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Chapter 5

Stability Analysis of linear system control systems.

5.1 Introduction.

The issue of ensuring the ~~ability~~ Stability of a closed-loop feedback system system is central to control system design. knowing that unstable closed loop system is generally of no practical value, thus we seek methods of to help us analyze and design stable systems.

Basically, the design of linear control system required as a problem of arranging the location of the poles and zeros of the closed loop transfer function such that the corresponding system will perform according to the prescribed specification.

For analysis of and design purpose, we can classify the stability of control systems as absolute stability and relative stability.

Absolute Stability refers to the cond^n whether the system is stable or unstable

Once the system is to be stable, it is of interest to determine how stable it is, and this degree of stability is a measure of relative stability.

5.2 Bounded input - Bounded - output (BIBO)

The system is said to be BIBO stable, or simply stable, if its output $c(t)$ is bounded to bounded input.

The BIBO stability is related to the location of the roots of the characteristics of eq^n of the system transfer function

$$T = \frac{G}{1+GH}$$

$\Rightarrow$ $1+GH$ is called the Char. eq^n

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Consider the convolution integral relating $r(t), c(t), g(t)$

$$r(t) \longrightarrow \boxed{g(t)} \longrightarrow c(t)$$

$$c(t) = \int_0^{\infty} r(t-\tau)\, g(\tau)\, d\tau \quad \text{--- } 1$$

Taking the absolute value of both side

$$|c(t)| = \left| \int_0^{\infty} r(t-\tau)\, g(\tau)\, d\tau \right| \quad \text{--- } 2$$

$$|c(t)| \le \int_0^{\infty} |r(t-\tau)|\, |g(\tau)|\, d\tau \quad \text{--- } 3$$

if $r(t)$ is bounded

$$|r(t)| \le M \quad \text{--- } 4$$

where M is a finite positive number

Then

$$|c(t)| \le M \int_0^{\infty} |g(\tau)|\, d\tau \quad \text{--- } 5$$

Thus if $c(t)$ is to be bounded

$$|c(t)| \le N \le \infty \quad \text{--- } 6$$

N is a positive finite number

The ff cond^{n} must hold

$$M \int_0^{\infty} |g(\tau)|\, d\tau \le N < \infty \quad \text{--- } 7$$

or for any positive finite value Q

$$\int_0^{\infty} |g(\tau)|\, d\tau \le Q < \infty \quad \text{--- } 8$$

$\Longrightarrow$ the area under $|g(\tau)|$ -versus- τ curve must be finite.

To show the relation-ship with the roots of the char. eq^{n} and cond of 8

$$G(s) = \mathcal{L}[g(t)] = \int_0^{\infty} g(t)\, e^{-st}\, dt \quad \text{--- } 9$$

taking abs

$$|G(s)| = \left| \int_0^{\infty} g(t)\, e^{-st} \right| \le \int_0^{\infty} |g(t)|\, |e^{-st}|\, dt \quad \text{-- } 10$$

Since $|e| = |e|$

where σ is the real part of s, where s assumes the value of the poles of G(s)

If $G(s) \to \infty$ and eqn (10) becomes

$$\infty \le \int_0^\infty |g(t)||e^{\sigma t}|dt \quad ---- 11$$

If one or more roots of the Char. eqn are in the right-half s-plane or on the jω-axis

$\sigma \ge 0$ then

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$$|e^{-\sigma t}| \le M = 1 \quad --- 12$$

Then eqn (11) becomes

$$\infty \le \int_0^\infty M|g(t)|dt = \int_0^\infty |g(t)|dt$$

which violates the BIBO stability

For BIBO stability, a necessary and sufficient condns for a system to be stable.

1- The roots of the Char. eqn or the poles of G(s), must ~~be~~ all lie on the left-half s-plane.

2. All the poles of the system transfer function have negative real parts.

5.3 Zero-input stability.

Zero input stability refers to the stability condn when the input is zero, and the system is driven only by its initial condns

The zero input stability defined as

If the zero input-response c(t), subjected to the finite initial condn F(to) reaches zero as t approaches infinity, the system is said to be zero input stable

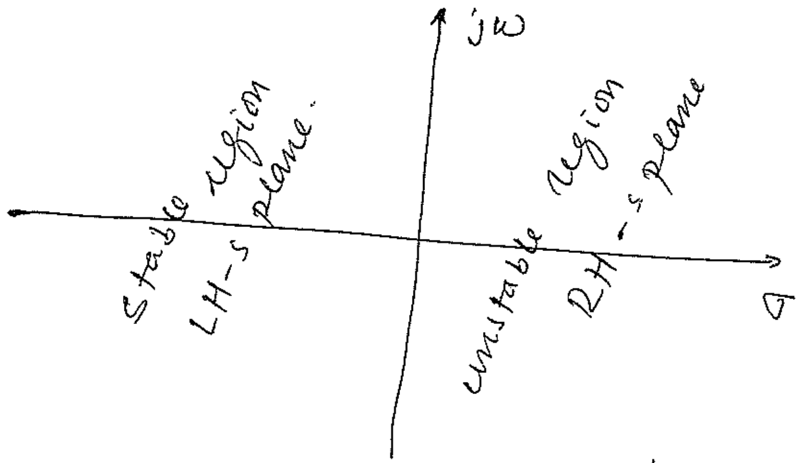
1) $|c(t)| \le M < \infty$ for all $t \ge t_0$

and 2) $\lim\limits_{t \to \infty} |c(t)| = 0$

Because of the condn in the last eqn requires that the magnitude of c(t) reaches zero as time approaches infinity, the zero-input stability is also known as the asymptotic Stability

When the char. eqn has roots on the $j\omega$ axis, and ~~non-in~~ non in the RH-s plane the system may be marginally stable or marginally unstable.

Stable and unstable regions on the s-plane



eg. consider the following transfer function and illustrate the Stability condn

a) $$T(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

The poles $s = -1, -2, -3$

All the poles lie in the left-half s-plane

b) $$T(s) = \frac{20(s+1)}{(s-1)(s^2+2s+2)}$$

the system is unstable due to the pole at $s=1$

c) $$T(s) = \frac{20(s+1)}{(s+2)(s^2+4)} = \frac{20(s+1)}{(s+2)(s+j2)(s-j2)}$$

the system is marginally stable/unstable due to poles on the $j\omega$ axis.

Methods of determining stability of linear control system.

The methods to be outlined for determination of the stability of linear control system :

1. Routh - Hurwitz criterion

The criterion tests whether any of the roots of the char. eqn lies on the right half s-plane.

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2. Root locus diagram

- is the loci of the char. eqn roots when certain system parameter varies.

- the root loci provides a clear picture of the stability with reference to the variable parameter

Chap. 6

3. Nyquist criterion and Bode diagram

Chap 7 and 8.

3.1 - Nyquist criterion is a semi-graphical method that gives information on the difference between the number of poles and zeros of the closed loop transfer function provided the stability of the system.

3.2 Bode diagram is a plot of the magnitude and phase of closed loop system and provides the stability of the system.

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5.4. Routh-Hurwitz stability criterion.

The Routh-Hurwitz Criterion is necessary and sufficient condn for stability of linear control systems based on the cofficent of the Char. eqn.

Consider the Char. eqn

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

The necessary condn are

1- all the coefficients of $F(s)$ have the same sign
2- None of the coefficents vanishes
3- All the roots lie in the left-half of s-plane

Routh's tabulation.

- The first step in the simplification of the Hurwitz criterion is to arrange the coefficients of $F(s)$ in two rows
- the first row consists $a_0 - a_2 - a_4 -$
  " second " " $a_1 - a_3 - a_5 -$

The Routh table

$$\begin{array}{l|llll}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & 0 \\
s^{n-2} & A & B & a_6 & - \\
s^{n-3} & C & D & & \\
\vdots & & & & \\
s^1 & & & & \\
s^0 & a_6 & & &
\end{array}$$

where

$$A = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{-(a_0 a_3 - a_1 a_2)}{a_1} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$B = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{-(a_0 a_5 - a_1 a_4)}{a_1} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

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$$C = \frac{-\begin{vmatrix} a_1 & a_3 \\ A & B \end{vmatrix}}{A} = \frac{-(a_1 B - a_3 A)}{A} = \frac{a_3 A - a_1 B}{A}$$

$$D = \frac{-\begin{vmatrix} a_1 & a_5 \\ A & a_6 \end{vmatrix}}{A} = \frac{-(a_1 a_6 - a_5 A)}{A} = \frac{a_5 A - a_1 a_6}{A}$$

The Routh-Hurwitz criterion states that the number of roots of F(s) with positive real part is equal to the number of changes in sign of the first column of the Routh array

* This criterion requires that there be no change in sign of the first column for stable system. This requirement is both necessary and sufficient

example

A control system has a char. eqn

$$F(s) = s^4 + 5s^3 + 20s^2 + 40s + 50 = 0$$

using the Routh-Hurwitz criterion Determine the stability of the syst

soln

The routh array is

s^4	1	20	50
s^3	5	40	0
s^2	$A = 12$	$B = 50$	0
s^1	$C = \frac{230}{12}$	0	0
s^0	50	0	0

w = [1 5 20 40 50]

roots(w)

−0.78 + 2.92i

−0.783 ~~+2~~.92i

−1.7161 ± j1.582i

$$A = \frac{-\begin{vmatrix} 1 & 20 \\ 5 & 40 \end{vmatrix}}{5} = \frac{20*5 - 40*1}{5} = 12 = \frac{60}{5}$$

$$B = \frac{-\cancel{\begin{vmatrix} 5 & 40 \\ 12 & 50 \end{vmatrix}}}{12} - \frac{\begin{vmatrix} 1 & 50 \\ 5 & 0 \end{vmatrix}}{5} = \frac{5*50 - 0*1}{5} = \frac{250}{5} = 50$$

$$C = \frac{-\begin{vmatrix} 5 & 40 \\ 12 & 50 \end{vmatrix}}{12} = \frac{40*12 - 5*50}{12} = \frac{480 - 250}{12} = \frac{230}{12}$$

There is no sign change on the first colum

⇒ the system is stable

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Eg 2.

The Char. eqn of a system is given by

$$F(s) = s^3 + s^2 + 2s + 24 = 0$$

⇒ all cofficient the same sign → it satisfies the necessary cond.n

using Routh - Hurwitz Criterion, Determine the stability of the system

sol/n

∴ The Routh array

sign chang	s^3	1	2	0
	s^2	1	24	0
sign chan	s^1	-22	0	
	s^0	24		

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There are two sign change in the first column.

therefore → there are two roots of F(s) in the right-Half s plane

Hence the system is unstable.

eg 3.

Consider the Char. eq

$$1+GH = F(s) = s^3 - 4s^2 + s + 6 = (s-2)(s+1)(s-3)$$

Determine the stability of the system

Roots $s = 2, 3, -1$ — +ve real part.

sol/n

1 - the Char. eqn has one negative coefficent, thus from the necessary cond.n not all the roots of the eqn in the LH-s plane. and the system is unstable

2 - Routh's tabulation

$$s^3 + 4s^2 + s + 6$$

sign change	s^3	1	1	
	s^2	-4	6	
sign	s^1	2.5	0	
	s^0	6		

Since there are two sign changes in the first column, i.e. there are two roots of F(s) in the RH-plane

⇒ the system is unstable.

b) Special Cases when Routh's Tabulation Terminates prematurely

⇒ Two special difficulties may arise while obtaining the Routh-array for a given characteristics eqⁿ

Case I :- Zero in the first column

i - If any row of the Routh-Array is zero, it should be replaced by a small-positive number ϵ in order to complete the array

ii - The sign of the elements of the first column is then examined as ϵ approaches zero.

Example.

consider the characteristics eqn

$$F(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 = 0$$

Determine whether the system is stable or unstable. If unstable, what is the number of roots on the RH-s plane.

soln

The Routh array is as follows

$$\frac{24\epsilon - \frac{72}{\epsilon} + 10\epsilon}{4 - \frac{12}{\epsilon}}$$

s^5	1	2	11	0
s^4	2	4	10	
s^3	$\cancel{0}\epsilon$	+6	0	
s^2	$\frac{4\epsilon - 12}{\epsilon}$	10		
s^1	$\frac{24\epsilon - 72 - 10\epsilon^2}{4\epsilon - 12}$	0		
s^0	10			

$4 - \frac{12}{\epsilon}$ $-\infty$

$$\Rightarrow \lim_{\epsilon \to 0} \left[\frac{4\epsilon - 12}{\epsilon}\right] = -\infty$$

$$\Rightarrow \lim_{\epsilon \to 0} \left[\frac{24\epsilon - 72 - 10\epsilon^2}{4\epsilon - 12}\right] = \frac{-72}{-12} = 6$$

As $\epsilon \to 0$, $\frac{4\epsilon - 12}{\epsilon} \Rightarrow$ large -ve

∴ There are two sign changes on the first column

⇒ the system is unstable

⇒ the system has two-roots on the RH s-plane.

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Using the Routh-Hurwitz Criterion, Determine the stability of the system whose Characteristic eqn is given by

$$F(s) = s^5 + 2s^4 + 5s^3 + 10s^2 + 8s + 24 = 0$$

soln:

$$\begin{array}{c|ccc}
s^5 & 1 & 5 & 8 \\
s^4 & 2 & 10 & 24 \\
s^3 & \cancel{0}\,\epsilon & -4 & \\
s^2 & \dfrac{10\epsilon+8}{\epsilon} & 24 & \\
s^1 & \dfrac{-40\epsilon-32-24\epsilon^2}{10\epsilon+8} & & \\
s^0 & 24 & &
\end{array}$$

$$\Rightarrow \lim_{\epsilon \to 0}\left[\frac{10\epsilon+8}{\epsilon}\right] \Rightarrow +\infty$$

$$\Rightarrow \lim \left[\frac{-40\epsilon-32-24\epsilon^2}{10\epsilon+8}\right] = -4$$

$\Rightarrow$ two sign change in the first column

~~this~~ System is unstable

the system has two roots in the RH-s-plane.

Case II All Elements on any one row are zero

- The situation with the entire row of zero can be remedied by using the auxiliary eqn $A(s)=0$, which is formed from the coefficents of the row just above the row of zeros on the Routh's tabulation.

Take the following steps.

1- ~~From~~ Form the Auxiliary eqn $A(s)=0$ by use of the coefficients from the row just preceding the row of zeros

2- Take the derivative of the auxilary eqn w.r.t 's', this gives $\frac{dA(s)}{ds}$ which

3- Replace the row of zeros, with coefficents of $\frac{dA(s)}{ds}$.

eg. Consider the system Char. eqn and determine whether the system is stable or not

Soln $F(s) = s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

Routh's Array.

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	
~~s^1~~	~~0~~	~~0~~	
s^1	8	0	
s^0	4		

$A(s) = 4s^2 + 4 = 0$

$\frac{dA(s)}{ds} = 8s + 0$

Roots $-1.5 \pm 1.32i$, $0 \pm 1i$

since there is no sign change in the first column

→ solve the auxilary eqn

$$A(s) = 4s^2 + 4 = 0$$
$$s = \pm j$$

The eqn has two roots on the $j\omega$ axis, and the system can be regarded as marginally stable.

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Eg 2: The Char. eqn of a system is given as

$$F(s) = s^6 + 4s^5 + 12s^4 + 16s^3 + 41s^2 + 36s + 72 = 0$$

Determine whether the system is stable or unstable

sol/n

The routh Array is as follows

s^6	1	12	41	72
s^5	4	16	36	0
s^4	8	32	72	
~~s^3~~	~~0~~	~~0~~		
s^3	32	64		
s^2	16	72		
s^1	-80			
s^0	72			

$\Rightarrow X(s) = 8s^4 + 32S + 72$

$-2 \pm 2i$
$\pm 0.707 \pm 1.58i$

There are two sign change in the first column

$\Rightarrow$ the system is unstable

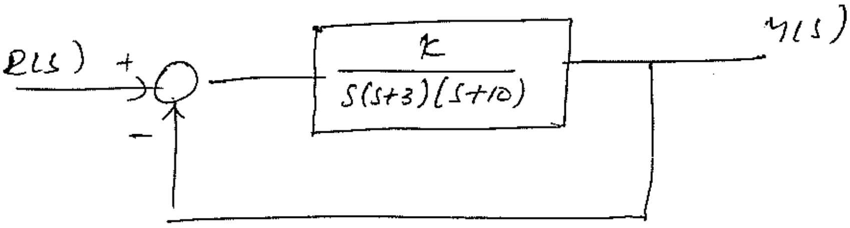
$\Rightarrow$ " " has two roots on the right-Half splane.

5-6 Application to design of Routh - Hurwitz criterion

Example: The forward path transfer function of unity feedback control system is given by

$$G(s) = \frac{K}{s(s+3)(s+10)}$$

Determine the range of K for the system to be stable



$$T(s) = \frac{G}{1+G} = \frac{K}{s(s+3)(s+10) + K} = \frac{N(s)}{D(s)}$$

the Char. eqn $D(s) = 1 + G = 0$

$$D(s) = s(s+3)(s+10) + K$$
$$= s^3 + 13s^2 + 30s + K = 0$$

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The Routh-array become

s^3	1	30
s^2	13	k
s^1	$30-\frac{k}{13}$	0
s^0	k	

cond^n for stability
$\Rightarrow$ not to have a sign change in the first column

① $30-\frac{k}{13} > 0$

② $k > 0$

i.e $k < 390$ and $k > 0$

for stable system to be $0 < k < 390$

$s^3 + 13s^2 + 30s + 390 = 0$
soln
$-13, \; 0 \pm 5.477i$

if we choose $k = 390$ (the largest gain)
$\Rightarrow$ row ③ becomes $30 - \frac{390}{13} = 0$ $\Rightarrow$ zero row

$\Rightarrow$ the auxilary char. eq^n becomes

$13s^2 + 390 = 0$

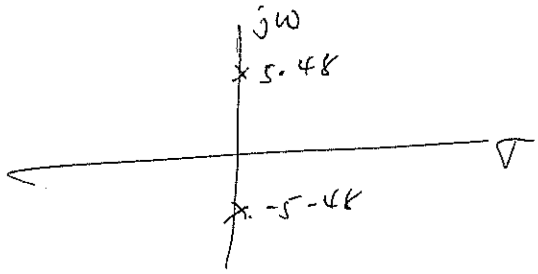
$s^2 = -30$ $\Rightarrow$ complex number $s = \sigma + j\omega$ ($\sigma = 0$)

$s^2 = j^2 30$

$\cancel{(j\omega)^2 = j^2 30}$

$s = \pm j\sqrt{30}$

$= \pm j\, 5.48$



the system will be marginally stable.

Exersize

1) For the ff. Char eq. ~~chec~~ sta

a) $F(s) = s^3 + 4s^2 + 6s + 6$ — unst

b) $F(s) = s^4 + s^3 + 2s^2 + 10s + 8$ — unst

c) $F(s) = s^5 + s^4 + 2s^3 + s + 5$

2) Determine the range of values of k for the system to be stable and whose char. eq^n is given as follows.

a) $F(s) = s^4 + s^3 + 3s^2 + 2s + K$ 14.

b) $F(s) = s^5 + s^4 + 2s^3 + s^2 + s + K$

a) $0 < K < 2$

b) $0 < K < 1$

3) For unity feedback control system with

$$G(s) = \frac{K(s^2+2s+2)}{s(s^2-2s+2)}$$

Determine the range of values of K for stability

Ans $K \geq 2.732$

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