Chapter 2: Discrete Time Systems Convolution

Sem. II, 2010 E.C Electrical and Computer Engineering By: Waltengus

2.1 Discrete Time Signals

- A discrete-time signal is an indexed sequence of **real** or **complex numbers**.
- A discrete-time signal is a function of an integer-valued variable, n, that is denoted by x(n).
- Although the independent variable n need not necessarily

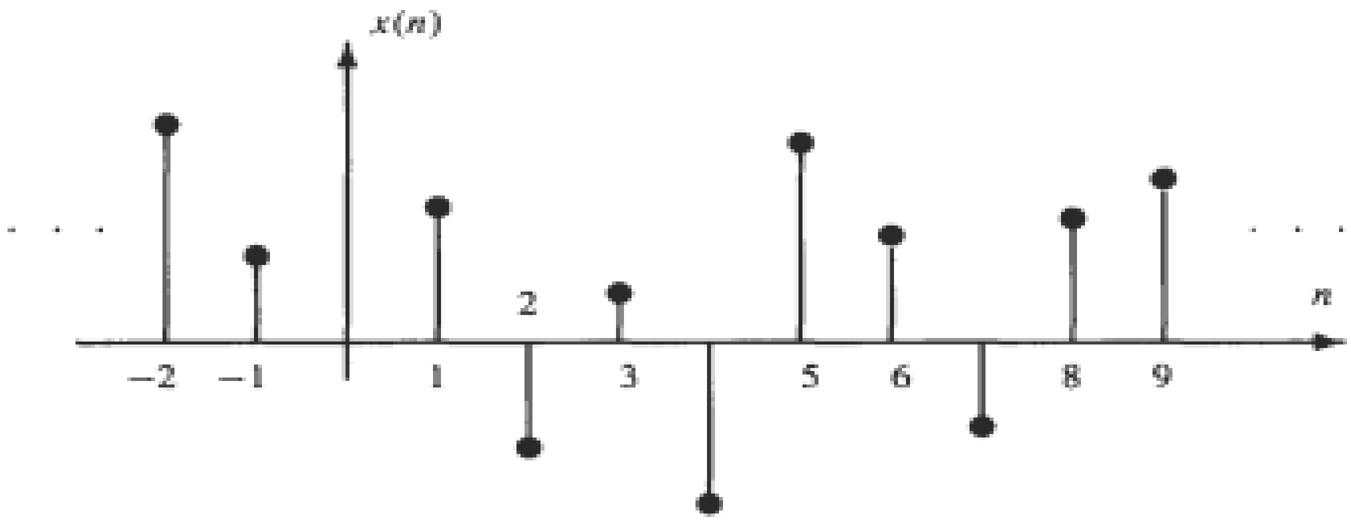


Fig. 2-1. The graphical representation of a discrete-time signal x(n)

- **Most information-bearing signals of practical interest are complicated functions of time, there are three simple, yet important, discrete-time signals that are frequently used in the representation and description of more complicated signals.**
- **These are Unit sample, Unit step and Exponential.**

Unit sample, denoted by δ(n), is defined by

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and plays the same role in discrete-time signal processing that the unit impulse plays in continuous-time signal processing. The *unit step*, denoted by $u(n)$, is defined by

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and is related to the unit sample by

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

Similarly, a unit sample may be written as a difference of two steps:

$$\delta(n) = u(n) - u(n-1)$$

Finally, an *exponential* sequence is defined by

$$x(n) = a^n$$

where a may be a real or complex number. Of particular interest is the exponential sequence that is formed when $a = e^{j\omega_0}$, where ω_0 is a real number. In this case, $x(n)$ is a complex exponential

$$e^{jn\omega_0} = \cos(n\omega_0) + j\sin(n\omega_0)$$

2.1.1 Signal Duration

- **Discrete-time signals may be conveniently classified in terms of their duration or extent.**
- **For example, a discrete- time sequence is said to be a finite-length sequence if it is equal to zero for all values of n outside a finite interval [N1, N2].**
- **Signals that are not finite in length, such as the unit step and the complex exponential, are said to be infinite-length sequences.**

- Infinite-length sequences may further be classified as either being right-sided, left-sided, or two-sided.
- A right-sided sequence is any infinite-length sequence that is equal to zero for all values of $n < n_o$ for some integer n_o. The unit step is an example of a right-sided sequence.
- Similarly, an infinite-length sequence x(n) is said to be left-sided if, for some integer n_o, $x(n) = 0$ for all $n > n_o$. An example of a left-sided sequence is

$$x(n) = u(n_0 - n) = \begin{cases} 1 & n \leq n_0 \\ 0 & n > n_0 \end{cases}$$

Which is a time-reversed and delayed unit step. An infinite-length signal that is neither right-sided nor left-sided, such as the complex exponential, is referred to as a two sided sequence.

2.1.2 Periodic and Aperiodic Sequences

- A discrete-time signal may always be classified as either being periodic or aperiodic.
- A signal x(n) is said to be periodic if, for some positive real integer N, $$x(n) = x(n + N)$$

- For all n. This is equivalent to saying that the sequence repeats itself every N samples.
- If a signal is periodic with period N, it is also periodic with period 2N, period 3N, and all other integer multiples of N.
- The fundamental period, which we will denote by N, is the smallest positive integer for which Eq. above is satisfied. If Eq. above is not satisfied for any integer N, x(n) is said to be an aperiodic signal.

Example 2.1 The signals

$$x_1(n) = a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

and

$$x_2(n) = \cos(n^2)$$

are not periodic, whereas the signal

$$x_3(n) = e^{j\pi n/8}$$

is periodic and has a fundamental period of $N = 16$.

If $x_1(n)$ is a sequence that is periodic with a period N_1, and $x_2(n)$ is another sequence that is periodic with a period N_2, the sum

$$x(n) = x_1(n) + x_2(n)$$

will always be periodic and the fundamental period is

$$N = \frac{N_1 N_2}{\gcd(N_1, N_2)}$$

Where gcd(N1, N2) means the greatest common divisor of N1 and N2.

The same is true for the product; that is,

$$x(n) = x_1(n)x_2(n)$$

will be periodic with a period N given by Eq. (above). However, the fundamental period may be smaller. Given any sequence x(n), a periodic signal may always be formed by replicating x(n) as follows:

$$y(n) = \sum_{k=-\infty}^{\infty} x(n - kN)$$

Where N is a positive integer. In this case, y(n) will be periodic with period N.

2.1.3 Symmetric Sequences

A discrete-time signal will often possess some form of symmetry that may be exploited in solving problems. Two symmetries of interest are as follows:

A real-valued signal is said to be *even* if, for all n,

$$x(n) = x(-n)$$

whereas a signal is said to be *odd* if, for all n,

$$x(n) = -x(-n)$$

Any signal $x(n)$ may be decomposed into a sum of its even part, $x_e(n)$, and its odd part, $x_o(n)$, as follows:

$$x(n) = x_e(n) + x_o(n)$$

To find the even part of $x(n)$ we form the sum

$$x_e(n) = \frac{1}{2}\{x(n) + x(-n)\}$$

whereas to find the odd part we take the difference

$$x_o(n) = \frac{1}{2}\{x(n) - x(-n)\}$$

For complex sequences the symmetries of interest are slightly different.

A complex signal is said to be conjugate symmetric if, for all n

$$x(n) = x^*(-n)$$

and a signal is said to be *conjugate antisymmetric* if, for all n,

$$x(n) = -x^*(-n)$$

- Any complex signal may always be decomposed into a sum of a conjugate symmetric signal and a conjugate anti symmetric signal.

2.1.4 Signal Manipulations

- Discrete-time signals and systems we will be concerned with the manipulation of signals. These manipulations are generally compositions of a few basic signal transformations.
- These transformations may be classified either as those that are transformations of the independent variable n or those that are transformations of the amplitude of x(n) (i.e., the dependent variable).

The most common transformations include shifting, reversal, and scaling, which are defined below.

Shifting : This is the transformation defined by $f(n) = n - n_o$. If $y(n) = x(n - n_o)$, x(n) is shifted to the right by n_o samples if **n_o is positive** (this is referred to as a delay), and it is shifted to the left by no samples if no is negative (referred to as an advance).

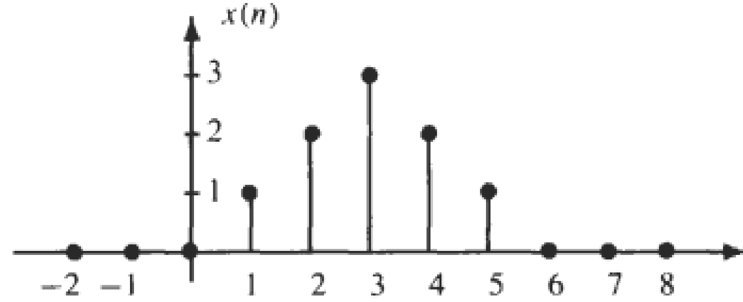
Reversal: This transformation is given by $f(n) = -n$ and simply involves "flipping" the signal x(n) with respect to the index n.

Time Scaling : This transformation is defined by $f(n) = Mn$ or $f(n) = n/N$ where M and N are positive integers.

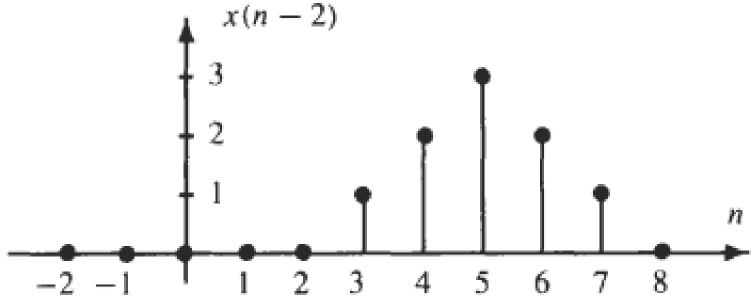
- In the case of $f(n) = Mn$, the sequence $x(Mn)$ is formed by taking every Mth sample of x(n) (this operation is known as **down-sampling**. With $f(n) = n/N$ the sequence $y(n) = x(f(n))$ is defined as follows:

$$y(n) = \begin{cases} x\left(\dfrac{n}{N}\right) & n = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise} \end{cases}$$

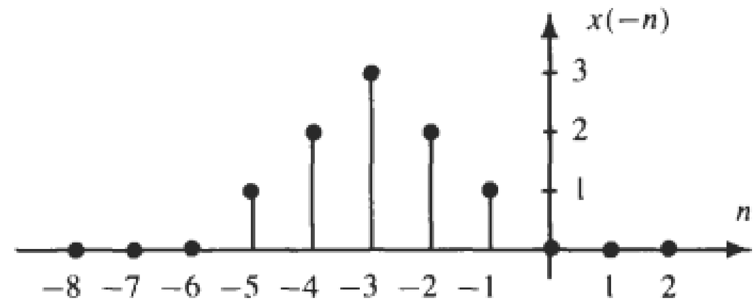
This operation is known as **up-sampling**.



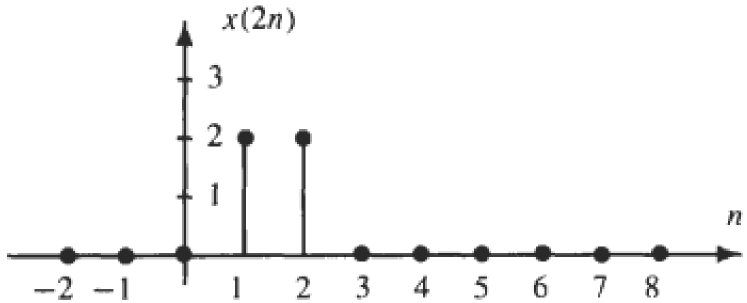
(*a*) A discrete-time signal.



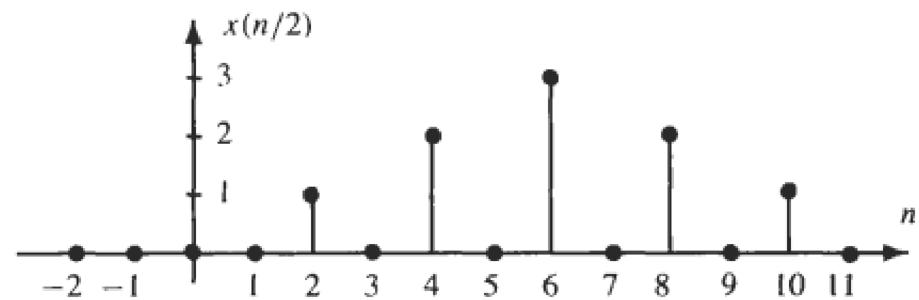
(*b*) A delay by $n_0 = 2$.



(*c*) Time reversal.



(*d*) Down-sampling by a factor of 2.



(*e*) Up-sampling by a factor of 2.

Addition, Multiplication, and Scaling

The most common types of amplitude transformations are addition, multiplication, and scaling. Performing these operations is straightforward and involves only pointwise operations on the signal.

Addition The sum of two signals

$$y(n) = x_1(n) + x_2(n) \qquad -\infty < n < \infty$$

is formed by the pointwise addition of the signal values.

Multiplication The multiplication of two signals

$$y(n) = x_1(n)x_2(n) \qquad -\infty < n < \infty$$

is formed by the pointwise product of the signal values.

Scaling Amplitude scaling of a signal $x(n)$ by a constant c is accomplished by multiplying every signal value by c:

$$y(n) = cx(n) \qquad -\infty < n < \infty$$

This operation may also be considered to be the product of two signals, $x(n)$ and $f(n) = c$.

2.2 Discrete Time Systems

- A discrete-time system is a mathematical operator or mapping that transforms one signal (the input) into another signal (the output) by means of a fixed set of rules or operations.
- The notation T[.] is used to represent a general system as shown in Fig. 2.2, in which an input signal x(n) is transformed into an output signal y(n) through the transformation T[.].
- The input-output properties of a system may be specified in any one of a number of different ways.
- The relationship between the input and output, for example, may be expressed in terms of a concise mathematical rule or function such as

$$y(n) = x^2(n)$$

or

$$y(n) = 0.5y(n-1) + x(n)$$

It is also possible, however, to describe a system in terms of an algorithm that provides a sequence of instructions or operations that is to be applied to the input signal, such as

$$y_1(n) = 0.5y_1(n-1) + 0.25x(n)$$
$$y_2(n) = 0.25y_2(n-1) + 0.5x(n)$$
$$y_3(n) = 0.4y_3(n-1) + 0.5x(n)$$
$$y(n) = y_1(n) + y_2(n) + y_3(n)$$

In some cases, a system may conveniently be specified in terms of a table that defines the set of all possible input-output signal pairs of interest.

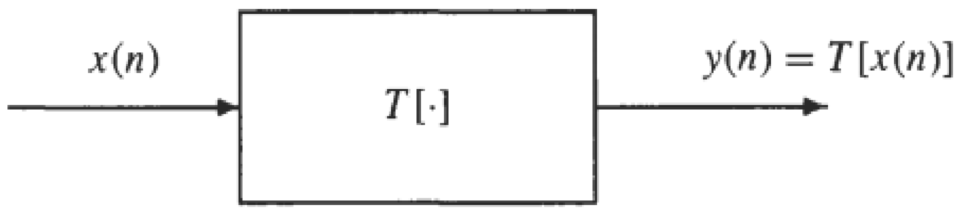


Fig. 2.2. The representation of a discrete-time system as a trans- formation T[.] that maps an input signal x(n) into an output signal y(n).

- **Discrete-time systems** may be classified in terms of the **properties that they possess.**
- The most common properties of interest include **memoryless, linearity, shift-invariance, causality, stability, and invertibility.**

2.2.1 Memoryless Systems

- A system is said to be **memoryless** if the **output** at any time $n = n_o$ depends only on the **input** at time $n = n_o$.
- In other words, a system is memory less if, for any n_o, we are able to determine the value of $y(n_o)$ given only the value of $x(n_o)$.

Example 2.2 The system is memoryless because $y(n_o)$ depends only on the value of $x(n)$ at time n_o.

$$y(n) = x^2(n)$$

The system on the other hand, is not memoryless because the output at time n_o depends on the value of the input both at time n_o and at time $n_o - 1$.

$$y(n) = x(n) + x(n - l)$$

2.2.2 Linear Systems

A system that is both additive and homogeneous is said to be linear. Thus, A system is said to be linear if

$$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$$

for any two inputs $x_1(n)$ and $x_2(n)$ and for any complex constants a_1 and a_2.

Linearity greatly simplifies the evaluation of the response of a system to a given input.

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} T[x(k)\delta(n-k)]$$

Because the coefficients $x(k)$ are constants, we may use the homogeneity property to write

$$y(n) = \sum_{k=-\infty}^{\infty} T[x(k)\delta(n-k)] = \sum_{k=-\infty}^{\infty} x(k)T[\delta(n-k)]$$

If we define $h_k(n)$ to be the response of the system to a unit sample at time $n = k$,

$$h_k(n) = T[\delta(n-k)]$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$$

which is known as the *superposition summation.*

A Nonlinear System

Consider the system defined by

$$w[n] = \log_{10}(|x[n]|).$$

The inputs $x_1[n] = 1$ and $x_2[n] = 10$ are a counterexample.
The output for the first signal is $w_1[n] = 0$, while for the second, $w_2[n] = 1$.
The scaling property of linear systems requires that, since $x_2[n] = 10x_1[n]$, if the system is linear, it must be true that $w_2[n] = 10w_1[n]$. Since this is not so for for this set of inputs and outputs, the system is *not* linear.

2.2.3 Time (Shift) Invariance Systems

- **If a system has the property that a shift (delay) in the input by n_o results in a shift in the output by n_o,**

- Let y(n) be the response of a system to an arbitrary input x(n).
- The system is said to be time-invariant if, for any delay n_0, the response to $x(n- n_0)$ produces the out put value $y(n - n_0)$.
- A system that is not shift-invariant is said to be shift varying.
- A system will be shift-invariant if its properties or characteristics do not change with time.
- To test for shift-invariance one needs to compare $y(n - n_0)$ to $T[x(n - n_0)]$.
- If they are the same for any input x(n) and for all shifts n_0, the system is shift-invariant.

Example 2.3 The system defined by

$$y(n) = x^2(n)$$

is shift-invariant, which may be shown as follows. If $y(n) = x^2(n)$ is the response of the system to $x(n)$, the response of the system to

$$x'(n) = x(n - n_0)$$

is

$$y'(n) = [x'(n)]^2 = x^2(n - n_0)$$

Because $y'(n) = y(n - n_0)$, the system is shift-invariant. However, the system described by the equation

$$y(n) = x(n) + x(-n)$$

is shift-varying. To see this, note that the system's response to the input $x(n) = \delta(n)$ is

$$y(n) = \delta(n) + \delta(-n) = 2\delta(n)$$

whereas the response to $x(n - 1) = \delta(n - 1)$ is

$$y'(n) = \delta(n - 1) + \delta(-n - 1)$$

Because this is not the same as $y(n - 1) = 2\delta(n - 1)$, the system is shift-varying.

2.2.4 Linear time-Invariant Systems

- A system that is both linear and shift-invariant is referred to as a linear shift-invariant (LSI) system.
- If h(n) is the response of an LSI system to the unit sample δ(n), its response to δ(n - k) will be h(n - k).

$$h_k(n) = h(n-k)$$

and it follows that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

The above Equation which is known as the convolution sum, is written as

$$y(n) = x(n) * h(n)$$

- ✓ Where * indicates the convolution operator. The sequence h(n), referred to as the unit sample response, provides a complete characterization of an LSI system.

2.2.5 Causality

- A system property that is important for real-time applications is causality.
- A system is said to be causal if, for any n_o, the response of the system at time n_o depends only on the input up to time $n = n_o$.
- For a causal system, changes in the output cannot precede changes in the input. Thus, if $x_1(n) = x_2(n)$ for $n \leq n_o$, $y_1(n)$ must be equal to $y_2(n)$ for $n \leq n_o$.
- Causal systems are therefore referred to as non anticipatory. An LSI system will be causal if and only if h(n) is equal to zero for $n < 0$.

Example 2.4 The system described by the equation $y(n) = x(n) + x(n - 1)$ is causal because the value of the output at any time $n = n_o$ depends only on the input $x(n)$ at time n_o and at time $n_o - 1$.

The system described by $y(n) = x(n) + x(n + 1)$, on the other hand, is non causal because the output at time $n = n_o$ depends on the value of the input at time $n_o + 1$.

4.2.6 Stability

- In many applications, it is important for a system to have a response, $y(n)$, that is bounded in amplitude whenever the input is bounded.
- A system with this property is said to be stable in the bounded input-bounded output (BIBO) sense. Specifically.

A system is said to be stable in the bounded input-bounded output sense if, for any input that is bounded, $|x(n)| \leq A < \infty$, the output will be bounded,

$$|y(n)| \leq B < \infty$$

For a linear shift-invariant system, stability is guaranteed if the unit sample response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

EXAMPLE .5 An LSI system with unit sample response $h(n) = a^n u(n)$ will be stable whenever $|a| < 1$, because

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} \qquad |a| < 1$$

The system described by the equation $y(n) = nx(n)$, on the other hand, is not stable because the response to a unit step, $x(n) = u(n)$, is $y(n) = nu(n)$, which is unbounded.

2.2.7 Invertibility

A system property that is important in applications such as channel equalization and deconvolution is invertibility.

- A system is said to be invertible if the input to the system may be uniquely determined from the output.
- In order for a system to be invertible, it is necessary for distinct inputs to produce distinct outputs.
- In other words, given any two inputs $x_1(n)$ and $x_2(n)$ with $x_1(n) \neq x_2(n)$, it must be true that $y_1(n) \neq y_2(n)$.

Example 2.6 The system defined by

$$y(n) = x(n)g(n)$$

is invertible if and only if $g(n) \neq 0$ for all n. In particular, given y(n) with g(n) nonzero for all n, x(n) may be recovered from y(n) as follows:

$$x(n) = \frac{y(n)}{g(n)}$$

2.2.8 Convolution

The relationship between the input to a linear shift-invariant system, x(n), and the output, y(n), is given by the convolution sum

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- **Convolution is fundamental to the analysis and description of LSI systems.**

Convolution Properties

Convolution is a linear operator and, therefore, has a number of important properties including the commutative, associative, and distributive properties.

Commutative Property

- The commutative property states that the order in which two sequences are convolved is not important.
- Mathematically, the commutative property is

$$x(n) * h(n) = h(n) * x(n)$$

- This property states that a system with a unit sample response h(n) and input x(n) behaves in exactly the same way as a system with unit sample response x(n) and an input h(n).

Associative Property

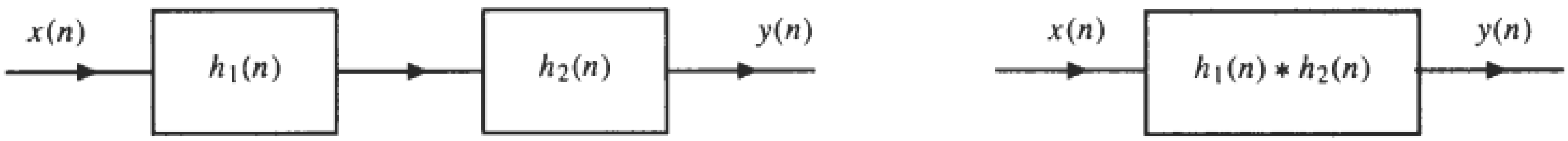
- The convolution operator satisfies the associative property, which is

$$\{x(n) * h_1(n)\} * h_2(n) = x(n) * \{h_1(n) * h_2(n)\}$$

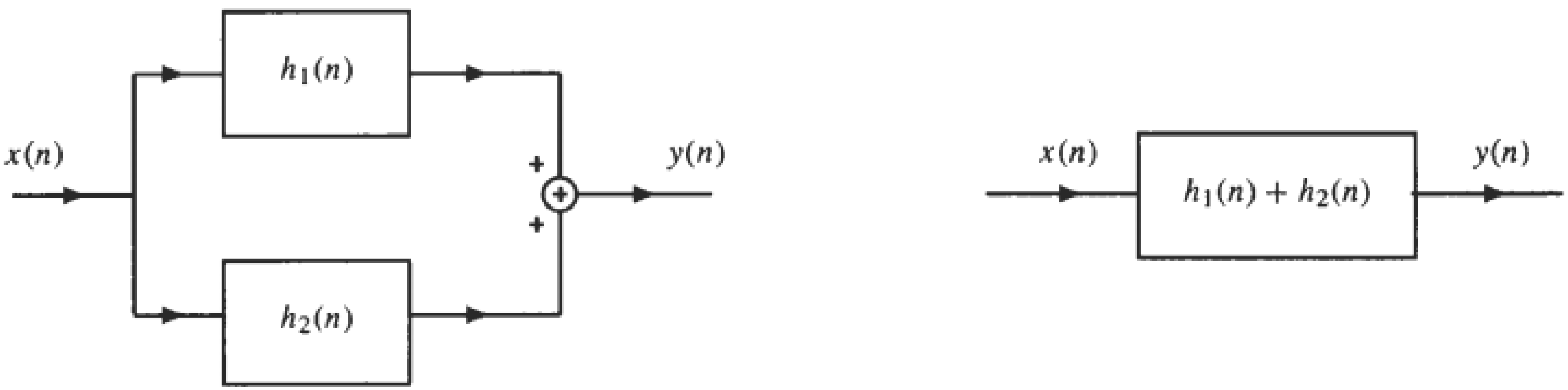
- The associative property states that if two systems with unit sample responses $h_1(n)$ and $h_2(n)$ are connected in cascade as shown in Fig. below (b).
- An equivalent system is one that has a unit sample response equal to the convolution of h_1 (n) and $h_2(n)$:

$$h_{eq}(n) = h_1(n) * h_2(n)$$

(*a*) The commutative property.



(*b*) The associative property.



(*c*) The distributive property.

Fig. 2.5. The interpretation of convolution properties from a systems point of view.

Distributive Property

The distributive property of the convolution operator states that

$$x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$$

- **This property asserts that if two systems with unit sample responses $h_1(n)$ and $h_2(n)$ are connected in parallel, as illustrated in Fig. 2.5(c).**
- **An equivalent system is one that has a unit sample response equal to the sum of $h_1(n)$ and $h_2(n)$:**

$$h_{eq}(n) = h_1(n) + h_2(n)$$

Example 2.6. Let us perform the convolution of the two signals

$$x(n) = a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$h(n) = u(n)$$

Example 2.7 Consider the convolution of the sequence

$$x(n) = \begin{cases} 1 & 10 \leq n \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

with

$$h(n) = \begin{cases} n & -5 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Because $x(n)$ is zero outside the interval [10, 20], and $h(n)$ is zero outside the interval [−5, 5], the nonzero values of the convolution, $y(n) = x(n) * h(n)$, will be contained in the interval [5, 25].

$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$	$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \lvert a \rvert < 1$
$\sum_{n=0}^{N-1} n a^n = \frac{(N-1)a^{N+1} - N a^N + a}{(1-a)^2}$	$\sum_{n=0}^{\infty} n a^n = \frac{a}{(1-a)^2} \quad \lvert a \rvert < 1$
$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$	$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N(N-1)(2N-1)$

With the direct evaluation of the convolution sum we find

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} a^k u(k)u(n-k)$$

Because $u(k)$ is equal to zero for $k < 0$ and $u(n-k)$ is equal to zero for $k > n$, when $n < 0$, there are no nonzero terms in the sum and $y(n) = 0$. On the other hand, if $n \geq 0$,

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Therefore,

$$y(n) = \frac{1-a^{n+1}}{1-a} u(n)$$

Examples

.1 Determine whether or not the signals below are periodic and, for each signal that is periodic, determine the fundamental period.

(a) $x(n) = \cos(0.125\pi n)$

(b) $x(n) = \text{Re}\{e^{jn\pi/12}\} + \text{Im}\{e^{jn\pi/18}\}$

(c) $x(n) = \sin(\pi + 0.2n)$

(d) $x(n) = e^{j\frac{\pi}{16}n}\cos(n\pi/17)$

(a) Because $0.125\pi = \pi/8$, and

$$\cos\left(\frac{\pi}{8}n\right) = \cos\left(\frac{\pi}{8}(n+16)\right)$$

$x(n)$ is periodic with period $N = 16$.

(b) Here we have the sum of two periodic signals,

$$x(n) = \cos(n\pi/12) + \sin(n\pi/18)$$

with the period of the first signal being equal to $N_1 = 24$, and the period of the second, $N_2 = 36$. Therefore, the period of the sum is

$$N = \frac{N_1 N_2}{\gcd(N_1, N_2)} = \frac{(24)(36)}{\gcd(24, 36)} = \frac{(24)(36)}{12} = 72$$

(c) In order for this sequence to be periodic, we must be able to find a value for N such that

$$\sin(\pi + 0.2n) = \sin(\pi + 0.2(n + N))$$

The sine function is periodic with a period of 2π. Therefore, $0.2N$ must be an integer multiple of 2π. However, because π is an irrational number, no integer value of N exists that will make the equality true. Thus, this sequence is aperiodic.

(d) Here we have the product of two periodic sequences with periods $N_1 = 32$ and $N_2 = 34$. Therefore, the fundamental period is

$$N = \frac{(32)(34)}{\gcd(32, 34)} = \frac{(32)(34)}{2} = 544$$

2 A linear discrete-time system is characterized by its response $h_k(n)$ to a delayed unit sample $\delta(n - k)$. For each linear system defined below, determine whether or not the system is shift-invariant.

(a) $h_k(n) = (n - k)u(n - k)$

(b) $h_k(n) = \delta(2n - k)$

(c) $h_k(n) = \begin{cases} \delta(n - k - 1) & k \text{ even} \\ 5u(n - k) & k \text{ odd} \end{cases}$

(a) Note that $h_k(n)$ is a function of $n - k$. This suggests that the system is shift-invariant. To verify this, let $y(n)$ be the response of the system to $x(n)$:

$$y(n) = \sum_{k=-\infty}^{\infty} h_k(n)x(k)$$

$$= \sum_{k=-\infty}^{\infty} (n-k)u(n-k)x(k) = \sum_{k=-\infty}^{n} (n-k)x(k)$$

The response to a shifted input, $x(n-n_0)$, is

$$y_1(n) = \sum_{k=-\infty}^{\infty} x(k-n_0)h_k(n) = \sum_{k=-\infty}^{\infty} (n-k)u(n-k)x(k-n_0)$$

$$= \sum_{k=-\infty}^{n} (n-k)x(k-n_0)$$

With the substitution $l = k - n_0$ this becomes

$$y_1(n) = \sum_{l=-\infty}^{n-n_0} (n-n_0-l)x(l)$$

$$y(n - n_0) = \sum_{k=-\infty}^{n-n_0} (n - n_0 - k)x(k)$$

which is the same as $y_1(n)$. Therefore, this system is shift-invariant.

(*b*) For the second system, $h_k(n)$ is *not* a function of $n - k$. Therefore, we should expect this system to be shift-varying. Let us see if we can find an example that demonstrates that it is a shift-varying system. For the input $x(n) = \delta(n)$, the response is

$$y(n) = h_0(n) = \delta(2n) = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

If we delay $x(n)$ by 1, the response to $x_1(n) = \delta(n - 1)$ is

$$y_1(n) = h_1(n) = \delta(2n - 1) = 0$$

Because $y_1(n) \neq y(n - 1)$, the system is shift-varying.

(*c*) Finally, for the last system, we see that although $h_k(n)$ is a function of $n - k$ for k even and a function of $(n - k)$ for k odd,

$$h_k(n) \neq h_{k-1}(n - 1)$$

In other words, the response of the system to $\delta(n - k - 1)$ is not equal to the response of the system to $\delta(n - k)$ delayed by 1. Therefore, this system is shift-varying.

3 If $x_1(n)$ is even and $x_2(n)$ is odd, what is $y(n) = x_1(n) \cdot x_2(n)$?

If $y(n) = x_1(n) \cdot x_2(n)$,

$$y(-n) = x_1(-n) \cdot x_2(-n)$$

Because $x_1(n)$ is even, $x_1(n) = x_1(-n)$, and because $x_2(n)$ is odd, $x_2(n) = -x_2(-n)$. Therefore,

$$y(-n) = -x_1(n) \cdot x_2(n) = -y(n)$$

and it follows that $y(n)$ is odd.

4 Given that $x(n)$ is the system input and $y(n)$ is the system output, which of the following systems are causal?

(a) $y(n) = x^2(n)u(n)$

(b) $y(n) = x(|n|)$

(c) $y(n) = x(n) + x(n-3) + x(n-10)$

(d) $y(n) = x(n) - x(n^2 - n)$

(a) The system $y(n) = x^2(n)u(n)$ is *memoryless* (i.e., the response of the system at time n depends only on the input at time n and on no other values of the input). Therefore, this system is causal.

(b) The system $y(n) = x(|n|)$ is an example of a noncausal system. This may be seen by looking at the output when $n < 0$. In particular, note that $y(-1) = x(1)$. Therefore, the output of the system at time $n = -1$ depends on the value of the input at a future time.

(c) For this system, in order to compute the output $y(n)$ at time n all we need to know is the value of the input $x(n)$ at times n, $n - 3$, and $n - 10$. Therefore, this system must be causal.

(d) This system is noncausal, which may be seen by evaluating $y(n)$ for $n < 0$. For example,

$$y(-1) = x(-1) - x(2)$$

Because $y(-1)$ depends on the value of $x(2)$, which occurs after time $n = -1$, this system is noncausal.

5 The first nonzero value of a finite-length sequence $x(n)$ occurs at index $n = -6$ and has a value $x(-6) = 3$, and the last nonzero value occurs at index $n = 24$ and has a value $x(24) = -4$. What is the index of the first nonzero value in the convolution

$$y(n) = x(n) * x(n)$$

and what is its value? What about the last nonzero value?

Because we are convolving two finite-length sequences, the index of the first nonzero value in the convolution is equal to the sum of the indices of the first nonzero values of the two sequences that are being convolved. In this case, the index is $n = -12$, and the value is

$$y(-12) = x^2(-6) = 9$$

Similarly, the index of the last nonzero value is at $n = 48$ and the value is

$$y(48) = x^2(24) = 16$$

The convolution of two finite-length sequences will be finite in length.

6 If the response of a linear shift-invariant system to a unit step (i.e., the step response) is

$$s(n) = n\left(\tfrac{1}{2}\right)^n u(n)$$

find the unit sample response, $h(n)$.

In this problem, we begin by noting that

$$\delta(n) = u(n) - u(n-1)$$

Therefore, the unit sample response, $h(n)$, is related to the step response, $s(n)$, as follows:

$$h(n) = s(n) - s(n-1)$$

Thus, given $s(n)$, we have

$$\begin{aligned} h(n) &= s(n) - s(n-1) \\ &= n\left(\tfrac{1}{2}\right)^n u(n) - (n-1)\left(\tfrac{1}{2}\right)^{n-1} u(n-1) \\ &= \left[n\left(\tfrac{1}{2}\right)^n - 2(n-1)\left(\tfrac{1}{2}\right)^n\right] u(n-1) \\ &= (2-n)\left(\tfrac{1}{2}\right)^n u(n-1) \end{aligned}$$

7 Prove the commutative property of convolution

$$x(n) * h(n) = h(n) * x(n)$$

Proving the commutative property is straightforward and only involves a simple manipulation of the convolution sum. With the convolution of $x(n)$ with $h(n)$ given by

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

with the substitution $l = n - k$, we have

$$x(n) * h(n) = \sum_{l=-\infty}^{\infty} x(n-l)h(l) = h(n) * x(n)$$

8 Prove the distributive property of convolution

$$h(n) * [x_1(n) + x_2(n)] = h(n) * x_1(n) + h(n) * x_2(n)$$

To prove the distributive property, we have

$$h(n) * [x_1(n) + x_2(n)] = \sum_{k=-\infty}^{\infty} h(k)[x_1(n-k) + x_2(n-k)]$$

Therefore,

$$\begin{aligned} h(n) * [x_1(n) + x_2(n)] &= \sum_{k=-\infty}^{\infty} h(k)x_1(n-k) + \sum_{k=-\infty}^{\infty} h(k)x_2(n-k) \\ &= h(n) * x_1(n) + h(n) * x_2(n) \end{aligned}$$

and the property is established.

Questions ?