# Chapter 3: Z-transforms and its Implementation

# 3 Z-transforms and its Implementation

## 3.1 Introduction

- The z-transform is a useful tool in the analysis of discrete-time signals and systems.
- Z-transform is the discrete-time counterpart of the Laplace transform for continuous-time signals and systems.
- Z-transform may be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input, and design linear filters.

## 3.2 Z-transforms

We saw that the discrete-time Fourier transform (DTFT) of a sequence x(n) is equal to the sum

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega}$$

Unfortunately, many of the signals that we would like to consider are not absolutely summable and, therefore, do not have a DTFT. Some examples include

$$x(n) = u(n) \qquad x(n) = (0.5)^n u(-n) \qquad x(n) = \sin n\omega_0$$

Z-transform is a generalization of the DTFT that allows one to deal with such sequences and is defined as follows:

**The z-transform of a discrete-time signal x(n) is defined by**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

**where $z = re^{j\omega}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the region of convergence (ROC).**

**Notationally, if x(n) has a z-transform X(z), we write**

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

The z-transform may be viewed as the DTFT of an exponentially weighted sequence. Specifically, note that with $z = re^{j\omega}$,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} [r^{-n} x(n)] e^{-jn\omega}$$

and we see that X(z) is the discrete-time Fourier transform of the sequence $r^{-n}x(n)$. Furthermore, the ROC is determined by the range of values of r for which

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

Because the z-transform is a function of a complex variable, it is convenient to describe it using the complex z-plane. With

$$z = \mathrm{Re}(z) + j\,\mathrm{Im}(z) = r e^{j\omega}$$

- The axes of the z-plane are the real and imaginary parts of z as illustrated in Fig. 3.1, and the contour corresponding to $|z| = 1$ is a circle of unit radius referred to as the **unit circle**.
- The z-transform evaluated on the unit circle corresponds to the DTFT,

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

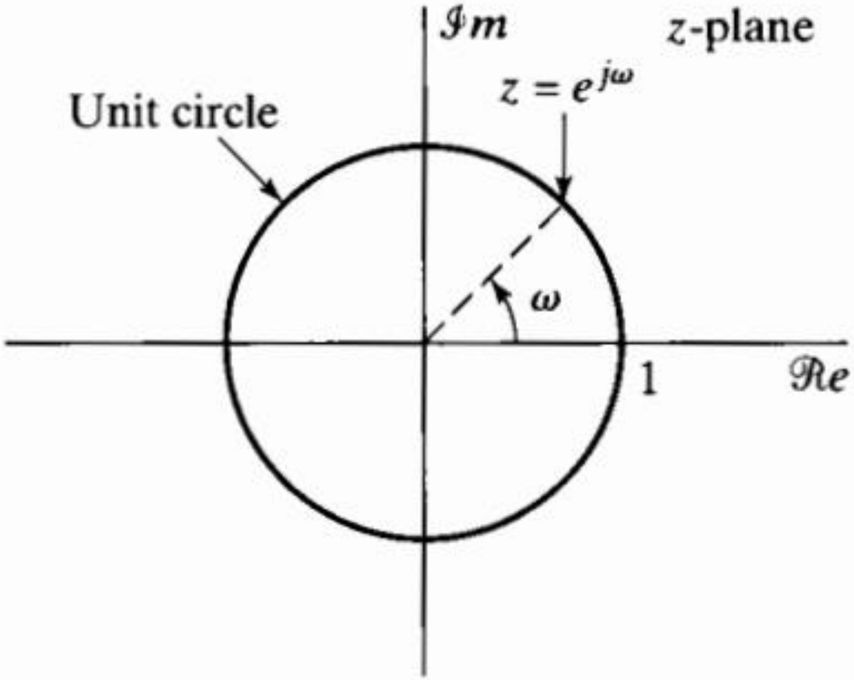



Fig. 3.1. The unit circle in the complex z-plane.

- If $\alpha = 0$, the ROC may also include the point $z = 0$, and if $\beta = \infty$, the ROC may also include infinity. For a rational X(z), the region of convergence will contain **no poles.**
- The **three properties** of the region of convergence:

1. A finite-length sequence has a $z$-transform with a region of convergence that includes the entire $z$-plane except, possibly, $z = 0$ and $z = \infty$. The point $z = \infty$ will be included if $x(n) = 0$ for $n < 0$, and the point $z = 0$ will be included if $x(n) = 0$ for $n > 0$.
2. A right-sided sequence has a $z$-transform with a region of convergence that is the *exterior* of a circle:

$$\text{ROC} : |z| > \alpha$$

3. A left-sided sequence has a $z$-transform with a region of convergence that is the *interior* of a circle:

$$\text{ROC} : |z| < \beta$$

## Example 3.2 **Right-Sided Exponential Sequence**

Consider the signal $x[n] = a^n u[n]$. Because it is nonzero only for $n \geq 0$, this is an example of a *right-sided* sequence. From Eq. (3.2),

$$X(z) = \sum_{n=-\infty}^{\infty} = a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that

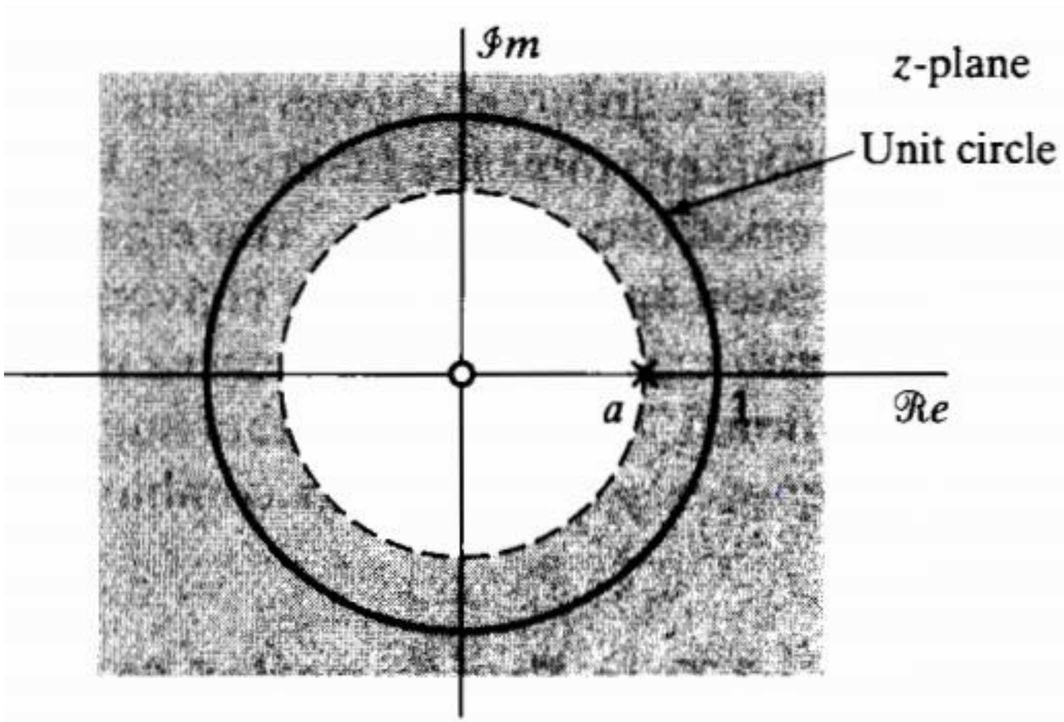
$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty.$$

Thus, the region of convergence is the range of values of $z$ for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| > |a|.$$

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1.$$

we see that $X(z)$ has a zero at $z = 0$ and a pole at $z = \alpha$. A pole-zero diagram with the region of convergence is shown in the figure below.




Note that if $|\alpha| < 1$, the unit circle is included within the region of convergence, and the DTFT of $x(n)$ exists.

# Example 3.3 Left-Sided Exponential Sequence

Now let $x[n] = -a^n u[-n-1]$. Since the sequence is nonzero only for $n \le -1$, this is a *left-sided* sequence. Then

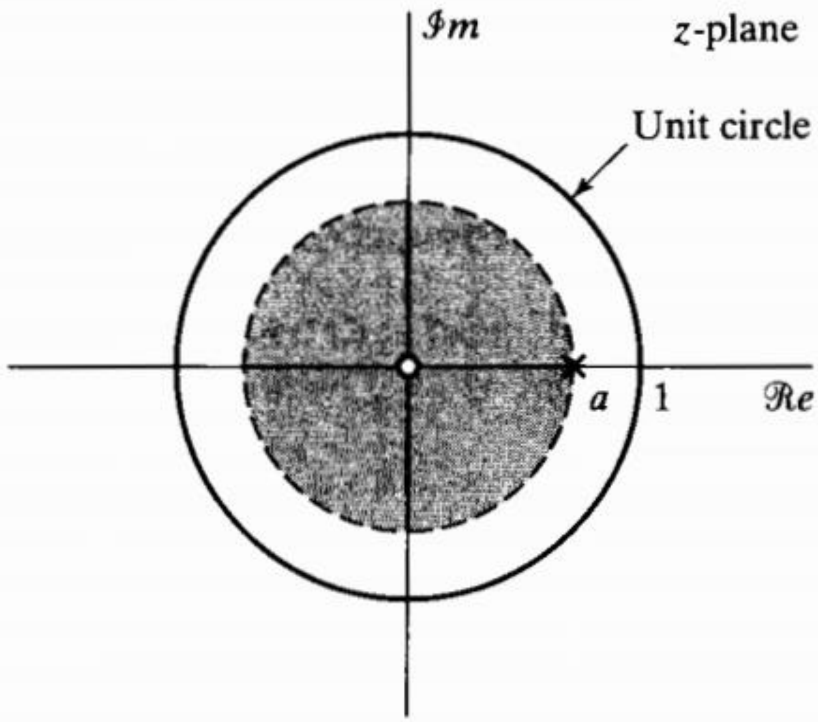
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n.$$

If $|a^{-1} z| < 1$ or, equivalently, $|z| < |a|$, the sum converges, and

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|.$$

with the sum converging if $|\alpha^{-1}z| < 1$ or $|z| < |\alpha|$. A pole-zero diagram with the region of convergence indicated is given in the figure below.




Note that for $|a| < 1$, the sequence $-a^n u[-n-1]$ grows exponentially as $n \to -\infty$, and thus, the Fourier transform does not exist.

# Table 3-1 Common z-Transform Pairs

| Sequence | $z$-Transform | Region of Convergence |
|---|---|---|
| $\delta(n)$ | $1$ | all $z$ |
| $\alpha^n u(n)$ | $\dfrac{1}{1-\alpha z^{-1}}$ | $\|z\| > \|\alpha\|$ |
| $-\alpha^n u(-n-1)$ | $\dfrac{1}{1-\alpha z^{-1}}$ | $\|z\| < \|\alpha\|$ |
| $n\alpha^n u(n)$ | $\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z\| > \|\alpha\|$ |
| $-n\alpha^n u(-n-1)$ | $\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ | $\|z\| < \|\alpha\|$ |
| $\cos(n\omega_0)u(n)$ | $\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |
| $\sin(n\omega_0)u(n)$ | $\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$ | $\|z\| > 1$ |

- Just as with the DTFT, there are a number of important and useful **z-transform properties**. A few of these properties are described below.

## 3.2.1 Linearity

As with the DTFT, the z-transform is a **linear operator**. Therefore, if x(n) has a z-transform X(z) with a region of convergence $R_x$ and if y(n) has a z-transform Y(z) with a region of convergence $R_y$

$$w(n) = ax(n) + by(n) \stackrel{Z}{\longleftrightarrow} W(z) = aX(z) + bY(z)$$

and the ROC of $w(n)$ will *include* the intersection of $R_x$ and $R_y$, that is,

$$R_w \text{ contains } R_x \cap R_y$$

- ❖ However, the region of convergence of $W(z)$ may be **larger.** For example, if $x(n) = u(n)$ and $y(n) = u(n - 1)$, the ROC of $X(z)$ and $Y(z)$ is $|z| > 1$.
- ❖ However, the z-transform of $\omega(n) = x(n) - y(n) = \delta(n)$ is the **entire** z-plane.

## 3.2.2 Shifting Property

- ✓ Shifting a sequence (delaying or advancing) multiplies the z-transform by a **power of z**. That is to say, if $x(n)$ has a z-transform $X(z)$,

$$x(n - n_0) \stackrel{Z}{\longleftrightarrow} z^{-n_0} X(z)$$

- ➢ Because shifting a sequence **does not affect** its absolute summability, shifting does **not change** the region of convergence.
- ➢ Therefore, the z-transforms of $s(n)$ and $x(n - no)$ have the **same region of convergence**, with the possible exception of adding or deleting the points $z = 0$ and $z = \infty$.

## 3.2.3 Time Reversal

If x(n) has a z-transform X(z) with a region of convergence $R_x$ that is the annulus $\alpha < |z| < \beta$, the z-transform of the time-reversed sequence x(- n) is

$$x(-n) \stackrel{Z}{\longleftrightarrow} X(z^{-1})$$

and has a region of convergence $1/\beta < |z| < 1/\alpha$, which is denoted by $1/R_x$.

## 3.2.4 Multiplication by an Exponential

If a sequence x(n) is multiplied by a complex exponential $\alpha^n$,

$$\alpha^n x(n) \stackrel{Z}{\longleftrightarrow} X(\alpha^{-1} z)$$

This corresponds to a scaling of the z-plane. If the region of convergence of X(z) is $r_- < |z| < r_+$ which will be denoted by $R_x$ the region of convergence of $X(\alpha^{-1}z)$ is $|\alpha|r_- < |z| < |\alpha|r_+$, which is denoted by $|\alpha| R_x$. As a special case, note that if x(n) is multiplied by a complex exponential. $e^{jn\omega_0}$, which corresponds to a rotation of the z-plane.

$$e^{jn\omega_0}x(n) \stackrel{Z}{\longleftrightarrow} X(e^{-j\omega_0}z)$$

### 3.2.5 Convolution Theorem

- The most important z-transform property is the **convolution theorem**, which states that convolution in the time domain is mapped into multiplication in the frequency domain, that is,

$$y(n) = x(n) * h(n) \stackrel{Z}{\longleftrightarrow} Y(z) = X(z)H(z)$$

- The region of convergence of Y(z) includes the intersection of $R_x$ and $R_y$

$$R_w \text{ contains } R_x \cap R_y$$

❖ However, the region of convergence of Y(z) may be **larger**, if there is a pole-zero cancellation in the **product X(z)H(z).**

**Example 3.3** Consider the two sequences

$$x(n) = \alpha^n u(n) \qquad h(n) = \delta(n) - \alpha\delta(n-1)$$

The $z$-transform of $x(n)$ is

$$X(z) = \frac{1}{1 - \alpha z^{-1}} \qquad |z| > |\alpha|$$

and the $z$-transform of $h(n)$ is

$$H(z) = 1 - \alpha z^{-1} \qquad 0 < |z|$$

However, the $z$-transform of the convolution of $x(n)$ with $h(n)$ is

$$Y(z) = X(z)H(z) = \frac{1}{1 - \alpha z^{-1}} \cdot (1 - \alpha z^{-1}) = 1$$

which, due to a *pole-zero* cancellation, has a region of convergence that is the entire $z$-plane.

## 3.2.6 Conjugation

❖ If X(z) is the z-transform of x(n), the z-transform of the complex conjugate of x(n) is

$$x^*(n) \overset{z}{\longleftrightarrow} X^*(z^*)$$

As a corollary, note that if $x(n)$ is real-valued, $x(n) = x^*(n)$, then

$$X(z) = X^*(z^*)$$

## 3.2.7 Derivative

If X(z) is the z-transform of x(n), the z-transform of nx(n) is

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

❖ These properties are summarized in Table 3-2. As illustrated in the following example, these properties are useful in simplifying the evaluation of z-transforms.

## Table 3-2 Properties of the z-Transform

| Property | Sequence | z-Transform | Region of Convergence |
|---|---|---|---|
| Linearity | $ax(n)+by(n)$ | $aX(z)+bY(z)$ | Contains $R_x \cap R_y$ |
| Shift | $x(n-n_0)$ | $z^{-n_0}X(z)$ | $R_x$ |
| Time reversal | $x(-n)$ | $X(z^{-1})$ | $1/R_x$ |
| Exponentiation | $\alpha^n x(n)$ | $X(\alpha^{-1}z)$ | $\lvert\alpha\rvert R_x$ |
| Convolution | $x(n) * y(n)$ | $X(z)Y(z)$ | Contains $R_x \cap R_y$ |
| Conjugation | $x^*(n)$ | $X^*(z^*)$ | $R_x$ |
| Derivative | $nx(n)$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$ |

*Note:* Given the $z$-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence $R_x$ and $R_y$, respectively, this table lists the $z$-transforms of sequences that are formed from $x(n)$ and $y(n)$.

| Sequence | Transform | ROC |
|---|---|---|
| $x[n]$ | $X(z)$ | $R_x$ |
| $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| $x[n - n_0]$ | $z^{-n_0} X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| $z_0^n x[n]$ | $X(z/z_0)$ | $\lvert z_0 \rvert R_x$ |
| $nx[n]$ | $-z \dfrac{dX(z)}{dz}$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ | Contains $R_x$ |
| $\mathcal{J}m\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ | Contains $R_x$ |
| $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| Initial-value theorem: | | |
| $x[n] = 0, \quad n < 0$ | $\lim_{z \to \infty} X(z) = x[0]$ | |

## Example 3.4

Let us find the $z$-transform of $x(n) = n\alpha^n u(-n)$. To find $X(z)$, we will use the time-reversal and derivative properties.

$$\alpha^n u(n) \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}} \qquad |z| > \alpha$$

Therefore,

$$\left(\frac{1}{\alpha}\right)^n u(n) \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha^{-1} z^{-1}} \qquad |z| > \frac{1}{\alpha}$$

and, using the time-reversal property,

$$\alpha^n u(-n) \stackrel{Z}{\longleftrightarrow} \frac{1}{1-\alpha^{-1} z} \qquad |z| < \alpha$$

Finally, using the derivative property, it follows that the $z$-transform of $n\alpha^n u(-n)$ is

$$-z\frac{d}{dz}\frac{1}{1-\alpha^{-1}z} = -\frac{\alpha^{-1}z}{(1-\alpha^{-1}z)^2} \qquad |z| < \alpha$$

❖ **A property that may be used to find the initial value of a causal sequence from its z-transform is the initial value theorem.**

## 3.2.8 Initial Value Theorem

If $x(n)$ is equal to zero for $n < 0$, the initial value, $x(0)$, may be found from $X(z)$ as follows:

$$x(0) = \lim_{z \to \infty} X(z)$$

This property is a consequence of the fact that if $x(n) = 0$ for $n < 0$,

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$

Therefore, if we let $z \to \infty$, each term in $X(z)$ goes to zero except the first.

## 3.3 The Inverse Z-Transform

- The z-transform is a useful tool in linear systems analysis.
- For finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence x(n) from X(z).
- Three possible approaches are described below

## 3.3.1 Partial Fraction Expansion

**For z-transforms that are rational functions of z,**

$$X(z) = \frac{\sum_{k=0}^{q} b(k) z^{-k}}{\sum_{k=0}^{p} a(k) z^{-k}} = C \frac{\prod_{k=1}^{q}(1 - \beta_k z^{-1})}{\prod_{k=1}^{p}(1 - \alpha_k z^{-1})}$$

a simple and straightforward approach to find the inverse $z$-transform is to perform a partial fraction expansion of $X(z)$. Assuming that $p > q$, and that all of the roots in the denominator are simple, $\alpha_i \neq \alpha_k$ for $i \neq k$, $X(z)$ may be expanded as follows:

$$X(z) = \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}$$

for some constants $A_k$ for $k = 1, 2, \ldots, p$. The coefficients $A_k$ may be found by multiplying both sides

by $(1 - \alpha_k z^{-1})$ and setting $z = \alpha_k$. The result is

$$A_k = \left[(1 - \alpha_k z^{-1}) X(z)\right]_{z=\alpha_k}$$

If $p \leq q$, the partial fraction expansion must include a polynomial in $z^{-1}$ of order $(p-q)$. The coefficients of this polynomial may be found by long division (i.e., by dividing the numerator polynomial by the denominator). For multiple-order poles, the expansion must be modified. For example, if $X(z)$ has a second-order pole at $z = \alpha_k$, the expansion will include two terms,

$$\frac{B_1}{1-\alpha_k z^{-1}} + \frac{B_2}{(1-\alpha_k z^{-1})^2}$$

where $B_1$ and $B_2$ are given by

$$B_1 = \alpha_k \left[ \frac{d}{dz}(1-\alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

$$B_2 = \left[ (1-\alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

## 3.3.2 Power Series

**The z-transform is a power series expansion,**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

where the sequence values $x(n)$ are the coefficients of $z^{-n}$ in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$ may be found by simply picking off the coefficients of $z^{-n}$.

### Example 3.3.2 Consider the z-transform

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

The power series expansion of this function is

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} a^n z^{-n}$$

Therefore, the sequence $x(n)$ having this $z$-transform is

$$x(n) = \begin{cases} \frac{1}{n}(-1)^{n+1} a^n & n > 0 \\ 0 & n \le 0 \end{cases}$$

### 3.3.3 Contour Integration

- Another approach that may be used to find the inverse z-transform of X(z) is to use **contour integration**.
- This procedure relies on Cauchy's integral theorem, which states that if **C** is a closed contour that encircles the origin in a counterclockwise direction,

$$\frac{1}{2\pi j}\oint_C z^{-k}dz = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

- Cauchy's integral theorem may be used to show that the coefficients **x(n)** may be found from X(z) as follows:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- Where C is a closed contour within the region of convergence of X(z) that encircles the origin in a counter clockwise direction.
- Contour integrals of this form may often by evaluated with the help of Cauchy's residue theorem,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \sum \left[\text{residues of } X(z) z^{n-1} \text{at the poles inside } C\right]$$

If $X(z)$ is a rational function of $z$ with a first-order pole at $z = \alpha_k$,

$$\text{Res}\left[X(z) z^{n-1} \text{ at } z = \alpha_k\right] = \left[(1 - \alpha_k z^{-1}) X(z) z^{n-1}\right]_{z=\alpha_k}$$

Contour integration is particularly useful if only a few values of $x(n)$ are needed.

# Table 3.3 Some common z-Transform pairs

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\frac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m > 0$) or $\infty$ (if $m < 0$) |
| 5. $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $[\cos\omega_0 n]u[n]$ | $\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $[\sin\omega_0 n]u[n]$ | $\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $[r^n\cos\omega_0 n]u[n]$ | $\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $[r^n\sin\omega_0 n]u[n]$ | $\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$ | $\frac{1-a^Nz^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

# Questions ?