DEBRE MARKOS UNIVERSITY

DEBRE MARKOS INSTITUTE OF TECHNOLOGY

SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING

Probability and Random Process (ECEg2103)

For 2012 Second Year ECE Regular Students;

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Chapter One
Introduction to Probability Theory

Outlines

➢ Introduction

➢ Discrete Vs Continuous random variables

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1.1 <u>Introduction</u>

• **Probability** deals with **unpredictability** and **randomness**, and **probability theory** is the branch of mathematics that is concerned with the **study of random phenomena**.

• A **random phenomenon** is one that, under **repeated observation**, yields **different outcomes** that are **not deterministically predictable**.

• However, these outcomes obey certain conditions of statistical regularity whereby the **relative frequency of occurrence** of the possible outcomes is **approximately predictable**.

• Probability allows us to **quantify the chance** that a certain event will occur.

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- The fundamental issue in random phenomena is the **idea of a repeated experiment** with **a set of possible outcomes** or **events.**

- Associated with each of these events is a **real number called the probability of the event** that is **related to the frequency of occurrence of the event** in a long sequence of **repeated trials** of the experiment.

- In this way it becomes obvious that the **probability of an event** is a value that lies **between zero** and **one**, and the **sum of the probabilities** of the events for a particular experiment should **sum** to **one**.

1.2 Discrete Vs Continuous random variables

- A random variable is a real-valued function of the outcome of the experiment.

❖ A random variable is called **discrete** if its range (the set of values that it can take) is **finite** or at most **countably infinite**. **Examples:-**

✓ In an experiment involving a sequence of 5 tosses of a coin, the number of heads in the sequence is a random variable.

✓ In an experiment involving two rolls of a die:

(i) The sum of the two rolls.

(ii) The number of sixes in the two rolls

Cont…

- In short a **discrete random variable** is a random variable whose value is obtained by **counting**.

Examples:- number of students present

-students grade level

❖**Continuous random variable** is a random variable whose value is obtained by **measuring;** its range contains an interval (either finite or infinite) of real numbers.

Examples:- height , weight of students in class

-time it takes to get to school.

- distance traveled between classes.

1.3 Some Applications of Probability

- The theory of probability deals with averages of mass phenomena occurring sequentially or simultaneously like; Electron emission, telephone calls, radar detection, quality control, system failure, games of chance, statistical mechanics, noise, birth and death rates, turbulence, etc.

- In these and other fields **certain averages approach constant** value as **the number of observations increase**.

- So, it is clear that; there are several science and engineering applications of probability.

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- To see some of these applications in some extent;

1. Reliability Engineering

- Reliability theory is **concerned with the duration of the useful life of components** and **systems of components.**

- System **failure times are unpredictable.**

- Thus, the time until a system fails, which is referred to as the time to failure of the system, is usually **modeled by a probabilistic** function.
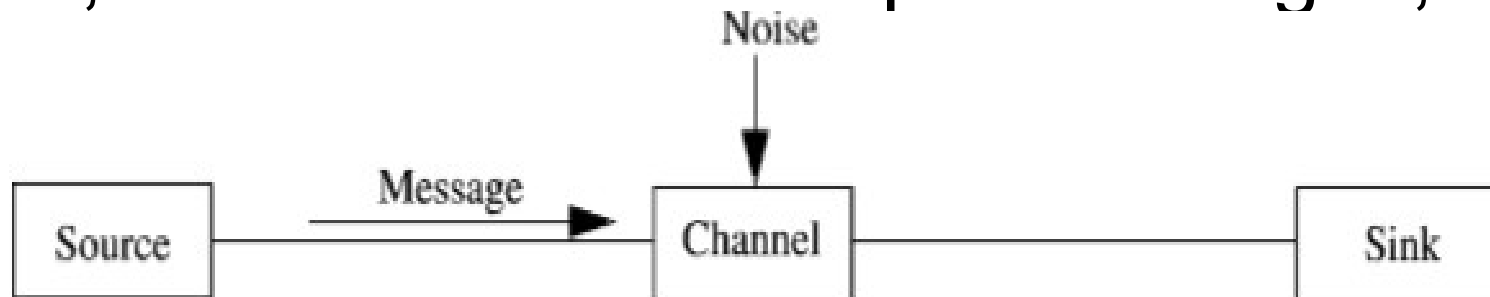
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2. Quality Control

- Quality control deals with the inspection of finished products to ensure that they meet the desired requirements and specifications.

- **One way** to perform the quality control function is to **physically test/inspect each product** as it comes off the production line. However, this is a **very costly way** to do it.

- The **practical method is to randomly select a sample** of the product from a lot and test each item in the sample; decision is on **probability**.

Cont…

3. Channel Noise

- Noise is an unwanted signal.

- A message transmitted from a source passes through a channel where it is subject to different kinds of random disturbances that can introduce errors in the message received at the sink.

- That is, channel noise corrupts messages, as show

Cont…

- Since **noise is a random signal**, one of the performance Isues is the **probability** that the received message was not corrupted by noise.

- Thus, **probability plays an important role in Evaluating the performance of noisy communication channels**.

4. System Simulation

- Sometimes it is **difficult to provide an exact solution** of physical problems involving random phenomena.

- The difficulty arises from the fact that such problems are **very complex.**

Cont…

- A **simulation model describes the operation of a system** in terms of individual events of the individual elements in the system.

- The **key** to a simulation model **is the generation of random numbers** that can be used to **represent events**.

- Since these **events are random in nature**, the **random numbers** are used to **drive** the **probability distributions** that characterize them.

- Thus, **knowledge of probability theory** is **essential for a meaningful simulation** analysis.

1.4 Sample Space and Events

• The **concepts of experiments and events** are very important in the study of probability.

• In probability, an **experiment is any process of trial and observation.**

❖ An experiment whose **outcome** is **uncertain** before it is performed is called a **random experiment.**

• In other word, **random experiment** is an experiment in which the **outcome varies in an unpredictable** fashion when the **experiment is repeated under the same conditions.**

-Some examples of a random experiment are the roll of a die, the toss of a coin, drawing a card from a deck, or selecting a message signal for transmission from several messages.

Cont…

❖ Sample space

- The **set of all possible outcomes** of a random experiment is called the **sample space (or universal set)**, and it is denoted by **S** or Ω.

- An **element** in S is called a **sample point**.

- Each outcome of a random experiment corresponds to a **sample point**.

- Note that the sample space S is the subset of itself, that is, S c S.

- Since **S** is the set of all possible outcomes, it is often called the **certain event**.

Cont…

- The sample space $S(\Omega)$ can be specified compactly by using set notation. It can be visualized by drawing tables, diagrams, intervals of the real line, or regions of the plane.

- Two basic ways to specify a set:

1. List all the elements, separated by commas, inside a pair of braces. For example: $A = \{1, 2, 3, \cdots, 13\}$

2. Give a property that specifies the elements of the set $A = \{x : x \text{ is an integer such that } 0 \leq x \leq 3\}$

Cont…

❖ Event

- Any subset of the sample space S is called an **even**t.
- A **sample point** of S is often referred to as an **elementary event.**
- An **event** is the occurrence of either a prescribed outcome or any one of a number of possible outcomes of an experiment.
- Thus, an **event** is a **subset** of the **sample space**.

Example 1.1:- if we toss a die, any number from 1 to 6 may appear.

➢ Therefore, in this experiment;

✓ the sample space is defined by S={1,2,3,4,5,6} .

✓ The event "the outcome of the toss of a die is an **even** number" is a subset of S and is defined by E={2,4,6}.

Cont…

EXAMPLE 1.2:- Find the sample space and event/s for the experiment of tossing a coin repeatedly and of **counting the number of tosses required until the first head appears.**

Solution

✓ Clearly all possible outcomes for this experiment are the terms of the sequence 1,2,3, ...

✓ Thus; sample space, **s** = {1, 2, 3, . . .}

• Note that there are an **infinite** number of outcomes.

Cont…

✓ For **event** it should be specified; Let:-

• A be the event that the number of tosses required until the first head appears is **even**.

• B be the event that the number of tosses required until the first head appears is **odd**.

• C be event that the number of tosses required until the first head appears is **less than 5**.

✓ Events A, B, and C; express as,

$$A = \{2, 4, 6, \ldots\}$$
$$B = \{1, 3, 5, \ldots\}$$
$$C = \{1, 2, 3, 4\}$$

Cont…

Example-1.3:

Consider a random experiment of flipping a fair coin twice.

i. Sample space

$\Omega = \{HH, \text{HT}, \text{TH}, \text{TT}\}$

ii.   Some possible events

▪ An event of getting exactly one head

$A = \{HT, \text{TH}\}$

▪ An event of getting at least one tail

$B = \{HT, \text{TH}, \text{TT}\}$

▪ An event of getting at most one tail

$C = \{HH, HT, \text{TH}\}$

1.5 <u>Revision of Set Algebra</u>

• Events and combinations of events play a central role in probability theory.

• The mathematics of events is closely tied to the theory of sets.

• Thus **Probability makes extensive use of set operations;** so it is better to introduce at the outset the relevant notation and terminology.

• A **set** is a collection of objects possessing some common properties.

• These objects are called **elements** of the set and they can be of any kind with any specified properties.

Cont...

- **Capital** letters A,B,C, Ω, ...shall be used to **denote sets**,and **lower-case** letters a,b,c **denote to** their **elements**

Remember:-

✓ **Empty** or **null** set containing **no elements** and denoted by ⊘ .

✓ finite sets containing a finite number of elements

✓ infinite sets having an infinite number of elements.

✓ An infinite set is called **enumerable** or **countable** if all of its **elements can be arranged** in such a way that there is a **one-to-one correspondence** between them and **all positive integers**. Ex,. A= {1,2,3,...}

Cont…

✓ A **nonenumerable** or **uncountable** set is one be where **one-to-one correspondence** cannot be established. Example; A={x: x≥0}

✓ If A={a, b, c, d, e}; **a** ϵ **A** to mean element a belongs to set A.

✓ **Subset:** If every element of a set A is also an element of a set B, the set **A is called a subset of B** and it represented symbolically A ⊂ B or B ⊃ A.

✓ Empty set is a subset of any set.

✓ Any set is a subset of itself.

Cont…

- Let a set A contain n elements a1,a2,...,an.labeled
- The number of possible subsets of A is 2^n,

- The **set of all subsets of a set A** is called the **power set of A and denoted by s(A)**. Thus, for the set A={a,b,c}, the power set of A is given by s(A)={∅,{a},{b},{c},{a,b},{a,c},{b,c},{a,b,c}}.

- The **number of members of a set A** is called the **cardinality** of A and denoted by |A|.

- Thus, if the cardinality of the set A is n, then the cardinality of the power set of A is |S(A)|= 2^n,

Cont…

- Let us now consider some algebraic operations of sets A,B,C,...that are subsets of space S.

I. Equality:

Two sets A and B are equal, denoted A = B, if and only if A c B and B c A.

2. Complementation : Suppose A c S;

The complement of set A, denoted Â, is the set containing all elements in S but not in A.

✓ Â = {x: x ∈ S ,but x is not belong to A)

Cont…

3. Union:

The union of sets A and B, denoted A u B, is the set containing all elements in either A or B or both.

4. Intersection:

The intersection of sets A and B, denoted A n B, is the set containing all elements in both A and B.

5. Difference:

The difference of two sets A and B, denoted by A−B, is the set containing all elements of A that are not in B.

6. Disjoint Sets:

Two sets A and B are called **disjoint or mutually exclusive** if they contain no common element.

Cont…

- The definitions of the **union** and **intersection** of two sets can be extended to any **finite** number of sets as follows:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= \{\zeta : \zeta \in A_1 \text{ or } \zeta \in A_2 \text{ or } \cdots \zeta \in A_n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$= \{\zeta : \zeta \in A_1 \text{ and } \zeta \in A_2 \text{ and } \cdots \zeta \in A_n\}$$

Cont…

- These definitions also can be extended to an infinite number of sets as:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots$$

- Every subset of S is an event, including sample space and the null set.

- N.B:

$$S = \text{the certain event}$$
$$\emptyset = \text{the impossible event}$$

Cont…

- If A and B are events in S, then

$$\bar{A} = \text{the event that } A \text{ did not occur}$$

$$A \cup B = \text{the event that either } A \text{ or } B \text{ or both occurred}$$

$$A \cap B = \text{the event that both } A \text{ and } B \text{ occurred}$$

- Similarly, if A,, A,, . . . , A, are a sequence of events in S, then

$$\bigcup_{i=1}^{n} A_i = \text{the event that at least one of the } A_i \text{ occurred;}$$
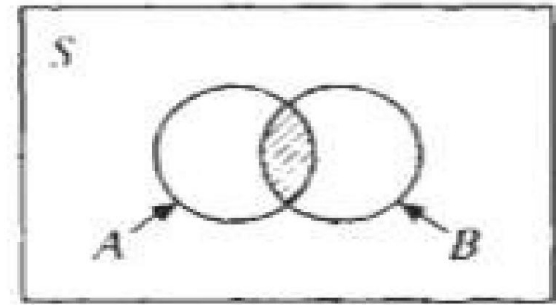
$$\bigcap_{i=1}^{n} A_i = \text{the event that all of the } A_i \text{ occurred.}$$
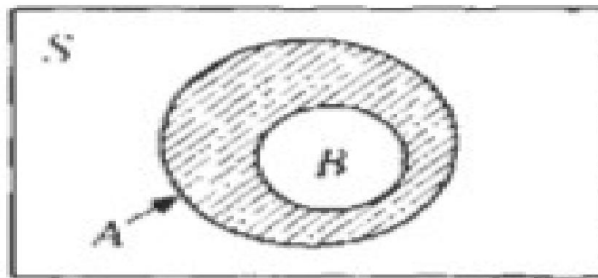
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- **Venn Diagram**: is a graphical representation that is very useful for illustrating set operation. For instance,
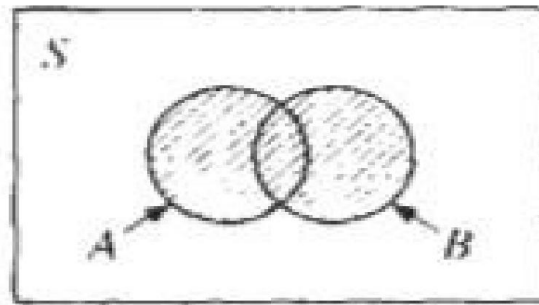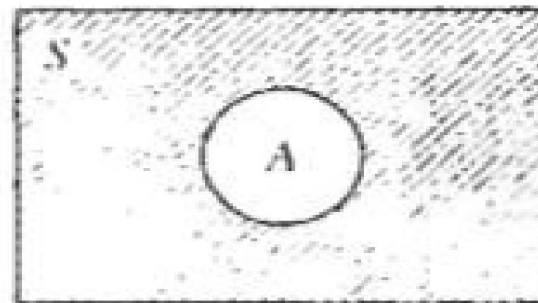

(a) Shaded region: $A \cup B$

(b) Shaded region: $A \cap B$

$B \subset A$
Shaded region: $A \cap \bar{B}$

(c) Shaded region: $\bar{A}$

Cont…

- **Identities**: By the above set definitions we obtain the following identities:

$$\overline{\overline{A}} = A \qquad \overline{S} = \varnothing \qquad S \cup A = S$$
$$\overline{\varnothing} = S \qquad S \cap A = A$$
$$A - B = A \cap \overline{B} \qquad A \cup \overline{A} = S$$
$$A \cap \overline{A} = \varnothing$$

- The union and intersection operations also satisfy the following laws:

1. **Commutative Laws**:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. **Associative Laws**:
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Cont…

3. Distributive Laws: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

It can be extended as;

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i)$$

4. De Morgan's Laws: It can be extended as;

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{\left(\bigcup_{i=1}^{n} A_i \right)} = \bigcap_{i=1}^{n} \bar{A}_i$$

$$\overline{\left(\bigcup_{i=1}^{n} A_i \right)} = \bigcap_{i=1}^{n} \bar{A}_i$$

1.6. <u>Definitions of Probability</u>

- There are several ways to define probability.

- we try to see three definitions: the axiomatic, the relative-frequency, and the classical definition.

1. Axiomatic Definition

- Consider a random experiment whose sample space is S.

- For each event A of S we assume that a number **P(A)**, called the probability of event A, is defined such that the following hold:

Cont...

i. **Axiom**1:**0≤P(A)≤1**,which means that the probability of A is some number between and including 0 and 1.

ii. **Axiom 2: P(S)=1**, which states that with

probability 1, the outcome will be a sample point in the sample space.

iii **Axiom 3**: For any set of n **mutually exclusive** events A1,A2,...,An defined on the same sample space,

P(A1 ∪A2∪···∪An)=P(A1)+P(A2)+···+P(An)

Cont…

2. Relative-Frequency Definition

- Consider a random experiment that is performed **n** times. If an event **A occurs nA times**, then the probability of event A, P(A), is defined as follows:

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

- The ratio **nA/n** is called the **relative frequency** of **event A**.

- Even it is intuitively satisfactory for many practical problems, **it has a few limitations**; especially when the **experiment may not be repeatable**, like testing of expensive and/or scarce resources.

Cont…

3. Classical Definition

- The probability **P(A)** of an event **A** is the **ratio of** the number of outcomes **NA** of an experiment that are **favorable to A** to the **total number N** of **possible outcomes** of the experiment. That is,

- This probability is **determined a priori without a** $P(A) = \dfrac{N_A}{N}$ **forming the experiment;** but it leads principles of insufficient reason in the **absence** of any **prior** knowledge.

- Thus **equally likely** consideration used to improve the above equation, means actually, equally probable. That is, for equally likely events; P1=P2=…=Pn

- For example, in a coin toss experiment, there are two possible outcomes: heads or tails. Thus, N=2, and if the coin is fair, the probability of the event that the toss comes up heads is 1/2.

1.7 Properties of Probability

- When we **combine** the results of **set identities with those of the axiomatic definition of probability**; we obtain the following results:
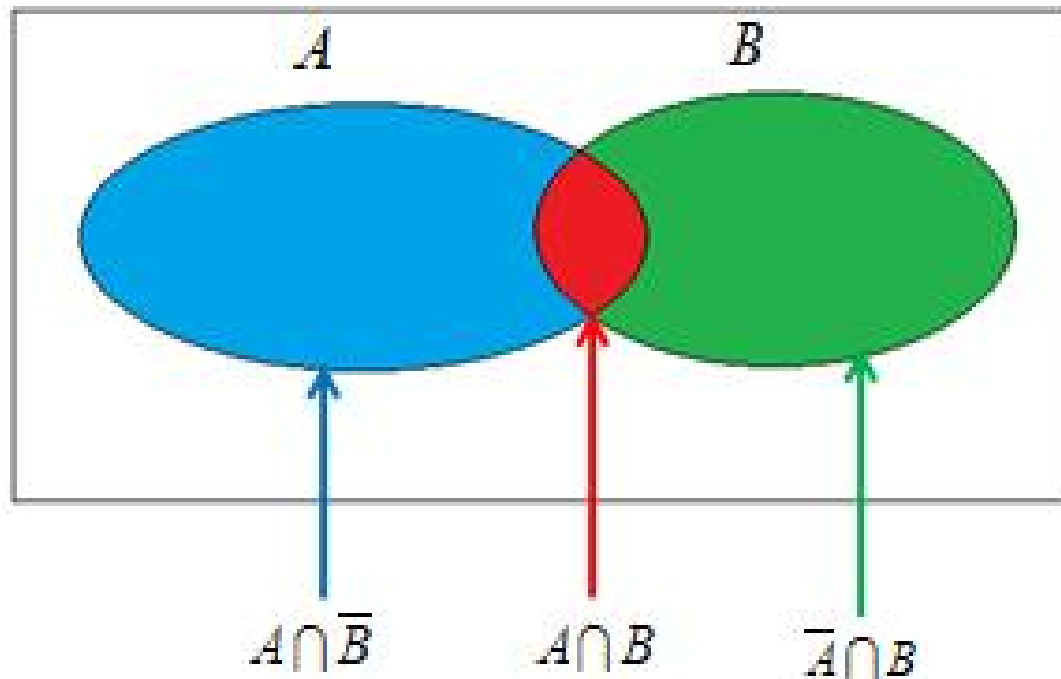
1. $P(\bar{A}) = 1 - P(A)$

Proof:

$$A \cap \bar{A} = \Phi \Rightarrow P(A \cup \bar{A}) = P(A) + P(\bar{A}), \text{ but } A \cup \bar{A} = s$$

$$\Rightarrow P(s) = P(A) + P(\bar{A}), \ P(s) = P(A \cup \bar{A})$$

$$\Rightarrow 1 = P(A) + P(\bar{A}), \ P(s) = 1$$

$$\therefore P(\bar{A}) = 1 - P(A)$$

Cont...

2. $P(\emptyset)=0$.

3. If $A \subset B$, then $P(A) \leq P(B)$.

4. $P(A) \leq 1$

5. If $A = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_1, A_2, ..., A_n$ are **mutually exclusive events**, then

$$P(A) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Cont…

- We can decompose the events **A**, **B** and **AUB** as unions of mutually exclusive (disjoint) events as follows.

We can see that $A \cap B$, $A \cap \bar{B}$ and $\bar{A} \cap B$ are disjoint events.

Cont…

6. From the above Venn diagram, we can write the following relations.

i. $A = (A \cap B) \cup (A \cap \bar{B})$

$\Rightarrow P(A) = P(A \cap B) + P(A \cap \bar{B})$

$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$

ii. $B = (A \cap B) \cup (\bar{A} \cap B)$

$\Rightarrow P(B) = P(A \cap B) + P(\bar{A} \cap B)$

$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$

iii. $A \cup B = A \cup (\bar{A} \cap B)$

$\Rightarrow P(A \cup B) = P(A) + P(\bar{A} \cap B)$

iv. $A \cup B = B \cup (A \cap \bar{B})$

$\Rightarrow P(A \cup B) = P(B) + P(A \cap \bar{B})$

v. $A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$

$\Rightarrow P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B)$

Cont…

- ## The other one is;

7. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof

$P(A \cup B) = P(A) + P(\overline{A} \cap B)$

But, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- We can generalize the above property for three events A, B and C as follows.

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$

$+ P(A \cap B \cap C)$

Cont…

- For **n** events $A_1, \ A_2, \ A_3, \ldots, A_n$ the above property can be generalized as:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k)$$
$$- \cdots (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

8. $P(A \cup B) \leq P(A) + P(B)$

Proof:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

But, $P(A \cap B) \geq 0$

$$\therefore P(A \cup B) \leq P(A) + P(B)$$

Cont…

Example-1.4:

A box contains 10 identical balls numbered 0, 1, 2,…,9. A single ball is selected from the box at random. Consider the following events.

A: number of ball selected is odd

B: number of ball selected is multiple of 3

C: number of ball selected is less than 5

Find the following probabilities.

a. $P(A)$ d. $P(A \cap B)$

b. $P(B)$ e. $P(A \cup B \cup C)$

c. $P(C)$

Cont…

Solution:

➢ The sample space(Ω) and the events are given by:

$\Omega = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $C = \{0, 1, 2, 3, 4\}$

$A = \{1, 3, 5, 7, 9\}$ $A \cap B = \{3, 9\}$

$B = \{3, 6, 9\}$ $A \cup B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$

➢ The number of elements in the sample space and

events are: $n(\Omega) = 10$ $n(C) = 5$

$n(A) = 5$ $n(A \cap B) = 2$

$n(B) = 3$ $n(A \cup B \cup C) = 9$

Cont…

Thus, probabilities of the given events are given by:

a. $P(A) = \dfrac{n(A)}{n(\Omega)} = \dfrac{5}{10} = \dfrac{1}{2}$

d. $P(A \cap B) = \dfrac{n(A \cap B)}{n(\Omega)} \quad \dfrac{2}{10} = \dfrac{1}{5}$

b. $P(B) = \dfrac{n(B)}{n(\Omega)} = \dfrac{3}{10}$

e. $P(A \cup B \cup C) = \dfrac{n(A \cup B \cup C)}{n(\Omega)} = \dfrac{9}{10}$

c. $P(C) = \dfrac{n(C)}{n(\Omega)} = \dfrac{5}{10} = \dfrac{1}{2}$

Example-1.5:

Given $P(A) = 0.9, P(B) = 0.8$ and $P(A \square B) = 0.75$, find :

a. $P(A \cup B)$ c. $P(\overline{A} \cap \overline{B})$ e. $P(\overline{A} \cup B)$

b. $P(A \cap \overline{B})$ d. $P(\overline{A} \cup \overline{B})$ f. $P(\overline{B})$

Cont…

Solution:

a. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = 0.9 + 0.8 - 0.75$

$\therefore P(A \cup B) = 0.95$

b. $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

$= 0.9 - 0.75$

$= 0.15$

c. $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$\Rightarrow P(\bar{A} \cap \bar{B}) = 1 - 0.95$

$\therefore P(\bar{A} \cup \bar{B}) = 0.05$

d. $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$

$\Rightarrow P(\bar{A} \cup \bar{B}) = 1 - 0.75$

$\therefore P(\bar{A} \cup \bar{B}) = 0.25$

e. $P(\bar{A} \cup B) = 1 - P(A) + P(A \cap B)$

$\Rightarrow P(\bar{A} \cup B) = 1 - 0.9 + 0.75$

$\therefore P(\bar{A} \cup B) = 0.85$

f. $P(\bar{B}) = 1 - P(B)$

$\Rightarrow P(\bar{B}) = 1 - 0.8$

$\therefore P(\bar{B}) = 0.2$

1.8 <u>Conditional Probability</u>

• The concept of conditional probability is a **very useful** one.

• Given two events A and B associated with a random Experiment, probability **P(A|B)** is defined as the conditional probability of A, given that B has occurred.

• The conditional probability of an event **A** given event **B**, denoted by **P(A | B)**, is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

where P(A n B) is the joint probability of A and B; $P(AB)$.

• Similarly, the conditional probability of an event B given event A is, $P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad P(A) > 0$

Cont…

- From the above equations we will get;

  $P(AB) = P(A \cap B) = P(A/B)P(B) = P(B/A)P(A)$

- Then using the above equation , we will get

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)} \quad OR \quad P(B/A) = \frac{P(A/B)P(B)}{P(A)} \quad \cdots 1$$

We know that

$$P(B) = P(A \cap B) + P(\overline{A} \cap B)$$

$$\Rightarrow P(B) = P(B/A)P(A) + P(B/\overline{A})P(\overline{A}) \quad \cdots\cdots\cdots 2$$

- Substituting equation (2) into equation (1), we will get
  $$P(A/B) = \frac{P(B/A)P(A)}{P(B/A)P(A) + P(B/\overline{A})P(\overline{A})} \quad \cdots 3$$
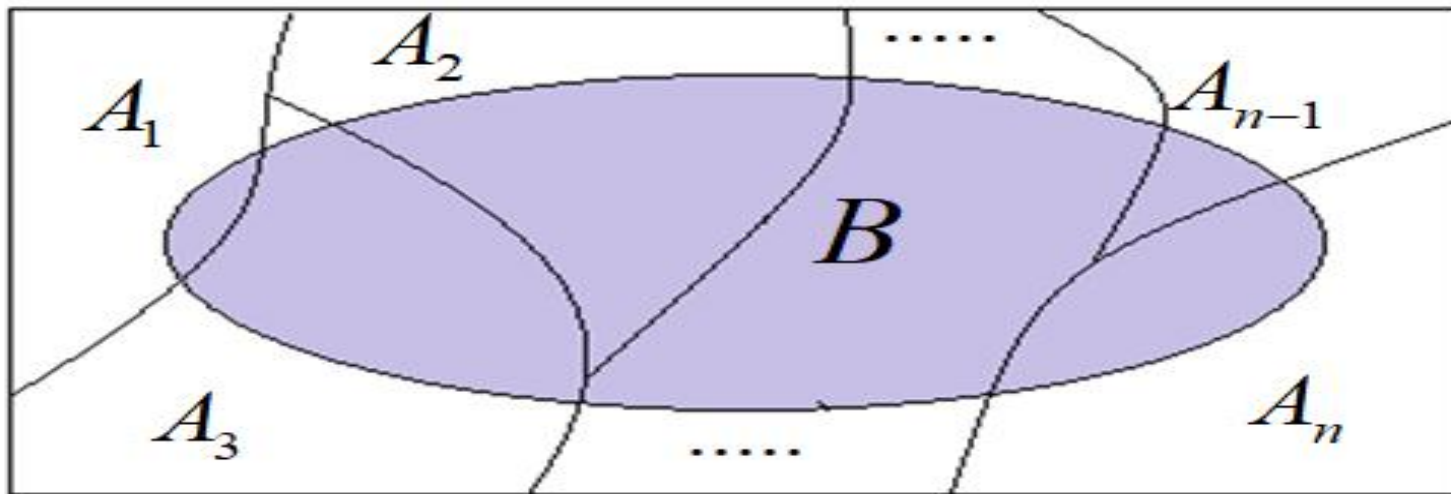
Cont…

- Similarly,

$$P(B/A) = \frac{P(A/B)P(B)}{P(A/B)P(B) + P(A/\bar{B})P(\bar{B})} \quad \dots 4$$

❖ Equations (3) and (4) are known as **_Baye's Rule_**.

▪ **_Baye's Rule_** can be extended for **n** events as follows;

Let events $A_1, A_2, A_3, \dots, A_n$ be pair wise **mutually exclusive** events and **_exhaustive_** (their union be the sample space $\boldsymbol{\Omega}$), i.e

$$A_i \cap A_j = \Phi \text{ and } \bigcup_{i=1}^{n} A_i = \Omega$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Cont…

❖For **Theorem of Total Probability** proof; Let B be any event in Ω as shown below:

$$B = B \cap (A_1 \cup A_2 \cup .... \cup A_n)$$

$$\Rightarrow B = (B \cap A_1) \cup (B \cap A_2) \cup ... \cup (B \cap A_n)$$

But, $A_i \cap A_j = \Phi \Rightarrow (B \cap A_i) \cap (B \cap A_j) = \Phi$

Cont…

- The events $B \cap A_i$ and $B \cap A_j$ are mutually exclusive.

$$\Rightarrow P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

$$\Rightarrow P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \ldots + P(B/A_n)P(A_n)$$

- This, called **theorem of total probability:** shortly;

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B/A_i)P(A_i)$$

- Then using equation (1), we can obtain:

$$P(A_i / B) = \frac{P(B/A_i)P(A_i)}{\sum_{i=1}^{n} P(B/A_i)P(A_i)}$$

Cont…

Example-1.6:

Show that $P(\overline{A}/B) = 1 - P(A/B)$

Solution:

$$P(B) = P(A \cap B) + P(\overline{A} \cap B)$$

$$\Rightarrow P(B) = P(A/B)P(B) + P(\overline{A}/B)P(B)$$

Dividing both sides by $P(B)$, we obtain

$$1 = P(A/B) + P(\overline{A}/B)$$

$$\therefore P(\overline{A}/B) = 1 - P(A/B)$$

Cont…

Example-1.7:

Let A and B be two events such that $P(A)=x$, $P(B)=y$ and $P(B/A)=z$. Find the following probabilities in terms of x, y and z.

   $a.\ \ P(A/B)$

   $b.\ \ P(\overline{A}\cup\overline{B})$

   $c.\ \ P(\overline{A}/B)$

Solution:

$$P(A\cap B)=P(B/A)P(A)=xz$$

a. $\quad P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{xz}{y}$

b. $\quad P(\overline{A}\cup\overline{B})=P(\overline{A\cap B})=1-P(A\cap B)=1-xz$

c. $\quad P(\overline{A}/B)=\dfrac{P(\overline{A}\cap B)}{P(B)}=\dfrac{P(B)-P(A\cap B)}{P(B)}=1-\dfrac{xz}{y}$

Cont…

Example-1.8:

A box contains two black and three white balls. Two balls are selected at random from the box without replacement. Find the probability that

a. both balls are black

b. the second ball is white

Solution:

First let us define the events as follows:

B_1 : the outcome in the first selection is a black ball

Cont…

B_2 : the outcome in the second selection is a black ball

W_1 : the outcome in the first selection is a white ball

W_2 : the outcome in the second selection is a black ball

$$P(B_1) = 2/5 \qquad P(B_2/B_1) = 1/4 \qquad P(W_2/B_1) = 3/4$$
$$P(W_1) = 3/5 \qquad P(B_2/W_1) = 2/4 \qquad P(W_2/W_1) = 2/4$$

a. $P(B_1 \cap B_2) = P(B_2/B_1)P(B_1) = (1/4)(2/5)$

$\therefore P(B_1 \cap B_2) = 1/10$

b. $P(W_2) = P(W_2 \cap B_1) + P(W_2 \cap W_1)$

$\qquad = P(W_2/B_1)P(B_1) + P(W_2/W_1)P(W_1)$

$\qquad = (3/4)(2/5) + (2/4)(3/5)$

$\therefore P(W_2) = 3/5$

Cont…

Example-1.9:

Box _A_ contains 100 bulbs of which 10% are defective. Box _B_ contains 200 bulbs of which 5% are defective. A bulb is picked from a randomly selected box.

a. Find the probability that the bulb is defective

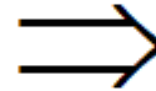b. Assuming that the bulb is defective, find the probability that it came from box _A_.

Cont…

Solution:

First let us define the events as follows.

A : Box A is selected

B : Box B is selected $\Longrightarrow$

D : Bulb is defective

$P(A) = P(B) = 1/2$

$P(D/A) = 1/10$

$P(D/B) = 1/20$

a. $P(D) = P(D/A)P(A) + P(D/B)P(B)$

$\quad = (1/10)(1/2) + (1/20)(1/20)$

$\therefore P(D) = 3/40$

b. $P(A/D) = \dfrac{P(D/A)P(A)}{P(D)} = \dfrac{1/20}{3/40} = (1/20)(40/3)$

$\therefore P(A/D) = 2/3$

Cont…

Example-1.10:

One bag contains 4 white and 3 black balls and a second bag contains 3 white and 5 black balls. One ball is drawn from the first bag and placed in the second bag unseen and then one ball is drawn from the second bag. What is the probability that it is a black ball?

Solution:

First let us define the events as follows.

B_1: black ball is drawn from the first bag

W_1: white ball is drawn from the first bag

Cont…

B_2 : black ball is drawn from the second bag

W_2 : white ball is drawn from the second bag

Then, we will have:

$$P(B_1) = 3/7 \qquad P(B_2/B_1) = 6/9 \qquad P(B_2/W_1) = 5/9$$

$$P(W_1) = 4/7 \qquad P(W_2/B_1) = 3/9 \qquad P(W_2/W_1) = 4/9$$

$$P(B_2) = P(B_2 \cap B_1) + P(B_2 \cap W_1)$$

$$\Rightarrow P(B_2) = P(B_2/B_1)P(B_1) + P(B_2/W_1)P(W_1)$$

$$\Rightarrow P(B_2) = (6/9)(3/7) + (5/9)(4/7)$$

$$\therefore P(B_2) = 28/63$$

1.9 <u>Statistically Independent</u>

❖ If we consider a special case for two events A&B; in which the occurrence or nonoccurrence of one **does not affect** the occurrence or nonoccurrence of the other.

• In this situation events A and B are called **statistically independent** or simply **independent.**

✓ Two events A and B are said to be independent if and only if: **P(AB)=P(A∩B)=P(A)P(B).**

✓ Similarly, three events A, B and C are said to be statistically independent if and only if

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Cont…

- Generally, if $A_1, A_2, \ldots, A_n$ are a sequence of independent events, then

$$A_i \cap A_j = \Phi \text{ and } \bigcup_{i=1}^{n} A_i = \Omega$$

$$\Rightarrow P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

- If A and B are independent, then we have;

i. $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)P(B)}{P(B)} = P(A)$

$\Rightarrow P(A/B) = P(A)$

ii. $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{P(A)P(B)}{P(A)} = P(B)$

$\Rightarrow P(B/A) = P(B)$

Cont…

Example-1.11:

If A and B are independent, then show that A and $\overline{B}$ are also independent.

Solution:

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

$$\Rightarrow P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$

$$\Rightarrow P(A \cap \overline{B}) = P(A)[1 - P(B)] = P(A)P(\overline{B})$$

$\therefore$ By the definition of independent events, A and $\overline{B}$ are independent.

Cont…

Example-1.12:

The probability that a husband and a wife will be alive 20 years from now are given by 0.8 and 0.9 respectively. Find the probability that in 20 years:

a. Both will be alive

b. Neither will be alive

c. At least one will be alive

Cont…

Solution:

- First let us define the events as follows.

  H : Husband will be alive

  W : Wife will be alive

- Then we will have

  $$P(H) = 0.8 \Rightarrow P(\overline{H}) = 1 - P(H) = 1 - 0.8 = 0.2$$
  $$P(W) = 0.9 \Rightarrow P(\overline{W}) = 1 - P(W) = 1 - 0.9 = 0.1$$

- The two events can be considered as independent.

Cont…

a. $P(both) = P(H \cap W) = P(H)P(W)$

$\Rightarrow P(both) = P(H \cap W) = (0.8)(0.9)$

$\therefore P(both) = P(H \cap W) = 0.72$

b. $P(neither) = P(\overline{H} \cap \overline{W}) = P(\overline{H})P(\overline{W})$

$\Rightarrow P(neither) = P(\overline{H} \cap \overline{W}) = (0.2)(0.1)$

$\therefore P(neither) = P(\overline{H} \cap \overline{B}) = 0.02$

c. $P(\text{at least one}) = 1 - P(\text{neither })$

$\Rightarrow P(\text{at least one}) = 1 - 0.02$

$\therefore P(\text{at leat one}) = 0.98$

THANK YOU!!!!!

After we will see some More Examples on a board; Chapter One be Ended!!!

If You have Any question??? Welcome!!!!!