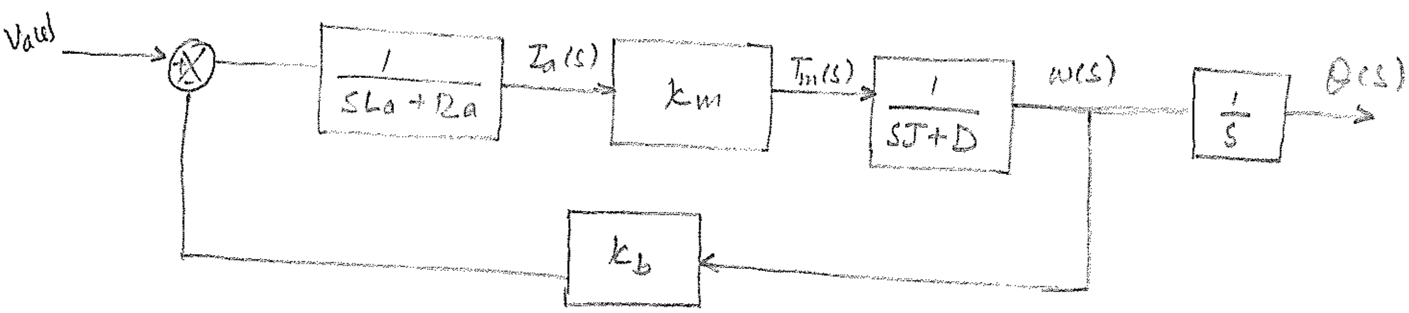


The angular position $\theta(s)$ is expressed as

$\omega(s) = s\,\theta(s)$

$\Rightarrow \theta(s) = \frac{1}{s}\omega(s)$

The above block diagram becomes.



Block diagram Reduction.

A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangment.

Simplification of the block diagram by rearrangments considerably reduces the labor needed for subsequent mathematical analyses.

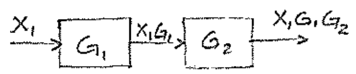
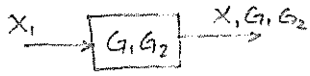
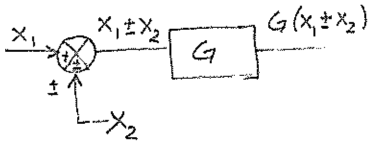
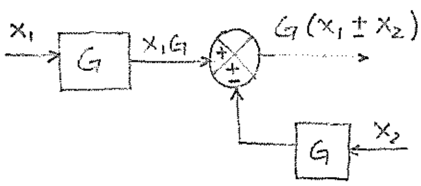
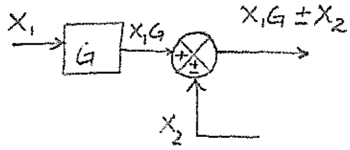
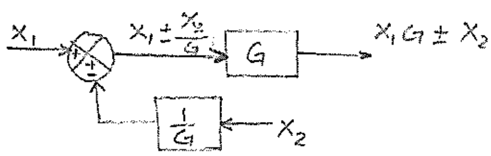
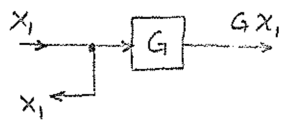
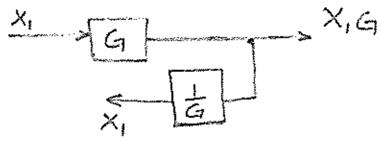
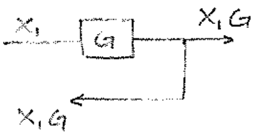
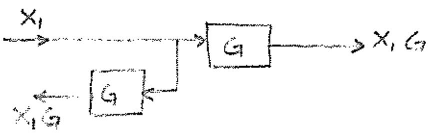
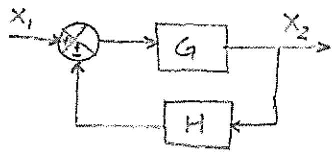
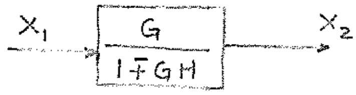
In simplifying a block diagram, remember the following.

1. The product of the transfer function in the feedforward direction must remain the same.
2. The product of the transfer function arround the loop must remain the same.

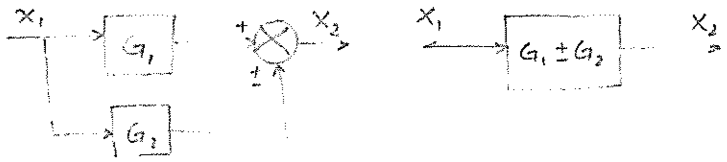
Rearrangments of the block diagrams is done using the rule of block diagram algebra. All these rules are drived by simple algebraic manipulation of the equation representing the block.

Some of the important are given below.

Rules of Block diagram Algebra

Rule	Origional Diagram	Equivalent diagram
1. Combining blocks in Cascade		
2. Moving a summing point after a block		
3. Moving a summing point a head of a block		
4. Moving a take off point after a block		
5. Moving a take off point a head of a block		
6. Eliminating a feedback loop.		

7. Combining blocks in parallel; or eliminating a forward loop



The following general steps may be used as a basic approach in the reduction of complicated block diagrams. Each step refers to specific transformation listed in the above table.

Step 1: Combine all cascaded blocks using transformation 1

2: Combine all parallel blocks using transformation 2

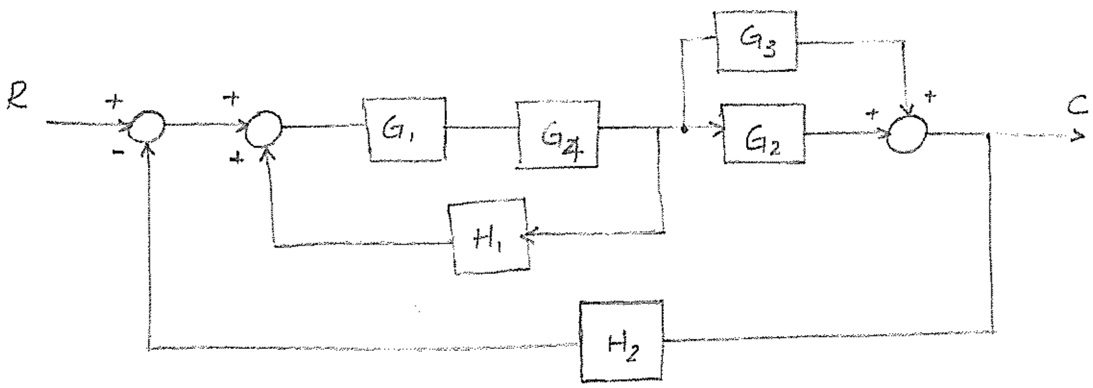
3: Eliminate all minor feedback loops using transformation 6

4: Shift summing points to the left and take off points to the right of the major loop, using transformation 3, 4.

5: Repeate step 1 to 4 until canonical form has been achived for a particular point.

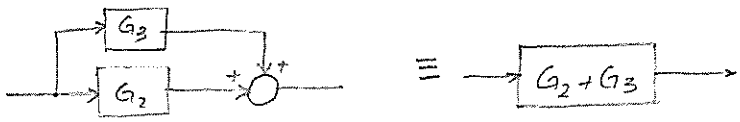
6: For multiple input, repeate 1 to 5 for each input, as required.

Eg. Reduce the block diagram to canonical form.

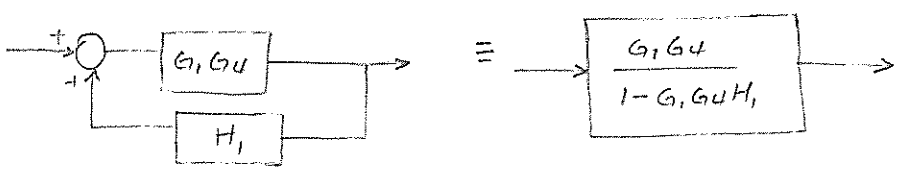


Step 1:

Step 2:

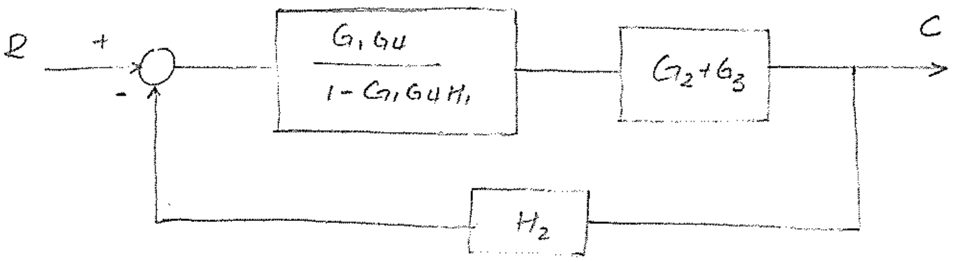


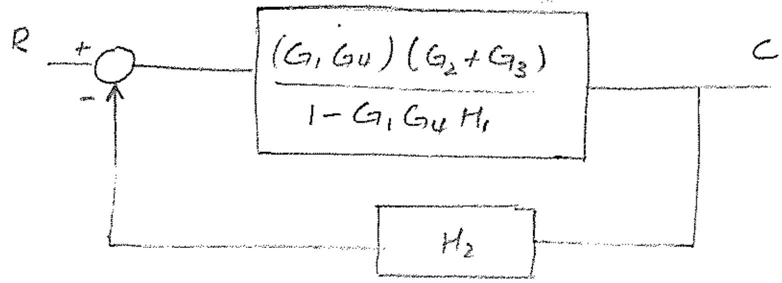
Step 3:



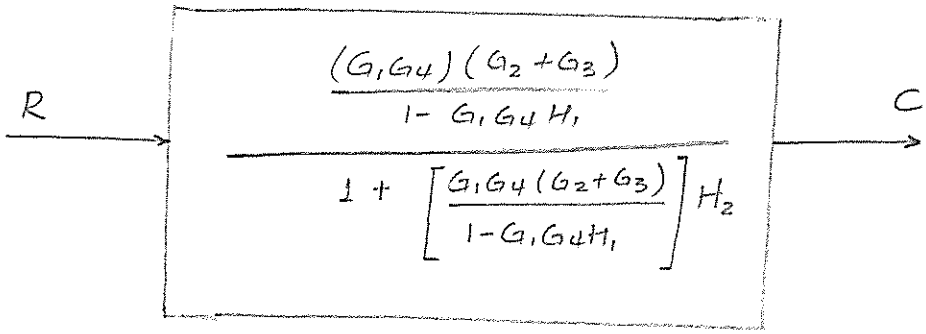
Step 4: Does not apply

Step 5:

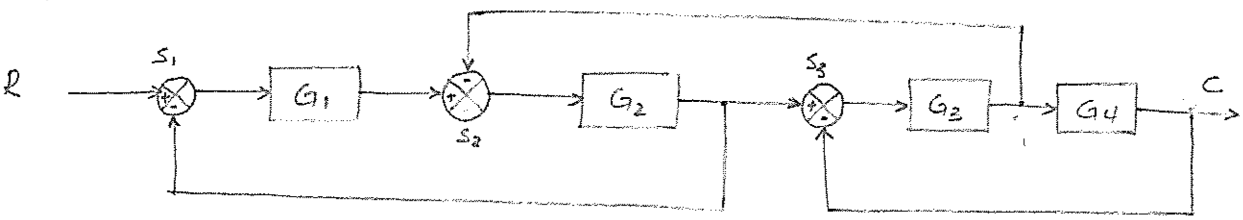




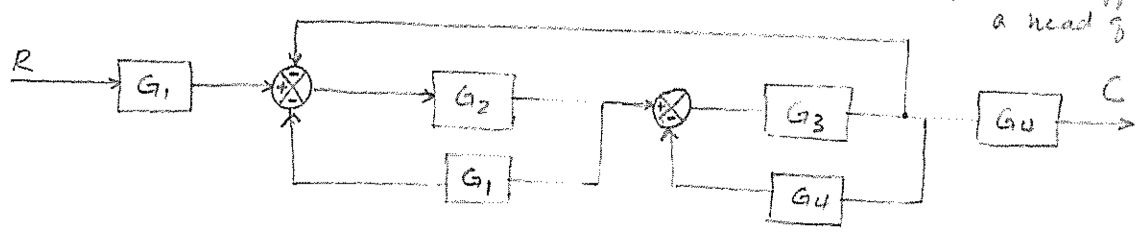
Cannonical form



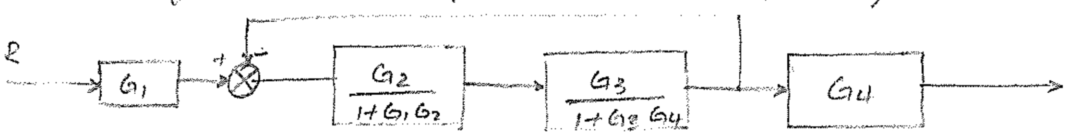
Eg. Simplify the block diagram shown below and obtain the closed loop system transfer function.



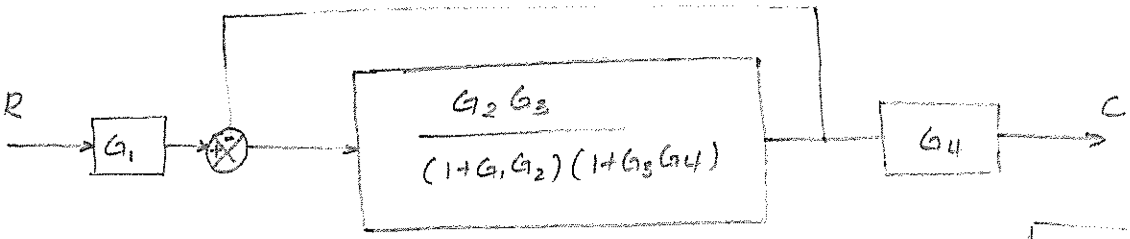
1. Moving a summing point (S_1) behind a block And moving a pick off point a head of a block.



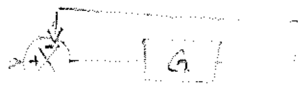
2. Eliminating feed back loop (G_1, G_2 and G_3, G_4)



3. combining cascaded blocks

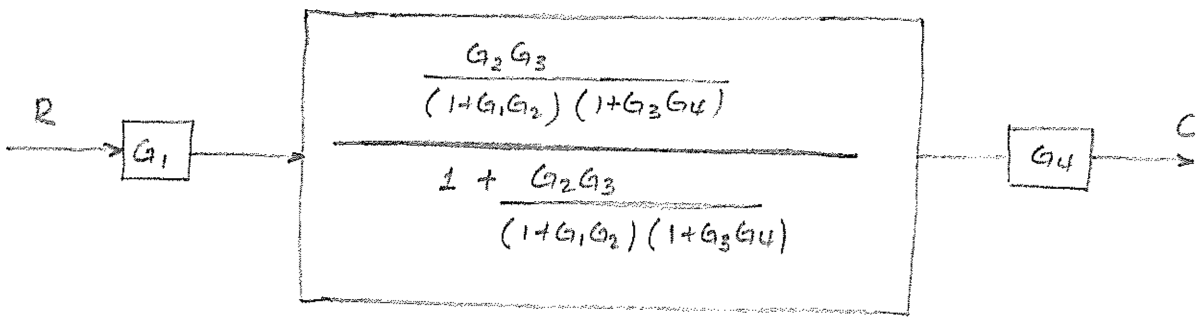


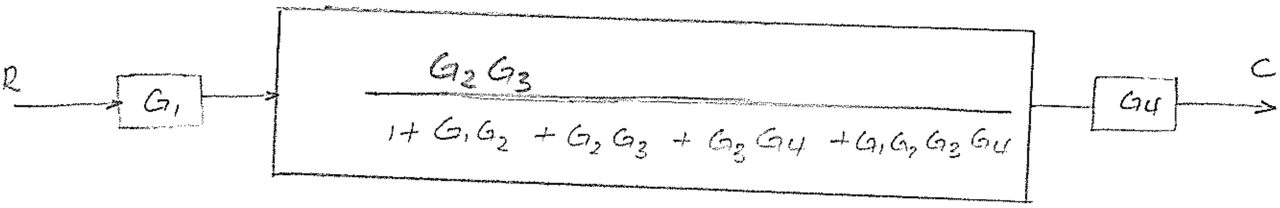
4. Eliminating the unity feedback

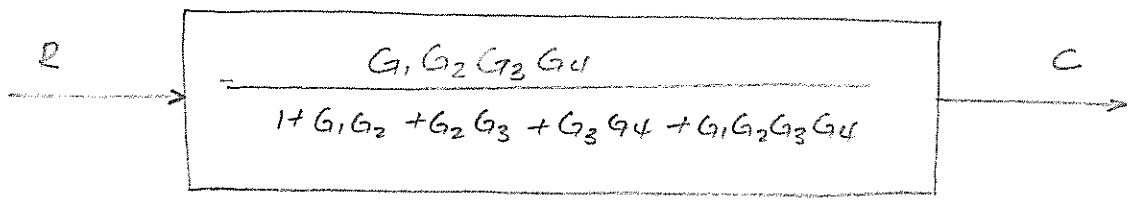


$$= \frac{G}{1+G}$$

5.





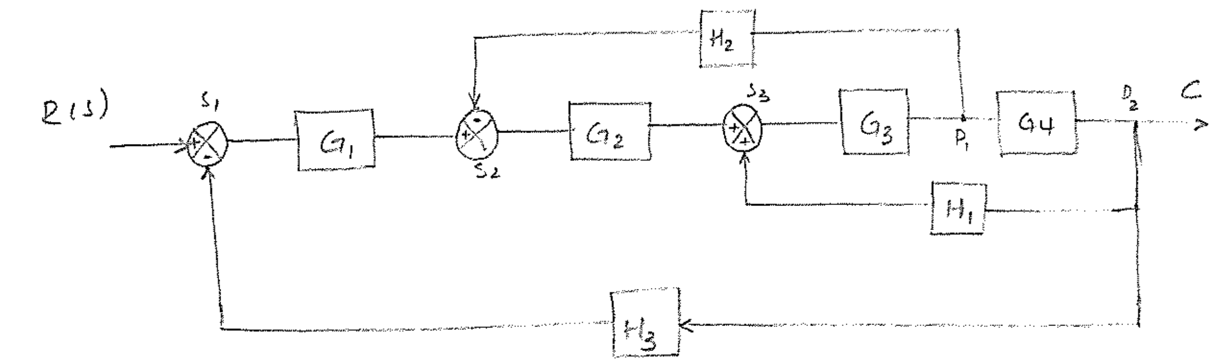


6. Transfer function

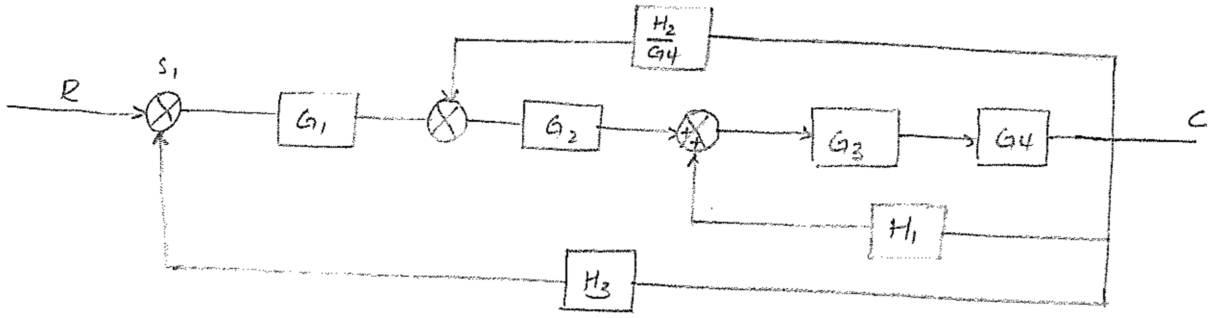
$$T = \frac{C}{R} = \frac{G_1 G_2 G_3 G_4}{1+G_1G_2+G_2G_3+G_3G_4+G_1G_2G_3G_4}$$

//

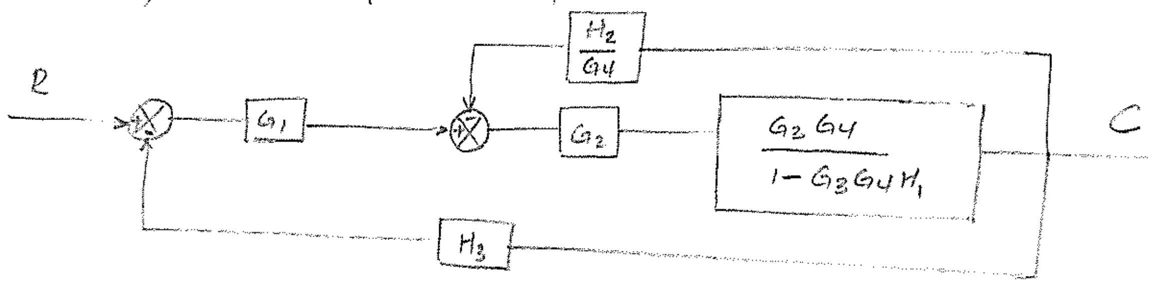
Eg. Simplify the multiple feedback control system block diagram shown below and obtain the closed-loop transfer function.



1. Moving the pick off point (P_1) after the Block G_4

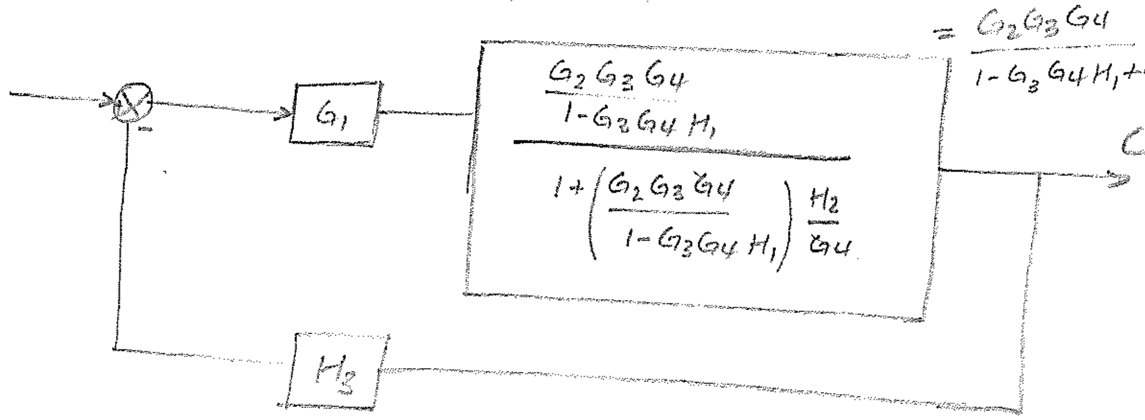


2. Combining blocks connected in cascade G_3 and G_4 and eliminating the positive feedback loop involving H_1

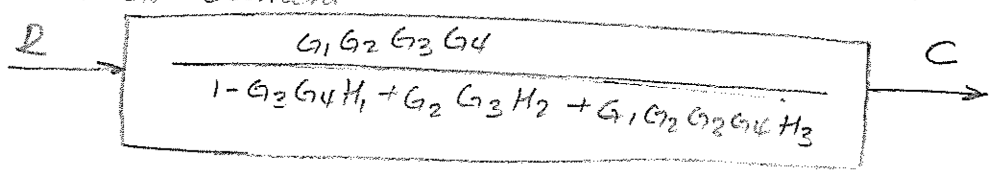


3. Combining the cascaded blocks G_2 and $\frac{G_3 G_4}{1-G_3G_4H_1}$ and eliminating the feedback element involving H_2/G_4

$$= \frac{G_2 G_3 G_4}{1-G_3G_4H_1+G_2G_3H_2}$$



4- Eliminating the feedback element



Signal-Flow Graphs (SFG) Models.

Block diagrams are very useful for representing control systems, but for complicated systems, the block diagram reduction process is tedious and time consuming. An alternate approach is that of signal flow graphs developed by S.J. Mason, which does not require any reduction process because of availability of a flow graph gain formula which relates the input and output system variables.

A signal flow graph is a graphical representation of the relationships between the variables of a set of linear algebraic equations.
It consists of a network in which nodes representing each of the system variables are connected by direct branches.

The basic elements of SFG are as follows.

1- Branch: it is a unidirectional path segment which relates the dependency of input and output variable in a manner of equivalent to a Block of a block-diagram.
A signal travels along a branch from one node to another in the direction of indicated by the branch arrow and in the process gets multiplied by the gain of the branch. $x_j = A_{ij} x_i$

2- Node: the input and output point or junction.
- it represents a system variable which is equal to the sum of incoming signals at the node.
- outgoing signals from the node do not affect the value of the node variable.

3- Path: it is the traversal of connected branches in the direction of the branch arrows such as no node is traversed more than once.

4- Input node or source: it is a node with only outgoing branches

5- Output node or sink: it is a node with only incoming branches.

6- Forward path: it is a path from the input node to the output node.

7- loop: It is a path which orginates and terminates at the same node

8. Non-touching loop: Loops are said to be non-touching if they do not posses any common node.

(Two-touching loops share one or more common nodes)

9. Forward path gain: It is the product of branch gains encountered in ~~trans~~ traversing a forward path.

10- Loop gain: It is the product of the branch gain encountered on traversing the loop.

The flow graph is simply a pictorial method of writing a system of algebraic eqns so that ~~the interdependencies~~ so as to indicate the enterdependencies of the Variables.

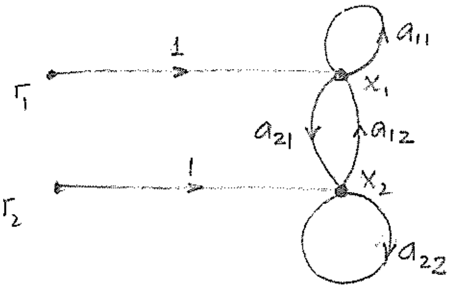
Eg. consider a set of simultaneous algebraic equations

input r_2
output x_2

$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1$$

$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2$$

Draw the signal-flow graph using the two inputs variables r_1 and r_2 and output variables are x_1 and x_2



SFG of the algebraic eqn

Description

1- r_1 and r_2 are input nodes, while x_1 and x_2 are output nodes

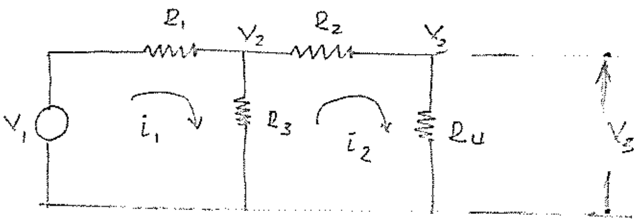
2- The unidirectional path segments b/n the nodes r_1 and x_1, r_2 and x_2, x_1 and x_2, are branches

3. The loop existing b/n the nodes x_1 and x_2 is $a_{21}a_{12}$. similarly a_{11} and a_{22} are the loops existing at the nodes x_1 and x_2 respectively

4. The loops $\left\{ \begin{array}{l} a_{11} \text{ and } a_{21}a_{12} \\ a_{22} \text{ and } a_{21}a_{12} \end{array} \right.$ are touching loops

a_{11} and a_{22} are non-touching.

Example construct a signal flow graph for the simple Resistance network given below.



1- There are five Variables
Voltage :- V_1 , V_2 and V_3
current :- i_1 and i_2

2. V_1 is known input quantity and write four independent equations using KCL and KVL

a) To find current i_1

$$i_1 = \frac{V_1 - V_2}{R_1} = \left(\frac{1}{R_1}\right) V_1 - \left(\frac{1}{R_1}\right) V_2 \quad \text{---} (1)$$

b) To find Voltage V_2

$$V_2 = (i_1 - i_2) R_3$$
$$(R_3)\, i_1 - (R_3)\, i_2 \quad \text{----} (2)$$

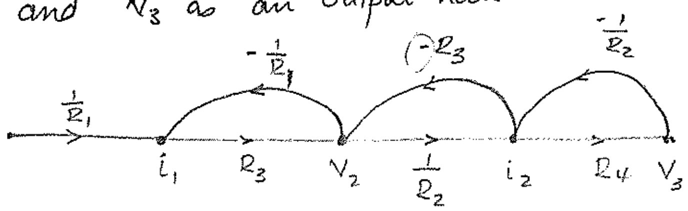
c) To find current i_2

$$i_2 = \frac{V_2 - V_3}{R_2} = \left(\frac{1}{R_2}\right) V_2 - \left(\frac{1}{R_2}\right) V_3 \quad \text{---} (3)$$

d) To find Voltage V_3

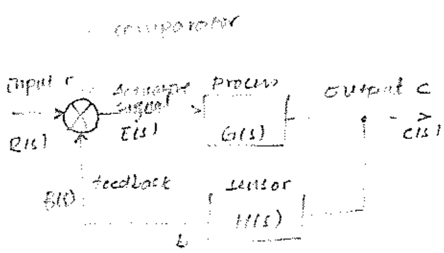
$$V_3 = R_4\, i_2 \quad \text{----} (4)$$

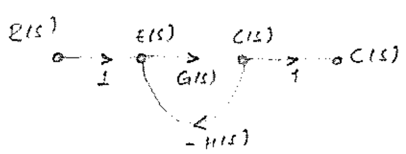
3. Laying out the five nodes in the same order with V_1 as an input Node and V_3 as an output node.



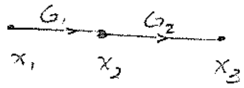
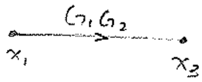
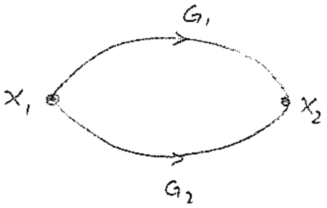
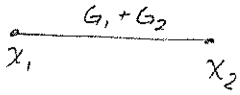
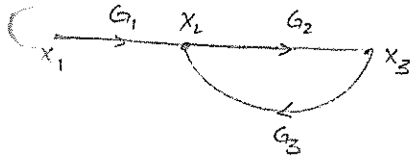
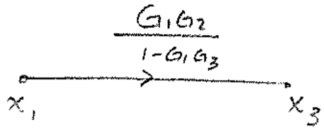
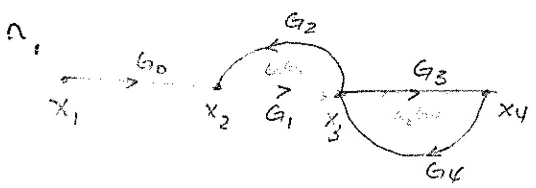
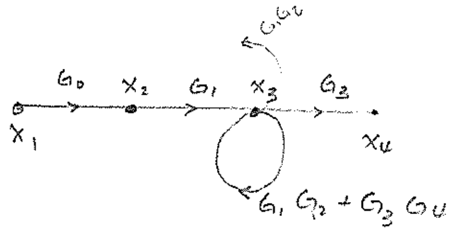
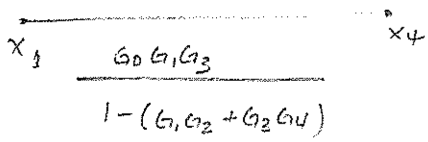
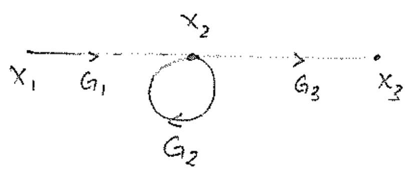
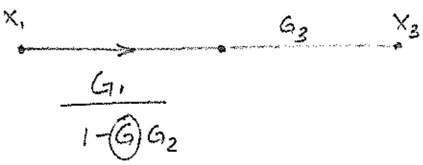
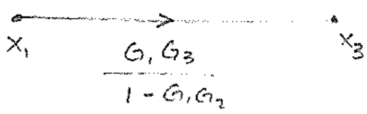
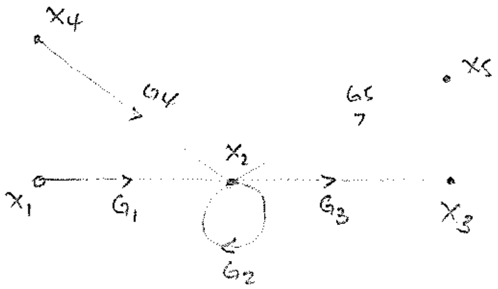
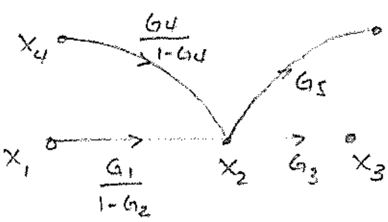
We have one forward-path and three feedback loops.

e.g.





Signal flow graph Reduction Rules.

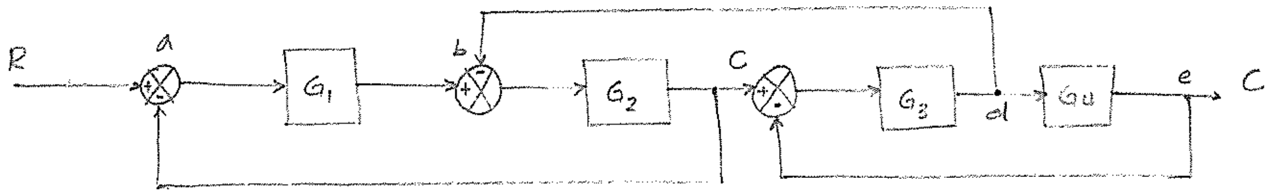
Origional Graph	Equivalent Reduced graph	Explantation
		Cascaded Transfer
		Parallel trans
		Elimination of a loop
	 	Elimination of cascaded loop
	 	Elimination of a self loop
		Elimination of self-loop

Example.

Consider the multiple-loop feed-back control system shown below.

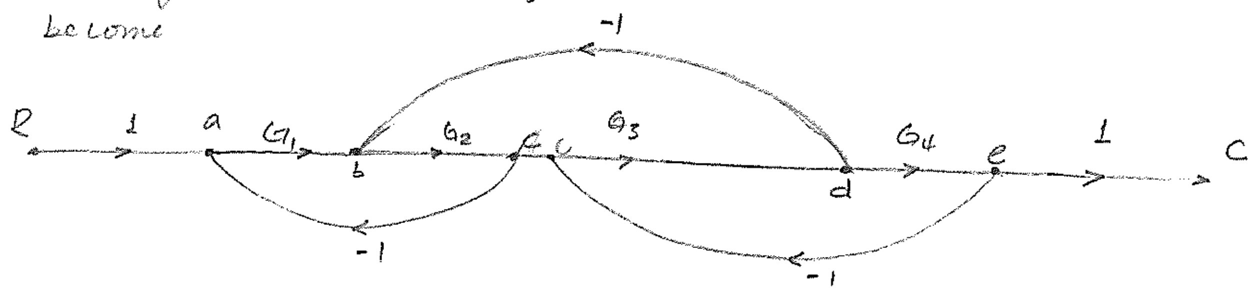
a) Convert the block-diagram into signal flow-diagram

b) Obtain the closed-loop transfer function.

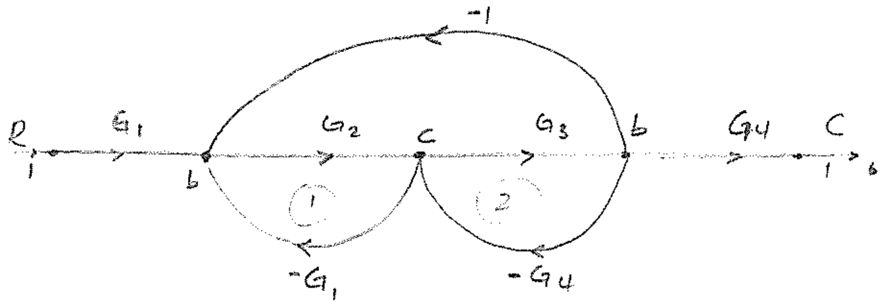


Sol/n

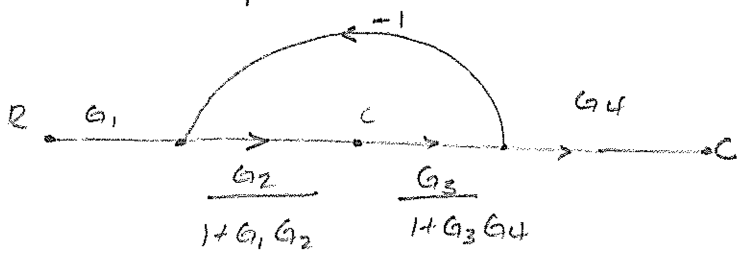
1- The signal-flow-graph of the multiple loop feedback control system become



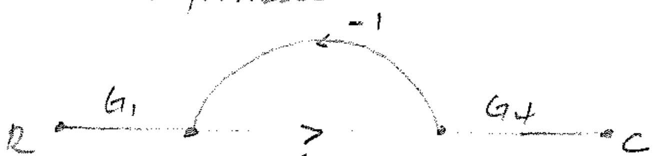
2- Move node a behind node b and also 'e' to 'd'



3- Eliminate loop (1) and (2)



4. cascade transformation



$$G = \frac{G_2 G_3}{(1+G_1G_2)(1+G_3G_4)}$$

5. Elimination of a loop

$$R \xrightarrow{G_1} \cdot \xrightarrow{\frac{G}{1+G}} \cdot \xrightarrow{G_4} C$$

6. cascaded transformation

where
$$\frac{G}{1+G} = \frac{\dfrac{G_2 G_3}{(1+G_1G_2)(1+G_3G_4)}}{1+\dfrac{G_2G_3}{(1+G_1G_2)(1+G_3G_4)}}$$

$$R \xrightarrow{G_1\left(\frac{G}{1+G}\right)G_4} C$$

$$\frac{C}{R} = \frac{G_1 G G_4}{1+G} = \frac{G_1 G_2 G_3 G_4}{1+G_1G_2+G_2G_3+G_3G_4+G_1G_2G_3G_4}$$

Mason's Gain Formula

The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between the input and output nodes and is known as the overall gain of the system

Mason's gain formula for the determination of the overall system gain is given

$$T(s) = \frac{C(s)}{R(s)} = \sum_{k=1}^{N} \frac{P_k \Delta_k}{\Delta}$$

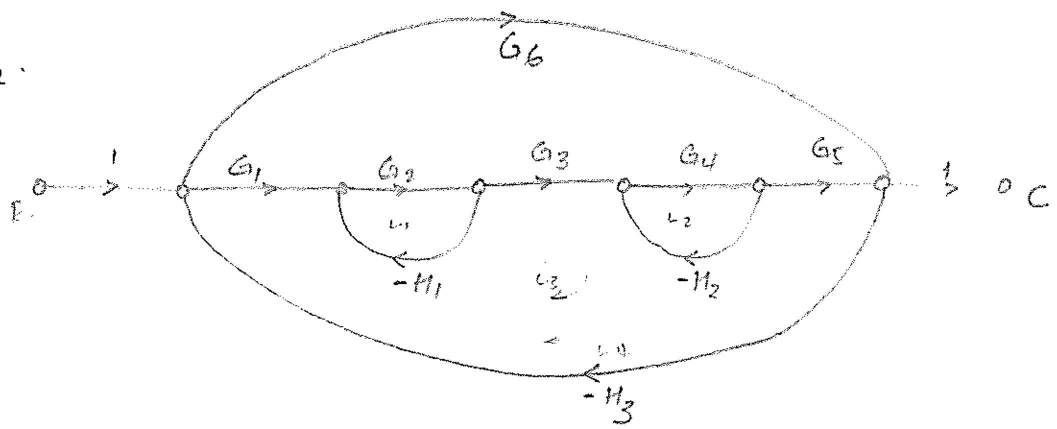
Where

P_k = Transfer function of the k^{th} forward path (from the input Node to Output Node)

Δ = determinant of the graph
= 1 − [sum of all individual loop gain]
+ [sum of the gain product of all combination of two non-touching loops]
− [sum of the gain product of all combination of three Non-touching loops]
+ - - - -

Δ_k = Cofactor of the path P_k
= All terms in Δ that does not have elements or paths common with an element or path in P_k

Example:

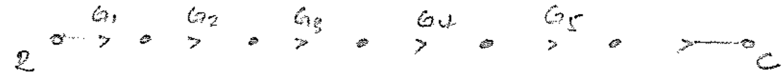


Determine the transfer function of the system shown above using Mason's gain formula.

Sol^n

1) There are two forward paths

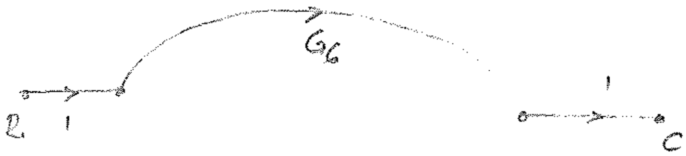
a)



Gain of forward path

$P_1 = G_1 G_2 G_3 G_4 G_5$

b)

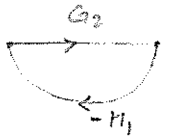


Gain of forward path

$P_2 = G_6$

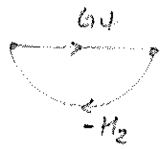
2) There are four loops

1) loop 1



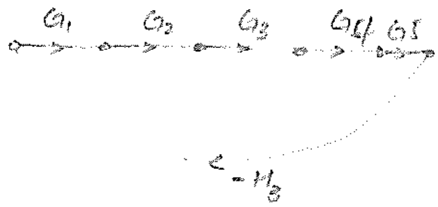
loop gain $L_1 = -G_2 H_1$

2) loop 2



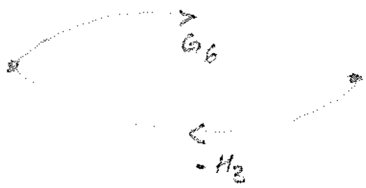
loop gain $L_2 = -G_4 H_2$

3) loop 3



loop gain $L_3 = -G_1 G_2 G_3 G_4 G_5 H_3$

4) loop 4



loop gain $L_4 = -G_6 H_3$

3) Gain-product of combination of two non touching loops

Out of Four loops loop 1, loop 2, and loop 4 are non touching combination of two loops

i) loop 1 and loop 2 : loop gain $L_{12} = L_1 * L_2 = G_2 G_4 H_1 H_2$

ii) loop 1 and loop 4 : loop gain $L_{14} = L_1 * L_4 = G_2 G_6 H_1 H_3$

iii) loop 2 and loop 4 : loop gain $L_{24} = L_2 * L_4 = G_4 G_6 H_2 H_3$

4) Gain product of all combination three non touching loops

loop 1, loop 2 and loop 4 are possible combination of three non touching loops.

$\rightarrow$ loop gain $L_{124} = L_1 \times L_2 * L_4$

$$= (-G_2H_1)(-G_4H_2)(-G_6H_3)$$

$$= -G_2G_4G_6H_1H_2H_3$$

5. Determinant of the graph (Δ)

$\Delta = 1 -$ [sum of all individual loop gains]

+ [sum of the gain product of all combination of two nontouching loops]

− [sum of the gain product of all combination of three nontouching loops]

$$= 1 - [L_1 + L_2 + L_3 + L_4] + [L_{12} + L_{14} + L_{24}] - [L_{124}]$$

$$= 1 - [-G_2H_1 - G_4H_2 - G_1G_2G_3G_4G_5H_3 - G_6H_3] + [G_2G_4H_1H_2 + G_2G_6H_1H_3 + G_4G_6H_2H_3] - [-G_2G_4G_6H_1H_2H_3]$$

6. To determine the cofactor of the path P_k (= All terms in Δ that do not have elements or common path in P_k)

i) considering forward path 1 = P_1,

loops 1, 2, 3 and 4 touches forward path P_1

Eliminating these loops from the determinant Δ

$$\Delta_1 = 1 - [\cancel{L_1 + L_2 + L_3 + L_4}]^{0} + [\cancel{L_1L_2 + L_1L_4 + L_2L_4}]^{0} - [\cancel{L_1L_2L_3}]^{0}$$

$$= 1 - 0$$

ii) considering forward path 2 = P_2

loops 1 and loop 2 do not touch P_2, but loop 3 and 4 touches it

Eliminating loop 3 and 4

$$\Delta_2 = 1 - [L_1 + L_2 + \cancel{L_3} + \cancel{L_4}] + [L_1L_2 + \cancel{L_1L_4} + \cancel{L_2L_4}] - [\cancel{L_1L_2L_3}]$$

$$= 1 - [L_1 + L_2] + [L_1L_2]$$

$$= 1 - (-G_2H_1 - G_4H_2) + (G_2G_4H_1H_2)$$

7. The transfer function of the system.

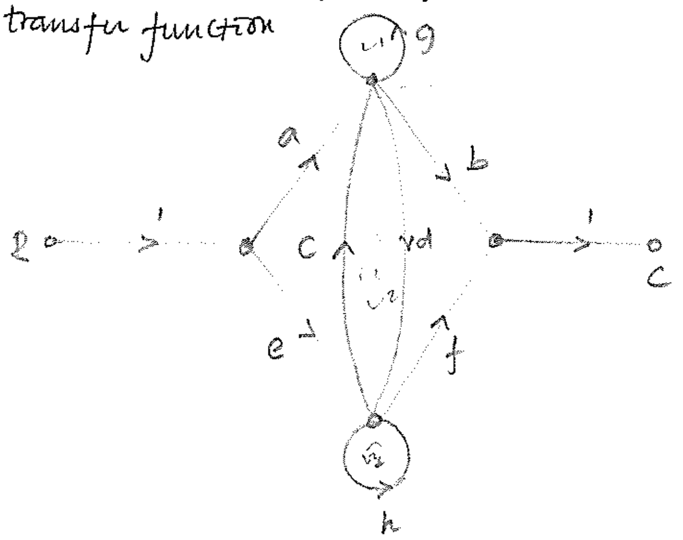
$$T(s) = \frac{C(s)}{R(s)} = \sum_{k=1}^{N} \frac{P_k \Delta_k}{\Delta} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{P_1\Delta_1}{\Delta} + \frac{P_2\Delta_2}{\Delta}$$

N = 2 two forward paths

$$= \frac{G_1G_2G_3G_4G_5 + G_6(1 + G_2H_1 + G_4H_2 + G_2G_4H_1H_2)}{1 + G_2H_1 + G_4H_2 + G_1G_2G_3G_4G_5H_3 + G_6H_3 + G_2G_4H_1H_2 + G_2G_6H_1H_3 + G_4G_6H_2H_3 + G_2G_4G_6H_1H_2H_3}$$

Example.

Consider the signal flow graph shown below and determine the transfer function



1) There are four forward path

$T_1 = ab$, $T_2 = ef$, $T_3 = adf$ $T_4 = ecb$

2) There are three loops

$L_1 = g$ $L_2 = cd$, $L_3 = h$

3) out of three loops loop 1 and loop 3 are two-non-touching loops

$L_{13} = L_1 L_3 = gh$

4. To find Δ

$$\Delta = 1 - \sum L + \sum L(\text{two-non-touching})$$
$$= 1 - (L_1 + L_2 + L_3) + (L_{13})$$
$$= 1 - (g + cd + h) + gh$$

5. To find Δ_k L_1, L_2

Tracing the forward path, eliminate the touching loop from Δ

$\Delta_1(T_1) = 1 - (L_1 + L_2 + L_3) + (L_1 L_3) = 1 - h$ only loop loop 3 do not touc

$\Delta_2(T_2) = 1 - (L_1 + L_2 + L_3) + (L_1 L_3)$ — only loop 1 do not touch

$= 1 - L_1 = 1 - g$

$\Delta_3(T_3) = 1 - (L_1 + L_2 + L_3) + (L_1 L_3)$ — all touch the path

$= 1$

$\Delta_4(T_4) = 1 - (L_1 + L_2 + L_3) + (L_1 L_3)$ — all touch the path

$= 1$

6. The transfer function becomes

$$T = \frac{C}{R} = \sum T_k \frac{\Delta_k}{\Delta} = T_1 \frac{\Delta_1}{\Delta} + T_2 \frac{\Delta_2}{\Delta} + T_3 \frac{\Delta_3}{\Delta} + T_4 \frac{\Delta_4}{\Delta}$$

$$= \frac{ab(1-h) + ef(1-g) + adf + ecb}{1 - (g + cd + h) + gh}$$