DEBRE MARKOS UNIVERSITY

DEBRE MARKOS INSTITUTE OF TECHNOLOGY
SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING

Probability and Random Process (ECEg2103)

For 2012 Second Year ECE Regular Students;

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Chapter Two

Random Variables and Random Distribution

Outlines

- Introduction
- Random Variables
- Events Defined by Random Variable
- The Cumulative Distribution Function (CDF)
- Types of Random Variables
- The Probability Mass Function (PMF)
- The Probability Density Function (PDF)
- Expected Value, Variance and Moments
- Some Special Distributions
- Multiple Random Variables

2.1 Introduction

- The **concept of a probability space** that completely describes the outcome of a **random experiment** has been developed in Chapter-1, **but a systematic and unified procedure is needed to facilitate making** these statements, which can be quite complex.
- One of the **immediate steps** that can be taken in this unifying attempt is to **require** that **each of the possible outcomes** of a random experiment be **represented** by a **real number**.
- In this way, when the experiment is performed, **each outcome is identified by its assigned real number rather than by its physical description**.

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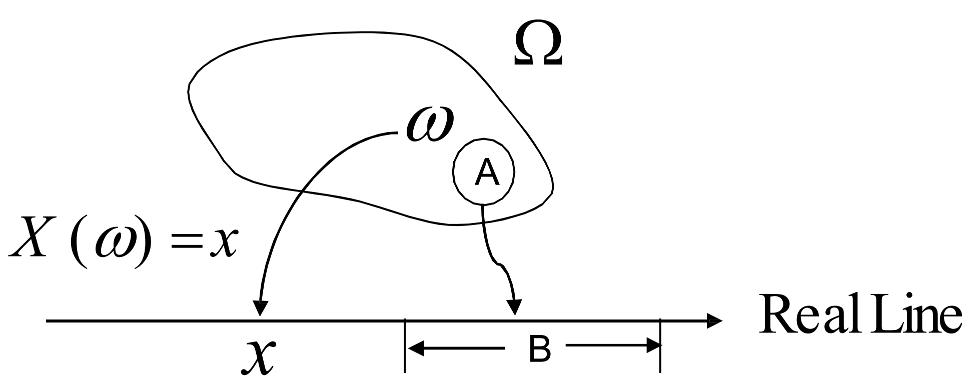
- This procedure not only permits **to replace a sample space** of arbitrary elements **by a new sample space having only real numbers** as its elements,
- But also; since most problems in **science** and **engineering** deal with **quantitative measures**; which leads sample spaces associated with **many random experiments of interest** and these are already themselves sets of real numbers; the real-number assignment procedure is thus **a natural unifying agent**.
- So, **introducing a variable** , which is **used to represent real numbers**, the **values of which are determined by the outcomes of a random experiment**; **leads** to the notion of **a random variable** is necessary.

2.2 RANDOM VARIABLES

- A random variable ***X*** is a **function** that assigns a real number ***X(ω)*** to each outcome ***ω*** in the sample space ***Ω*** of a random experiment.
- Generally a random variable is represented by a single letter X instead of the function X(w).
- The sample space ***Ω*** is the **domain** of the random variable and the set $\boldsymbol{R_X}$ of all values taken on by ***X*** is the **range** of the random variable.
- Thus, $\boldsymbol{R_X}$ is the subset of all real numbers.



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- If X is a random variable, then $\{\omega: X(\omega) \leq x\} = \{X \leq x\}$ is an event for every X in R_X.

Example 2.1: Consider a random experiment of tossing a fair coin three times. The sequence of heads and tails is noted and the sample space **Ω** is given by:

$$\Omega = \{HHH, HHT, HTH, THH, THT, HTT, TTH, \mathrm{TTT}\}$$

- Let X be the number of heads in three coin tosses. X assigns each possible outcome ω in the sample space Ω a number from the set $R_X = \{0, 1, 2, 3\}$.

$$\begin{array}{lcccccccc} \omega: & HHH & HHT & HTH & THH & THT & HTT & TTH & TTT \\ X(\omega): & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \end{array}$$

2.3 Events Defined by Random Variable

- Let **X** be a **random variable** and **x** be a **fixed real value**, and event **Ax** define the subset of **S** that consists of all real sample points to which the random variable X assigns the number **x**. That is,

$$\mathbf{Ax = \{w \mid X(w)=x\}=[X = x]}$$

- Since **Ax** is an event, it will have a probability, which we define as: **p=P[Ax]**
- Other types of events can define in terms of a random variable.
- For fixed numbers **x, a**, and **b**, we can define the following:

$$\mathbf{[X \leq x]=\{w \mid X(w)\leq x\}}$$

$$\mathbf{[X > x]=\{w \mid X(w)>x\}}$$

$$\mathbf{[a < X < b]=\{w \mid a < X(w)<b\}}$$

Cont…

- These events have probabilities that are denoted by;

- ✓ $P[X \leq x]$ is the probability that X takes a value less than or equal to x.
- ✓ $P[X > x]$ is the probability that X takes a value greater than x; this is equal to $1-P[X \leq x]$.
- ✓ $P[a < X < b]$ is the probability that X takes a value that strictly **lies between a** and **b**.

Example 2.2 :- Consider an experiment in which a fair coin is tossed twice.

- ➢ The sample space consists of four equally likely sample points: S={HH,HT,TH,TT}

Cont…

- Let X denote the random variable that counts the **number** of **heads** in each sample point.
- Thus X has the range$\{0,1,2\}$.
- If we consider$[X \leq 1]$, which is the event that the **number** of **heads is at most 1**, we obtain $[X \leq 1]=\{TT,TH,HT\}$
- $P[X \leq 1]=P[TT]+(P[TH]+P[HT])$

$$=P[X=0]+P[X=1]$$

$$= 1/4 + 1/2 = 3/4$$

2.4 <u>The Cumulative Distribution Function</u>

- The distribution function (or cumulative distribution function (cdf)) of a random variable ***X*** is defined as the probability of the event ***{X≤x}***. $F_X(x) = P(X \le x)$; $-\infty < x < \infty$

Properties of the cdf, $F_X(x)$:

- The cdf has the following properties.

$i.$ $F_X(x)$ is a non-negative function, i.e.,

$$0 \le F_X(x) \le 1$$

ii. $\lim_{x\to\infty} F_X(x) = 1$

$iii.$ $\lim_{x\to-\infty} F_X(x) = 0$

Cont...

iv. $F_X(x)$ is a non-decreasing function of X, i.e.,

If $x_1 < x_2$, then $F_X(x_1) \le F_X(x_2)$

v. $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

vi. $P(X > x) = 1 - P(X \le x) = 1 - F_X(x)$

Example 2.3: Find the cdf & draw its graph, of the random variable X which is defined as the number of heads in three tosses of a fair coin.

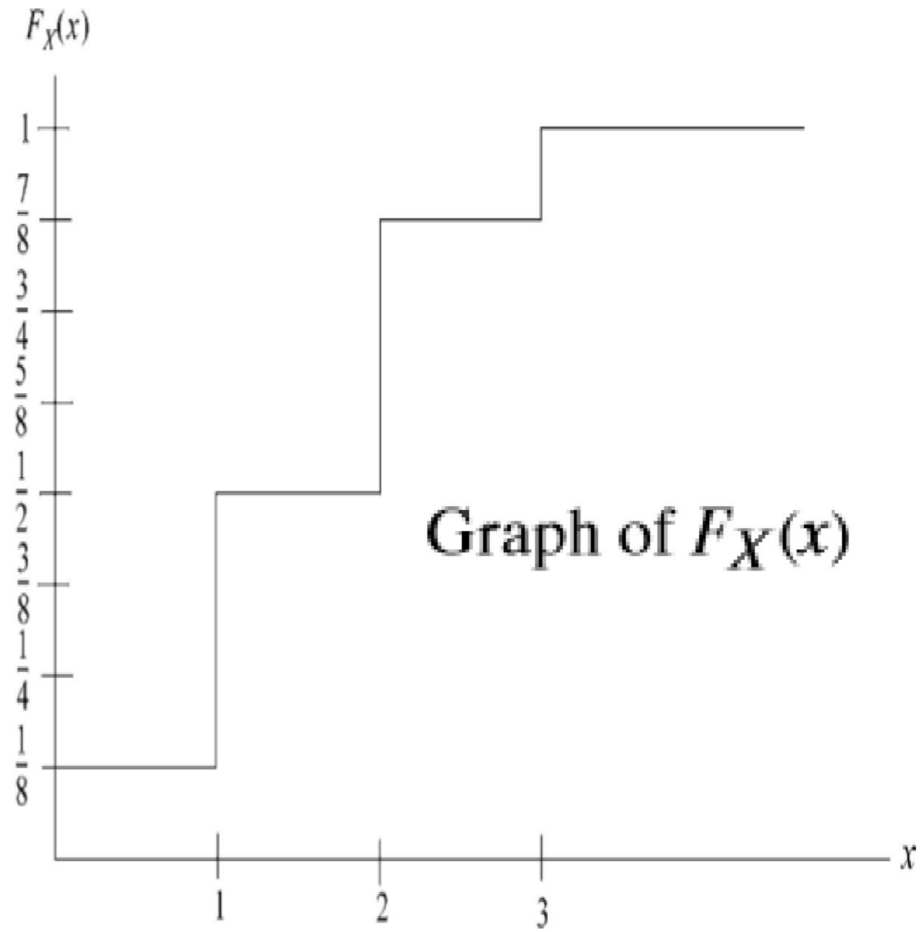
Solution:

- ✓ We know that ***X*** takes on only the values 0, 1, 2 and 3 with probabilities 1/8, 3/8, 3/8 and 1/8 respectively.

Cont…

- Thus, $F_X(x)$ is simply the sum of the probabilities of the outcomes from the set {0, 1, 2, 3} that are less than or equal to $\boldsymbol{x}$.

$$\therefore F_X(x) = \begin{cases} 0, & x < 0 \\ 1/8, & x \leq 0 \\ 1/2, & x \leq 1 \\ 7/8, & x \leq 2 \\ 1, & x \geq 3 \end{cases}$$



Fx(-∞)=0
Fx(∞)=1

Cont…

Example 2.4:-The CDF of the random variable X is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x + \frac{1}{2} & 0 \le x \le \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

(a) Draw the graph of the CDF

(b) Compute P[X > 1/ 4]

Solution

(a) The graph of the CDF is drawn next slide.

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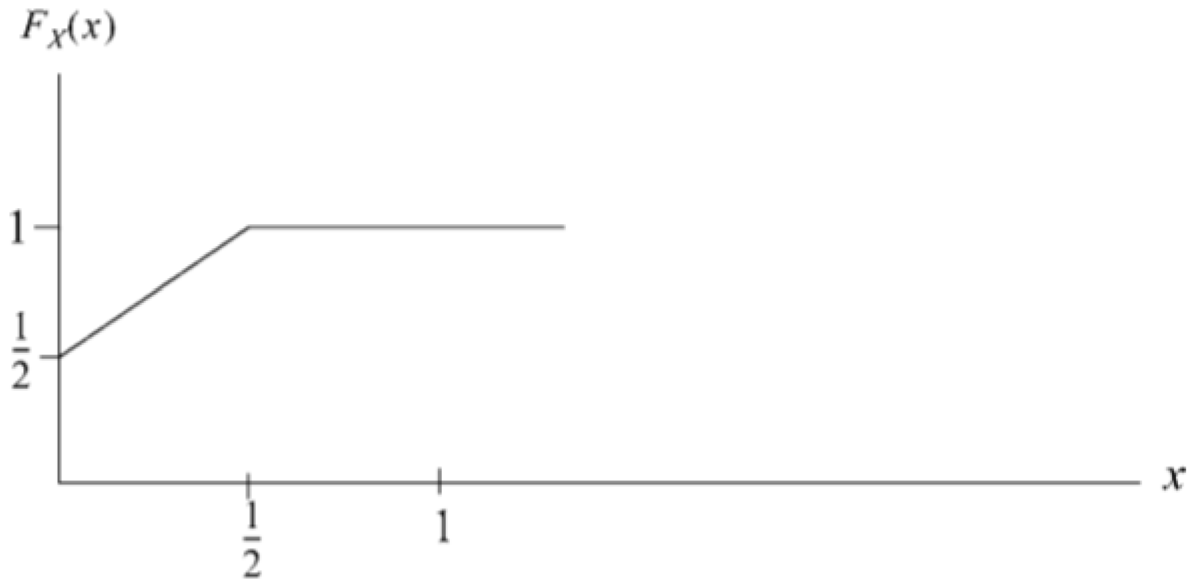


(b) The probability that X is greater than ¼ is

$$P\left[X > \frac{1}{4}\right] = 1 - P\left[X \le \frac{1}{4}\right] = 1 - F_X\left(\frac{1}{4}\right)$$

$$= 1 - \left(\frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

2.5 Types of Random Variables

- There are two basic types of random variables.

i. Discrete Random Variable

- ✓ A discrete random variable is a random variable that can take on at most a countable number of possible values, either finite or countably infinite.
- ✓ A discrete random variable is defined as a random variable whose **cdf**, ***$F_X(x)$***, is a **right continuous**, staircase function of ***X*** **with jumps at a countable set** of points ***$x_0, x_1, x_2, \ldots\ldots$***
- ✓ The **cdf** of a discrete random variable ***X*** can be obtained by using the formula:

$$F_X(x) = \sum_{x_k \leq x} P_X(x_k) U(x - x_k)$$

Cont…

ii. Continuous Random Variable

- A Continuous random variable is a random variable that can an **uncountable** set of possible values.
- It is defined as a random variable whose **cdf**, $\boldsymbol{F_X(x)}$, is **continuous every** where and can be written as an integral of some non-negative function $\boldsymbol{f(x)}$, i.e.,

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du$$

2.6 The Probability Mass Function (PMF)

- The **probability** of a random variable **equal to a number** is called the **probability mass function (pmf).**
- The probability mass function (pmf) of a discrete random variable ***X*** is defined as:

$$P_X(X = x_i) = P_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

- Since **P(X=x)=0** for any **x** for **continuous** random variables, pmf **does not exist** in the **case of the continuous** random variable.

Properties of the pmf, $P_X(x_i)$:

$$i. \quad 0 \le P_X(xk) \le 1, \; k = 1, 2, \ldots..$$

$$ii. \quad P_X(x) = 0, \text{ if } x \ne x_k, k = 1, 2, \ldots..$$

$$iii. \quad \sum_k P_X(x_k) = 1$$

Cont…

<u>Example 2.5 :</u>- If PMF of X is given as bellow; find its CDF?

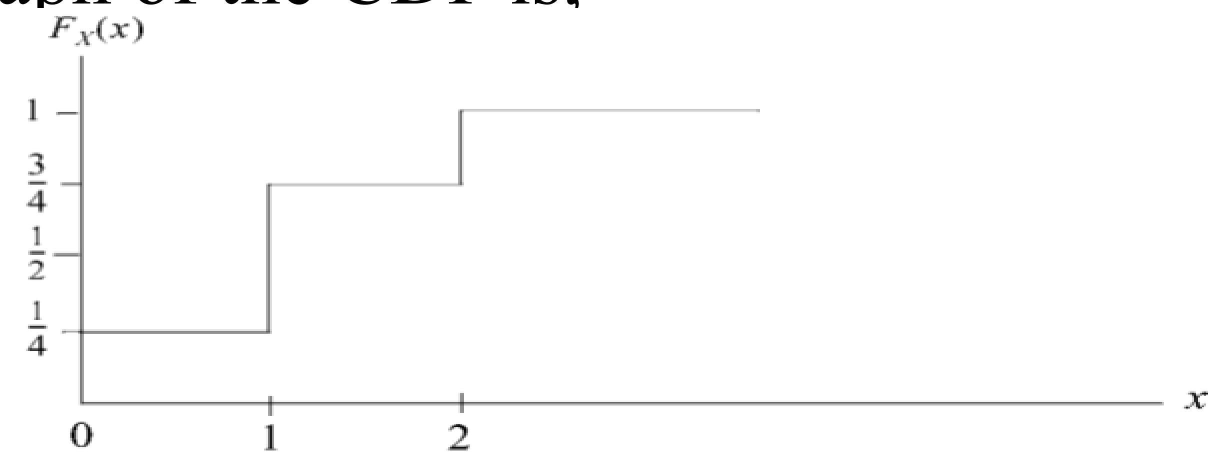
$$p_X(x)\begin{cases} 1/4 & x=0 \\ 1/2 & x=1 \\ 1/4 & x=2 \end{cases}$$

<u>Solution</u>

- Its CDF is given by;

$$F_X(x)\begin{cases} 0 & x<0 \\ 1/4 & 0\le x<1 \\ 3/4 & 1\le x<2 \\ 1 & x\ge 2 \end{cases}$$

- Thus, the graph of the CDF is;



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<u>Example 2.6</u> Let the random variable X denote the sum obtained in rolling a pair of fair dice. Determine the PMF of X.

<u>Solution</u>

- Let the pair (a, b) denote the outcomes of the roll, where a is the outcome of one die and b is the outcome of the other.
- Thus, the sum of the outcomes is X =a+b. The different events defined by the random variable X are as follows:

Cont…

$$[X=2]=\{(1,1)\}$$

$$[X=3]=\{(1,2),(2,1)\}$$

$$[X=4]=\{(1,3),(2,2),(3,1)\}$$

$$[X=5]=\{(1,4),(2,3),(3,2),(4,1)\}$$

$$[X=6]=\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

$$[X=7]=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$[X=8]=\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

$$[X=9]=\{(3,6),(4,5),(5,4),(6,3)\}$$

$$[X=10]=\{(4,6),(5,5),(6,4)\}$$

$$[X=11]=\{(5,6),(6,5)\}$$

$$[X=12]=\{(6,6)\}$$

Cont…

- Since there are 36 equally likely sample points in the sample space, the PMF of X is given by

$$p_X(x) = \begin{cases} 1/36 & x = 2 \\ 2/36 & x = 3 \\ 3/36 & x = 4 \\ 4/36 & x = 5 \\ 5/36 & x = 6 \\ 6/36 & x = 7 \\ 5/36 & x = 8 \\ 4/36 & x = 9 \\ 3/36 & x = 10 \\ 2/36 & x = 11 \\ 1/36 & x = 12 \end{cases}$$

Cont…

- **Obtaining the PMF from the CDF**
- We know for a discrete random variable X with PMF pX(x), the CDF is given by $F_X(x) = \sum_{k \le x} p_X(k)$
- Sometimes we are given the CDF of a discrete random variable and are required to obtain its PMF.
- CDF of a discrete random variable has the staircase plot with **jumps at those values of the random variable where the PMF has a nonzero value.**
- The **size of a jump at a value** of a random variable is equal to the **value of the PMF** at the value.

Cont…

<u>Example 2.7</u> Find the PMF of a discrete random variable X whose CDF is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1/6 & 0 \le x < 2 \\ 1/2 & 2 \le x < 4 \\ 5/8 & 4 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

<u>Solution</u>

- changes values at X = 0, X = 2, X = 4, and X = 6,
- px(0)=1/6. At X =2, px(2)= 1/2−1/6=1/3, px(4)= 5/8−1/2= 1/8, and px(6)= 1−5/8=3/8.
- Therefore, the PMF of X is

$$p_X(x) = \begin{cases} 1/6 & x = 0 \\ 1/3 & x = 2 \\ 1/8 & x = 4 \\ 3/8 & x = 6 \\ 0 & \text{otherwise} \end{cases}$$

2.7 <u>The Probability Density Function (PDF)</u>

- We define a random variable X to be a **continuous random variable** if there exists a **nonnegative** function fX(x), defined for all real $x \in (-\infty, \infty)$, having the property that for **any set A of real numbers**,

$$P(X \in A) = \int_A f_X(x)dx$$

- The function fX(x) is called the **probability density function (PDF)** or simply density function, of the random variable X and is defined by, $f_X(x) = \frac{dF_X(x)}{dx}$

Cont…

❖ **Properties of the pdf, $f_X(x)$:**

i. For all values of X, $f_X(x) \geq 0$

ii. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

iii. $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx$

- The **pdf does not exist for a discrete random variable since** its associated PDF **has discrete jumps** and is **not differentiable** at **these points of discontinuity.**
- Using the mass distribution analogy, the **pdf of a continuous random variable** plays exactly the **same role as the pmf of a discrete random variable.**

Cont…

- The function fx(x) can be interpreted as the **mass density (mass per unit length)**.

Example 2.8 :- Assume that X is a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} A(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of A?

(b) Find P[X > 1].

Solution

(a) Since fX(x) is a PDF, we have that

Cont…

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{0} 0dx + \int_{0}^{2} A\left(2x - x^2\right)dx + \int_{2}^{\infty} 0dx = \int_{0}^{2} A\left(2x - x^2\right)dx = 1$$

Thus, we obtain

$$A\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$A\left(4 - \frac{8}{3}\right) = \frac{4A}{3} = 1$$

$$A = \frac{3}{4}$$

Cont…

(b) Therefore,

$$P[X > 1] = \int_1^\infty f_X(x)dx$$

$$= \frac{3}{4}\int_1^2 (2x - x^2)dx = \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{3}{4}\left[\frac{4}{3} - \frac{2}{3}\right]$$

$$= \frac{1}{2}$$

Cont…

Example 2.9:- Is the following function a legitimate PDF?

$$f(x) = \begin{cases} \frac{x^2}{9} & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Solution

- For f(x) to be a legitimate PDF, we need to check to see if $\int_{-\infty}^{\infty} f(x)dx = 1.$

Thus;

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^3 \frac{x^2}{9}dx = \left[\frac{x^3}{27}\right]_0^3 = 1$$

- Therefore, f(x) is a legitimate PDF.

Cont…

Example 2.10 Consider the function

$$g(x) = \begin{cases} c & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

(a) For what value of c is g(x) a legitimate PDF?

(b) Find the CDF of the random variable X with the above PDF.

Solution

(a) For g(x) to be a legitimate PDF, we must have that

$$\int_{-\infty}^{\infty} g(x)dx = 1.$$

$$\int_a^b cdx = [cx]_a^b = c(b-a) = 1$$

This implies that $c = \frac{1}{b-a}$.

Cont…

(b) The CDF is given by

$$G_X(x) = \int_{-\infty}^{x} g(u)du = \int_{a}^{x} \frac{du}{b-a}$$

$$= \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

Cont…

Example 2.11 Consider the function

$$f(x) = \begin{cases} 2x & 0 \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

(a) For what value of b is f(x) a legitimate PDF?

(b) Find the CDF of the random variable X with the above PDF.

Solution

(a) For f(x) to be a valid PDF in the specified range,

$$\int_0^b 2x\,dx = [x^2]\Big|_0^b = b^2 = 1$$

- Thus, b=1.

Cont…

(b) The CDF of X is given by

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Example 2.12 The PDF of the time T it takes a bank teller to serve a customer is defined by

$$f_T(t) = \begin{cases} \frac{1}{6} & 2 \le t \le 8 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the CDF of T?

(b)What is the probability that a customer is served in less than 5 minutes?

Cont…

Solution

(a) The CDF of T is given by

$$F_T(t) = P[T \le t] = \int_{-\infty}^{t} f_T(u)du$$

$$= \int_2^t \frac{1}{6}du = \left[\frac{u}{6}\right]_2^t$$

$$= \begin{cases} 0 & t < 2 \\ \frac{t-2}{6} & 2 \le t < 8 \\ 1 & t \ge 8 \end{cases}$$

(b) Since T is a continuous random variable, the probability that a customer is served in less than 5 minutes is given by

$$P[T < 5] = F_T(5) = \frac{5-2}{6} = 0.5$$

Cont…

Example 2.13 The CDF of the random variable X is defined by

$$F_X(x) = \begin{cases} 0 & x < 2 \\ A(x-2) & 2 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

(a) What is the value of A?

(b) With the above value of A, what is P[X > 4]?

(c) With the above value of A, what is P[3≤X ≤5]?

Solution

(a) To find A, we know that FX(6)=1.

- Thus, from the definition of the CDF we have that

$$F_X(6) = A(6-2) = 4A = 1 \Rightarrow A = \frac{1}{4}$$

Cont…

(b) The probability that X is greater than 4 is given by

$$P[X > 4] = 1 - P[X \leq 4] = 1 - F_X(4) = 1 - \frac{1}{4}(4-2) = \frac{1}{2}$$

(c)The probability that X lies between 3 & 5 is given by

$$P[3 \leq X \leq 5] = F_X(5) - F_X(3) = \frac{1}{4}\{(5-2)-(3-2)\} = \frac{1}{2}$$

- We can also solve the problem by first finding the PDF of X as follows: $f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} A & 2 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$

- Then using the PDF; $\int_{-\infty}^{\infty} f_X(x)dx = \int_2^6 A dx = 1 \Rightarrow A = \frac{1}{4}$

$$P[X > 4] = \int_4^6 A dx = \frac{1}{2}$$

$$P[3 \leq X \leq 5] = \int_3^5 A dx = \frac{1}{2}$$

Cont…

<u>Example 2.14</u> The CDF of the random variable Y is defined by $F_Y(y) = \begin{cases} 0 & y < 0 \\ K\{1 - e^{-2y}\} & y \geq 0 \end{cases}$

(a) For what value of K is the function a valid CDF?

(b) With the above value of K, what is FY(3)?

(c) With the above value of K, what is P[2 < Y <∞]?

<u>Solution</u>

(a) To find K, we know that FY(∞)=1.

$$F_Y(\infty) = K\{1 - e^{-\infty}\} = K(1 - 0) = 1 \Rightarrow K = 1$$

(b) $F_Y(3) = K\{1 - e^{-6}\} = 1 - e^{-6} = 0.9975.$

(c) $P[2 < Y < \infty] = P[Y > 2] = 1 - P[Y \leq 2] = 1 - F_Y(2).$

Cont…

- Thus; $P[2 < Y < \infty] = 1 - \{1 - e^{-4}\} = e^{-4} = 0.0183$

▪ This problem can also be solved by first obtaining the PDF of Y and then integrating over the appropriate intervals as follows:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = 2Ke^{-2y}, \, y \geq 0$$

$$\int_0^\infty f_Y(y)dy = 1 = K[-e^{-2y}]_0^\infty \Rightarrow K = 1$$

$$F_Y(3) = \int_0^3 f_Y(y)dy = [-e^{-2y}]_0^3 = 1 - e^{-6}$$

$$P[2 < Y < \infty] = \int^2 f_Y(y)dy = [-e^{-2y}]_2^\infty = e^{-4}$$

2.8 Expected Value, Variance and Moments

- While a probability distribution [FX (x), pX (x), or f X(x)] contains a complete description of a random variable X, it is often of interest to seek a **set of simple numbers** that gives the random variable some of its **dominant** features.

- Given the set of data X1,X2,...,XN, we know that the arithmetic average (or arithmetic mean) is given

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_N}{N}$$

- When the above numbers occur with different frequencies, we usually assign weights w1,w2,...,wN to them and the so-called weighted arithmetic mean becomes

$$\overline{X} = \frac{w_1 X_1 + w_2 X_2 + \cdots + w_N X_N}{w_1 + w_2 + \cdots + w_N}$$

- The average is a value that is representative or typical of a set of data and tends to lie centrally within a set of data that are arranged according to their magnitudes.

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- Thus, it is usually called a measure of **central tendency**.
- The term expectation is used for the process of averaging when a random variable is involved.
- It is a number used to locate the "**center**" of the distribution of a random variable.
- In many situations we are primarily interested in the central tendency of a random variable, and as will be seen later, the expectation (or mean or average) of a random variable can be likened to the weighted arithmetic average defined above.
- Another measure of central tendency of a random variable is its **variance**, which measures the **degree to which a random variableis spread out**.

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i. Expected Value (Mean)

- Mean represents the average value of the random variable in a very large number of trials.
- The expectation (or expected value or mean) of a continuous random variable ***X***, denoted by $\boldsymbol{\mu_X}$ or ***E(X)***, is defined as:

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Similarly, the expected value of a discrete random variable ***X*** is :

$$\mu_X = E(X) = \sum_k x_k P_X(x_k)$$

- Thus, the expected value of X is a weighted average of the possible values that X can take, where each value is weighted by the probability that X takes that value.

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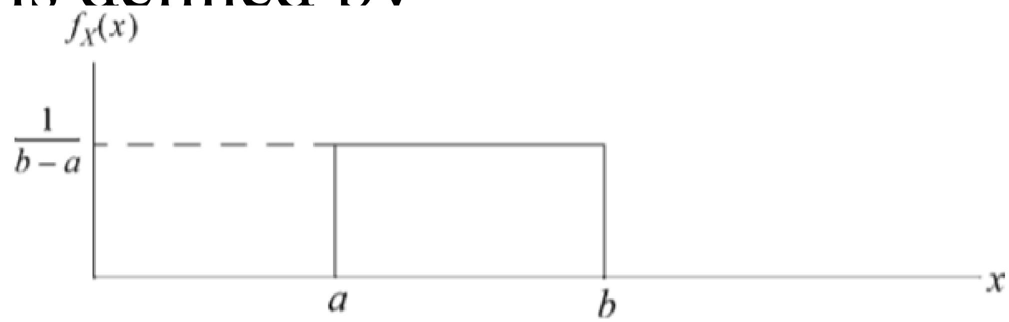
Example 2.15 Find the expected value of the random variable X whose PDF is defined by

$$f_X(x) = \begin{cases} 0 & x < a \\ \dfrac{1}{b-a} & a \le x \le b \\ 0 & x > b \end{cases}$$



Solution

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_a^b \frac{x}{b-a} dx = \left[\frac{x^2}{2(b-a)} \right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{b+a}{2}$$

Cont…

<u>Example 2.16</u> Find the expected value of the discrete random variable X with the following PMF:

$$p_X(x) = \begin{cases} \frac{1}{3} & x = 0 \\ \frac{2}{3} & x = 2 \end{cases}$$

<u>Solution</u>

$$E[X] = 0\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right) = \frac{4}{3}$$

<u>Example 2.17</u> Find the expected value of the random variable K with the following PMF:

$$p_K(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, \ldots$$

<u>Solution</u>

E[K]is given by

Cont…

$$E[K] = \sum_{k=0}^{\infty} k p_K(k) = \sum_{k=0}^{\infty} k\left(\frac{\lambda^k}{k!} e^{-\lambda}\right)$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

Since

$$\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

we obtain

$$E[K] = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Cont…

Example 2.18 Find the expected value of the random variable X whose PDF is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Solution

- The expected value of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^{\infty} x\lambda e^{-\lambda x} dx$$

Let $dv = \lambda e^{-\lambda x} dx$ and $u = x$. This means that $v = -e^{-\lambda x}$ and $du = dx$. Thus, integrating by parts, we obtain

$$\begin{aligned} E[X] &= [uv]_0^{\infty} - \int_0^{\infty} v du \\ &= [-xe^{-\lambda x}]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 - \left[\frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty} \\ &= \frac{1}{\lambda} \end{aligned}$$

Cont…

ii. Moments of Random Variables and the Variance

- The **nth moment** of the random variable X, denoted by E[Xn], is defined by

$$E[X^n] = \overline{X^n} = \begin{cases} \sum_i x_i^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^n f_X(x)dx & X \text{ continuous} \end{cases}$$

for n=1,2,3,...

- The **first moment, E[X],** is the expected value of X.
- We can also define the central moments (or moments about the mean) of a random variable.
- These are the moments of the difference between a random variable and its expected value.

Cont…

- The **nth central moment** is defined by

$$E\left[(X-\overline{X})^n\right] = \overline{(X-\overline{X})^n} = \begin{cases} \sum_i (x_i-\overline{X})^n p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\overline{X})^n f_X(x)dx & X \text{ continuous} \end{cases}$$

❖ The **central moment** for the case of **n=2** is very important and carries a special name, the **variance**, which is usually denoted by σ2 X. Thus,

$$\sigma_X^2 = E\left[(X-\overline{X})^2\right] = \overline{(X-\overline{X})^2} = \begin{cases} \sum_i (x_i-\overline{X})^2 p_X(x_i) & X \text{ discrete} \\ \int_{-\infty}^{\infty} (x-\overline{X})^2 f_X(x)dx & X \text{ continuous} \end{cases}$$

Cont…

- Let X be a random variable with the PDF fX(x) and mean E[X], and a and b be constants.
- Then, if Y is the random variable defined by Y=aX+b, the expected value of Y is given by E[Y]=aE[X]+b. Why?

Proof.

- Since Y is a function of X, its expected value is given by

$$E[Y] = E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)dx = \int_{-\infty}^{\infty} ax f_X(x)dx + \int_{-\infty}^{\infty} b f_X(x)dx$$

$$= a\int_{-\infty}^{\infty} x f_X(x)dx + b\int_{-\infty}^{\infty} f_X(x)dx$$

$$= aE[X] + b$$

Cont…

- Let X be a random variable with the PDF fX(x) and mean E[X], g1(X) and g2(X) be two functions of the random variable X, and let g3(X) be defined by g3(X)=g1(X)+g2(X).

- The expected value of g3(X) is E[g1(X)]+ E[g2(X)].

Proof

- Since g3(X) is a function of X, its expected value is given by

$$E[g_3(X)] = \int_{-\infty}^{\infty} g_3(X) f_X(x)dx = \int_{-\infty}^{\infty} \{g_1(X) + g_2(X)\} f_X(x)dx$$

$$= \int_{-\infty}^{\infty} g_1(X) f_X(x)dx + \int_{-\infty}^{\infty} g_2(X) f_X(x)dx$$

$$= E[g_1(X)] + E[g_2(X)]$$

Cont…

❖ Using above properties, and noting that E(X) is a constant, we obtain the variance of X as follows:

$$\sigma_X^2 = E\left[(X-\overline{X})^2\right] = E\left[X^2 - 2X\overline{X} + (\overline{X})^2\right] = E\left[X^2\right] - 2E[X]\overline{X} + (\overline{X})^2$$

$$= E\left[X^2\right] - 2\overline{X}\,\overline{X} + (\overline{X})^2 = E\left[X^2\right] - 2(\overline{X})^2 + (\overline{X})^2$$

$$= E\left[X^2\right] - (\overline{X})^2 = E\left[X^2\right] - (E[X])^2$$

❑ The **square root** of the **variance, σX**, is called the **standard deviation**.

- **Variance** is a measure of the "**spread"** of a PDF or PMF.
- If a random variable has a **concentrated** PDF or PMF, it will have a small variance.
- Similarly, if it has a **widely spread** PDF or PMF, it will have a **large** variance.

Cont..

Example-2.19:

The pdf of a continuous random variable is given by:

$$f_X(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

a. Determine the value of k.

b. Find the corresponding cdf of X.

c. Find $P(1/4 \le X \le 1)$

d. Evaluate the mean and variance of X.

Cont…

Solution:

$$a. \quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^1 kx\,dx = 1$$

$$\Rightarrow k\left(\frac{x^2}{2}\right)\Bigg|_0^1 = 1$$

$$\Rightarrow \frac{k}{2} = 1$$

$$\therefore k = 2$$

$$\Rightarrow f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

Solution:

$b.\quad \text{The cdf of } X \text{ is given by:}$

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

$\text{Case1:}\quad \text{for } x < 0$

$$F_X(x) = 0, \quad \text{since } f_X(x) = 0, \text{ for } x < 0$$

$\text{Case 2:}\quad \text{for } 0 \le x < 1$

$$F_X(x) = \int_0^x f_X(u)\,du = \int_0^x 2u\,du = u^2 \Big|_0^x = x^2$$

Cont…

Solution:

$$\text{Case 3:} \quad \text{for} \quad x \geq 1$$

$$F_X(x) = \int_0^1 f_X(u)\,du = \int_0^1 2u\,du = u^2 \Big|_0^1 = 1$$

$$\therefore \text{The cdf is given by}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Cont…

Solution:

$c.\quad P(1/4 \le X \le 1)$

$i.$ Using the pdf

$$P(1/4 \le X \le 1) = \int_{1/4}^{1} f_X(x)\,dx = \int_{1/4}^{1} 2x\,dx$$

$$\Rightarrow P(1/4 \le X \le 1) = x^2 \Big|_{1/4}^{1} = 15/16$$

$$\therefore P(1/4 \le X \le 1) = 15/16$$

$ii.$ Using the cdf

$$P(1/4 \le X \le 1) = F_X(1) - F_X(1/4)$$

$$\Rightarrow P(1/4 \le X \le 1) = 1 - (1/4)^2 = 15/16$$

$$\therefore P(1/4 \le X \le 1) = 15/16$$

Cont…

Solution:

d. Mean and Variance

i. Mean

$$\mu_X = E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 2x^2\,dx$$

$$\Rightarrow \mu_X = \frac{2x^3}{3}\Bigg|_0^1 = 2/3$$

ii. Variance

$$o_{X\,2} = Var(X) = E(X_{\,2}) - [E(X)]_2$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 2x^3\,dx = 1/2$$

$$\Rightarrow \sigma_{X\,2} = Var(x) = 1/2 - (2/3)_{\,2} = 1/18$$

Cont…

Example-2.20:

Consider a discrete random variable ***X*** whose pmf is given by:

$$P_X(x_k) = \begin{cases} 1/3, & x_k = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Cont…

Solution:

i. Mean

$$\mu_X = E(X) = \sum_{k=-1}^{1} x_k P_X(x_k) = 1/3(-1+0+1) = 0$$

ii. Variance

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{k=-1}^{1} x_k^2 P_X(x_k) = 1/3[(-1)^2 + (0)^2 + (1)^2] = 2/3$$

$$\Rightarrow \sigma_X^2 = Var(x) = 2/3 - (0)^2 = 2/3$$

Cont…

Example 2.21 Let X be a continuous random variable with the PDF

$$f_X(x) = \begin{cases} \frac{1}{4} & 2 \le x \le 6 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value and variance of X.

Solution

- The expected value of X is given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_2^6 \frac{x}{4} dx$$

$$= \left[\frac{x^2}{8}\right]_2^6 = \frac{36}{8} - \frac{4}{8} = 4$$

Cont…

$$\sigma_X^2 = E[(X-\overline{X})^2] = \int_{-\infty}^{\infty}(x-\overline{X})^2 f_X(x)dx = \int_2^6 \frac{(x-4)^2}{4}dx$$

$$= \frac{1}{4}\int_2^6 (x^2-8x+16)dx = \frac{1}{4}\left[\frac{x^3}{3}-4x^2+16x\right]_2^6$$

$$= \frac{1}{4}\left\{(96-72)-\left(32-16+\frac{8}{3}\right)\right\}$$

$$= \frac{4}{3}$$

- an alternative method

$$\sigma_X^2 = E[X^2]-(\overline{X})^2 = E[X^2]-4^2 = E[X^2]-16$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_2^6 \frac{x^2}{4}dx = \left[\frac{x^3}{12}\right]_2^6 = \frac{216-8}{12}$$

$$= \frac{52}{3}$$

Thus, $\sigma_X^2 = \frac{52}{3} - 16 = \frac{4}{3}$, which is the same result obtained earlier.

Cont…

Example 2.22 A test engineer discovered that the CDF of the lifetime of an equipment in years is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/5} & 0 \le x < \infty \end{cases}$$

a. What is the expected lifetime of the equipment?

b. What is the variance of the lifetime of the equipment?

Solution

- From the definition of its CDF, we can see that X is a random variable that takes only nonnegative values. Thus,

Cont…

(a) The expected lifetime of the equipment is given by

$$E[X] = \int_0^\infty P[X > x]dx = \int_0^\infty [1 - F_X(x)]dx$$

$$= \int_0^\infty e^{-x/5} dx = 5$$

(b) To find the variance, we first evaluate the PDF:

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} \frac{1}{5}e^{-x/5} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the second moment of X is given by

$$E[X^2] = \int_{-\infty}^\infty x^2 f_X(x)dx = \frac{1}{5}\int_0^\infty x^2 e^{-x/5}dx$$

Let $u = x^2 \Rightarrow du = 2xdx$, and let $dv = e^{-x/5}dx \Rightarrow v = -5e^{-x/5}$.

Cont…

Thus, $$E[X^2] = \left\{-\frac{5x^2 e^{-x/5}}{5}\right\}_0^{\infty} + 10\int_0^{\infty} \frac{xe^{-x/5}}{5}dx$$

$$= 0 + 2\int_0^{\infty} xe^{-x/5}dx = 2\int_0^{\infty} xe^{-x/5}dx$$

Let $u = x \Rightarrow du = dx$, and let $dv = e^{-x/5}dx \Rightarrow v = -5e^{-x/5}$. Then we have that

$$E[X^2] = 2\{-5xe^{-x/5}\}_0^{\infty} + 10\int_0^{\infty} e^{-x/5}dx = 0 + 10[-5e^{-x/5}]_0^{\infty} = 50$$

Finally, the variance of X is given by

$$\sigma_X^2 = E[X^2] - \{E[X]\}^2 = 50 - 25 = 25$$

Cont…

<u>Example 2.23</u> A shopping cart contains ten books whose weights are as follows: There are four books with a weight of 1.8 lbs each, one book with a weight of 2 lbs, two books with a weight of 2.5 lbs each, and three books with a weight of 3.2 lbs each.

a. What is the mean weight of the books?

b. What is the variance of the weights of the books?

<u>Solution</u>

The total number of books is 10. The fractions of books in each weight category are as follows:

- ✓ Fraction of books with weight 1.8 lbs is 4/10=0.4
- ✓ Fraction of books with weight 2.0 lbs is 1/10=0.1
- ✓ Fraction of books with weight 2.5 lbs is 2/10=0.2
- ✓ Fraction of books with weight 3.2 lbs is 3/10=0.3

Cont…

- Let Y be a random variable that denotes the weights of the books.
- Since these fractions are essentially the probabilities of occurrence of these weights, we have that

$E[Y]=(0.4\times1.8)+(0.1\times2.0)+(0.2\times2.5)+(0.3\times3.2)=2.38$

$$\sigma_Y^2 = \sum_{k=1}^{4}\left(y_k - E[Y]\right)^2 p_Y(y_k)$$

$$= \left\{(1.8-2.38)^2 \times 0.4\right\} + \left\{(2.0-2.38)^2 \times 0.1\right\} + \left\{(2.5-2.38)^2 \times 0.2\right\}$$

$$+ \left\{(3.2-2.38)^2 \times 0.3\right\}$$

$$= 0.3536$$

Cont…

Example 2.24 the lifetime of the equipment can be modeled by a random variable X that has the PDF

$$f(x) = \begin{cases} \dfrac{xe^{-x/10}}{100} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a. Show that f(x) is a valid PDF.

b. What is the probability that the lifetime of the equipment exceeds 20?

c. What is the expected value of X?

Cont…

Solution

(a) For f(x) to be a valid PDF,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} \frac{xe^{-x/10}}{100}dx$$

Let $u = x \Rightarrow du = dx$, and let $dv = e^{-x/10}dx \Rightarrow v = -10e^{-x/10}$. Thus,

$$\int_{0}^{\infty} \frac{xe^{-x/10}}{100}dx = \frac{1}{100}\left\{\left[-10xe^{-x/10}\right]_0^{\infty} + 10\int_{0}^{\infty} e^{-x/10}dx\right\}$$

$$= \frac{1}{100}\left\{0 - \left[100e^{-x/10}\right]_0^{\infty}\right\} = 1$$

This proves that $f(x)$ is a valid PDF.

Cont…

(b)The probability that the lifetime of the equipment exceeds 20 is

$$P[X > 20] = \int_{20}^{\infty} f(x)dx = \frac{1}{100}\left\{[-10xe^{-x/10}]_{20}^{\infty} + 10\int_{20}^{\infty} e^{-x/10}dx\right\}$$

$$= \frac{1}{100}\left\{200e^{-2} - [100e^{-x/10}]_{20}^{\infty}\right\} = 2e^{-2} + e^{-2} = 3e^{-2}$$

$$= 0.4060$$

(c) The expected value of X is given by

$$E[X] = \int_{-\infty}^{\infty} xf(x)ds = \int_{0}^{\infty} \frac{x^2e^{-x/10}}{100}dx$$

Let $u = x^2 \Rightarrow du = 2xdx$, and let $dv = e^{-x/10}dx \Rightarrow v = -10e^{-x/10}$. Thus,

$$E[X] = \left\{-\frac{10x^2e^{-x/10}}{100}\right\}_{0}^{\infty} + 20\int_{0}^{\infty} \frac{xe^{-x/10}}{100}dx = 0 + 20 = 20$$

2.9. Some Special Probability Distributions

- Random variables with **special probability distributions** are **encountered in different fields of science and engineering**.
- These include the Bernoulli distribution, binomial distribution, geometric distribution, Pascal distribution, hypergeometric distribution, Poisson distribution, exponential distribution, Erlang distribution, uniform distribution, and normal distribution.
- These can also divided as continuous and discrete.
- To see some of these;

Cont…

i. Continuous Probability Distributions

1. Normal (Gaussian) Distribution

- The random variable ***X*** is said to be normal or Gaussian random variable if its pdf is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

- The corresponding distribution function is given by:

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y-\mu)^2/2\sigma^2} dy = G\left(\frac{x-\mu}{\sigma}\right)$$

where

$$G(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

Cont…

- The normal or Gaussian distribution is the most common continuous probability distribution.

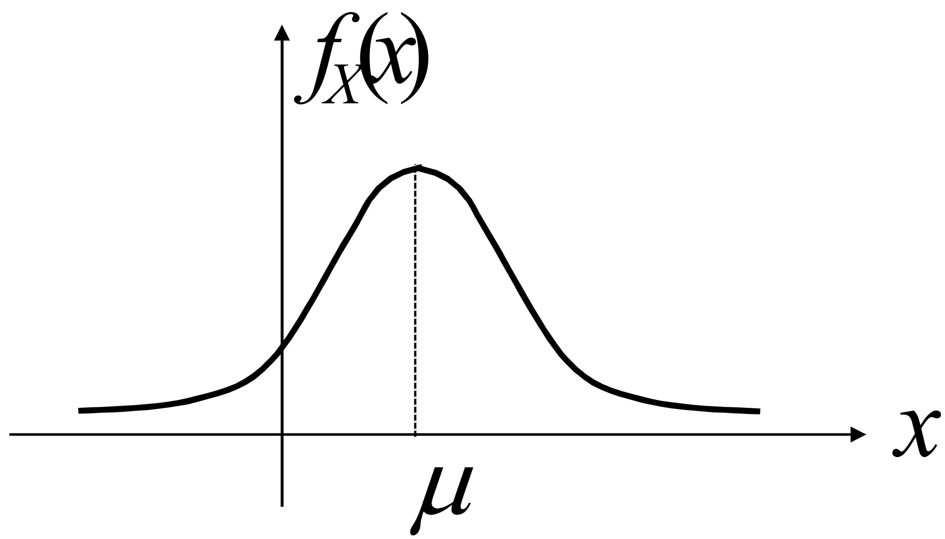


Fig. *Normal or Gaussian Distribution*

Cont…

2. Uniform Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

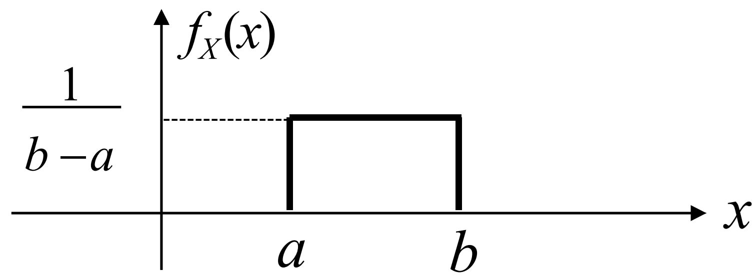


Fig. *Uniform Distribution*

3. Exponential Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{\lambda} e^{-x/\lambda}, & x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

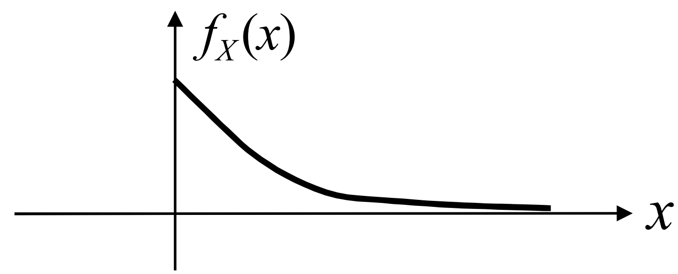


Fig. *Exponential Distribution*

Cont…

4. Gamma Distribution

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-x/\beta}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

5. Beta Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

where

$$\beta(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,du.$$

Cont…

6. Rayleigh Distribution

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

7. Cauchy Distribution

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + (x-\mu)^2}, \quad -\infty < x < +\infty.$$

8. Laplace Distribution

$$f_X(x) = \frac{1}{2\lambda} e^{-|x|/\lambda}, \quad -\infty < x < +\infty.$$

Cont...

i. Discrete Probability Distributions

1. Bernoulli Distribution ;a r.v is called a Bernoulli r.v with parameters p if its pmf is given by

Px(K)=P(x=K)=$p^k(1-p)^{1-k}$ k=0,1

2. Binomial Distribution

$$P(X=k)=\binom{n}{k}p^k q^{n-k}, \quad k=0,1,2,\ldots,n.$$

3. Poisson Distribution

$$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\ldots,\infty.$$

Cont...

4. Hypergeometric Distribution

$$P(X=k)=\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \qquad \max(0, m+n-N) \le k \le \min(m,n)$$

5. Geometric Distribution

$$P(X=k)=pq^k, \quad k=0,1,2,\square,\infty, \qquad q=1-p.$$

6. Negative Binomial Distribution

$$P(X=k)=\binom{k-1}{r-1}p^r q^{k-r}, \qquad k=r,\ r+1,\cdots.$$

Cont…

- The **Poisson distribution** has many applications in science and engineering.
- For example, the number of telephone calls arriving at a switchboard during various intervals of time and the number of customers arriving at a bank during various intervals of time are usually modeled by Poisson random variables.
- As already seen; a discrete random variable K is called a Poisson random variable with parameter λ, where λ>0, if its PMF is given by

$$p_K(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, \ldots$$

Cont…

- The CDF of K is given by

$$F_K(k) = P[K \leq k] = \sum_{r=0}^{k} \frac{\lambda^r}{r!} e^{-\lambda}$$

- The expected value of K is given by

$$E[K] = \sum_{k=0}^{\infty} k p_K(k) = \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

Cont…

- The second moment of K is given by

$$E[K^2] = \sum_{k=0}^{\infty} k^2 p_K(k) = \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} e^{-\lambda}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^{k-1}}{(k-1)!}$$

- But

$$\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{\lambda}$$

$$\frac{d}{d\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \sum_{k=1}^{\infty} \frac{d}{d\lambda}\left[\frac{\lambda^k}{(k-1)!}\right] = \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} = \frac{d}{d\lambda}\left(\lambda e^{\lambda}\right) = e^{\lambda}(1+\lambda)$$

Cont…

- Thus, the second moment is given by

$$E[K^2] = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} e^{\lambda}(1+\lambda)$$

$$= \lambda^2 + \lambda$$

- The variance of K is given by

$$\sigma_K^2 = E[K^2] - (E[K])^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

Cont…

<u>Example 2.25</u> Messages arrive at a switchboard in a Poisson manner at an average rate of six per hour. Find the probability for each of the following events:

(a) Exactly two messages arrive within one hour.

(b) No message arrives within one hour.

(c) At least three messages arrive within one hour.

<u>Solution</u>

- ✓ Let K be the random variable that denotes the number of messages arriving at the switchboard within a one-hour interval. The PMF of K is given by

$$p_K(k) = \left(\frac{6^k}{k!}\right)e^{-6} \quad k = 0, 1, 2, \ldots$$

Cont…

a. The probability that exactly two messages arrive within one hour is

$$p_K(2) = \left(\frac{6^2}{2!}\right)e^{-6} = 18e^{-6} = 0.0446$$

b. The probability that no message arrives within one hour is $p_K(0) = \left(\frac{6^0}{0!}\right)e^{-6} = e^{-6} = 0.0024$

c. The probability that at least three messages arrive within one hour is

$$\begin{aligned}
P[K \geq 3] &= 1 - P[K < 3] \\
&= 1 - \{p_K(0) + p_K(1) + p_K(2)\} \\
&= 1 - e^{-6}\left\{\frac{6^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!}\right\} = 1 - e^{-6}\{1 + 6 + 18\} = 1 - 25e^{-6} \\
&= 0.9380
\end{aligned}$$

2.10 Multiple Random Variables

- In many applications, it is very important to study **two or more random variable** defined on the **same sample space**.
- In this lecture, we will consider only two random variables and this concept can be extended to three or more random variables.
- Let $\boldsymbol{\Omega}$ be the sample space of a random experiment and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two random variables.
- Then, the pair $\boldsymbol{(X, Y)}$ is called a two dimensional random variable if each of $\boldsymbol{X}$ and $\boldsymbol{Y}$ associates a real number with every element of $\boldsymbol{\Omega}$.
- Thus, a two dimensional random variable $\boldsymbol{(X, Y)}$ is a function that assigns a point $\boldsymbol{(x, y)}$ in the xy-plane to each possible outcome ω in the sample space.

Cont…

2.10.1 <u>The Joint Cumulative Distribution Function</u>

- The joint **cdf** of two random variables ***X*** and ***Y*** denoted by ***F_{XY}(x, y)*** is a function defined by:

$$F_{XY}(x, y) = P[X(\omega) \le x \text{ and } Y(\omega) \le y]$$

$$\Leftrightarrow F_{XY}(x, y) = P(\, X \le x,\, Y \le y)$$

where x and y are arbitrary real numbers.

Properties of the Joint cdf, *F_{XY}(x, y):*

i. $0 \le F_{XY}(x, y) \le 1$

ii. $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F_{XY}(x, y) = F_{XY}(\infty, \infty) = 1$

iii. $\lim\limits_{\substack{x \to -\infty \\ y \to -\infty}} F_{XY}(x, y) = F_{XY}(-\infty, -\infty) = 0$

Cont…

$$iv.\quad \lim_{x\to-\infty} F_{XY}(x, y) = F_{XY}(-\infty, y) = 0$$

$$v.\quad \lim_{y\to-\infty} F_{XY}(x, y) = F_{XY}(x,-\infty) = 0$$

$$vi.\quad P(x_1 \le X \le x_2, Y \le y) = F_{XY}(x_2, y) - F_{XY}(x_1, y)$$

$$vii.\quad P(X \le x, y_1 \le Y \le y_2) = F_{XY}(x, y_2) - F_{XY}(x, y_1)$$

$$vii.\quad P(x_1 \le X \le x_2, y_1 \le Y \le y_2) = F_{XY}(x_2, y_2)$$

$$- F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)$$

Cont…

2.10.2 The Joint Probability Density Function

- The joint probability function (pdf) of two continuous random variables ***X*** and ***Y*** is defined as:

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Thus, the joint cumulative distribution function (cdf) is given by:

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(x, y) dx dy$$

Cont…

Properties of the Joint pdf, $f_{XY}(x, y)$:

1. $f_{XY}(x, y) \geq 0$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy = 1$

3. $P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{XY}(x, y)dxdy$

4. fXY(x,y) is continuous for all except possibly finitely values of x or of y.

Cont…

- When both X and Y are **discrete random** variables, we define their **joint PMF** as follows:

$$p_{XY}(x,y)=P[X=x, Y=y]$$

The properties of the joint PMF include the following:

1. As a probability, the PMF can neither be negative nor exceed unity, which means that $0 \leq p_{XY}(x, y) \leq 1$.
2. $\sum_{x}\sum_{y} p_{XY}(x, y) = 1$
3. $\sum_{x \leq a}\sum_{y \leq b} p_{XY}(x, y) = F_{XY}(a, b)$

Cont…

2.10.3 Marginal Statistics of Two Random Variables

- In the case of two or more random variables, the statistics of each individual variable are called marginal statistics.

i. Marginal cdf of ***X*** and ***Y***

$$F_X(x) = \lim_{y\to\infty} F_{XY}(x, y) = F_{XY}(x,\infty)$$

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x, y) = F_{XY}(\infty, y)$$

ii. Marginal pdf of ***X*** and ***Y***

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

Cont...

iii. Marginal pmf of ***X*** and ***Y***

$$P(X = x_i) = P_X(x_i) = \sum_{y_j} P_{XY}(x_i, y_i)$$

$$P(Y = y_j) = P_Y(y_j) = \sum_{x_i} P_{XY}(x_i, y_i)$$

Cont…

2.10.4 Independence of Two Random Variables

- If two random variables ***X*** and ***Y*** are independent, then

i. from the joint cdf

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

ii. from the joint pdf

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

iii. from the joint pmf

$$P_{XY}(x_i, y_j) = P_X(x_i)P_Y(y_j)$$

Cont…
2.10.5 Condition Distributions

i. Conditional Probability Density Functions

- If ***X*** and ***Y*** are two continuous random variables with joint pdf $f_{XY}(x, y)$, then the conditional pdf of ***Y*** given that ***X=x*** is defined by:

$$f_{Y/X}(y/x) = \frac{f_{XY}(x, y)}{f_X(x)}, \quad f_X(x) > 0$$

- Similarly, the conditional pdf of ***X*** given that ***Y=y*** is defined by:

$$f_{X/Y}(x/y) = \frac{f_{XY}(x, y)}{f_Y(y)}, \quad f_Y(y) > 0$$

Cont…

ii. Conditional Probability Mass Functions

- If X and Y are two discrete random variables with joint pmf $P_{XY}(x_i, y_j)$, then the conditional pmf of X given that $Y=y_j$ is defined by:

$$P_{X/Y}(x_i / y_j) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_j)}, \quad P_Y(y_j) > 0$$

- Similarly, the conditional pmf of Y given that $X=x_i$ is defined by:

$$P_{Y/X}(y_j / x_i) = \frac{P_{XY}(x_i, y_j)}{P_X(x_i)}, \quad P_X(x_i) > 0$$

Cont…
2.10.6 Correlation and Covariance

i. Correlation

$$R_{XY} = Cor(X,Y) = E(XY)$$

ii. Covariance

$$o_{XY} = Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$\Rightarrow \sigma_{XY} = Cov(X,Y) = E(XY) - E(X)E(Y)$$

iii. Correlation Coefficient

$$\rho_{XY} = \frac{Cov(X,Y)}{o_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Cont…

Example 2.26 The joint PMF of two random variables X and Y is given by $p_{XY}(x,y) = \begin{cases} k(2x+y) & x=1,2;\ y=1,2 \\ 0 & \text{otherwise} \end{cases}$

where k is a constant.

a. What is the value of k?

b. Find the marginal PMFs of X and Y.

c. Are X and Y independent?

d. What is the conditional PMF of Y given X?

e. What is the conditional PMF of X given Y?

Solution

(a) To evaluate k, we remember that

Cont…

$$\sum_x \sum_y p(x,y) = \sum_{x=1}^{2} \sum_{y=1}^{2} k(2x+y) = 1.$$

$$\sum_{x=1}^{2} \sum_{y=1}^{2} k(2x+y) = k\sum_{x=1}^{2}\{(2x+1)+(2x+2)\}$$

Thus,

$$= k\{(2+1)+(2+2)+(4+1)+(4+2)\}$$
$$= 18k = 1$$

This gives k=1/18.

(b) The marginal PMFs are

$$p_X(x) = \sum_y p_{XY}(x,y) = \frac{1}{18}\sum_{y=1}^{2}(2x+y) = \frac{1}{18}\{(2x+1)+(2x+2)\}$$

$$= \frac{1}{18}(4x+3) \quad x = 1, 2$$

Cont…

$$p_Y(y) = \sum_x p_{XY}(x,y) = \frac{1}{18}\sum_{x=1}^{2}(2x+y) = \frac{1}{18}\{(2+y)+(4+y)\}$$

$$= \frac{1}{18}(2y+6) \quad y = 1,2$$

(c) Since $p_X(x)p_Y(y) \neq p_{XY}(x,y)$, we conclude that X and Y are not independent.

d , e ; the conditional PMFs are given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{2x+y}{2y+6}$$

$$p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)} = \frac{2x+y}{4x+3}$$

Cont…

Example 2.27 X and Y are two continuous random variables whose joint PDF is given by

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & 0 \leq x < \infty, 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

Solution

To answer the question, we first evaluate the marginal PDFs of X and Y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^{\infty} e^{-(x+y)}dy = e^{-x}\int_0^{\infty} e^{-y}dy$$

$$= \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Cont…

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{\infty} e^{-(x+y)}dx = e^{-y}\int_0^{\infty} e^{-x}dx$$

$$= \begin{cases} e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x)f_Y(y) = e^{-x}e^{-y} = e^{-(x+y)} = f_{XY}(x,y),$$

which means that X and Y are independent.

Cont…

Example-2.28:

The joint pdf of two continuous random variables ***X*** and ***Y*** is given by:

$$f_{XY}(x, y) = \begin{cases} kxy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

a. Find the value of k.

b. Find the marginal pdf of X and Y.

c. Are X and Y independent?

d. Find $P(X + Y < 1)$

e. Find the conditional pdf of X and Y.

Cont…

Solution:

$$a.\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1\int_0^1 kxy\,dx\,dy = 1$$

$$\Rightarrow k\int_0^1 y\left(\frac{x^2}{2}\right)\Big|_0^1 = 1$$

$$\Rightarrow \frac{k}{2}\int_0^1 y\,dy = k\left(\frac{y^2}{4}\right)\Big|_0^1 = \frac{k}{4} = 1$$

$$\therefore k = 4$$

Cont…

$b.$ Marginal pdf of X and Y

$i.$ Marginal pdf of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^1 4xydy$$

$$\Rightarrow f_X(x) = 4x\left(\frac{y^2}{2}\right)\Bigg|_0^1 = 2x$$

$$\therefore f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

$b.$ Marginal pdf of X and Y

$ii.$ Marginal pdf of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_0^1 4xydx$$

$$\Rightarrow f_Y(y) = 4y\left(\frac{x^2}{2}\right)\Bigg|_0^1 = 2y$$

$$\therefore f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

$$c. \quad f_{XY}(x, y) = f_X(x) f_Y(y)$$

$$\therefore X \text{ and } Y \text{ are independen t}$$

$$d. \quad P(X+Y<1) = \int_0^1 \int_0^{1-y} 4xy\,dx\,dy = \int_0^1 4y \left(\frac{x^2}{2} \right) \Bigg|_0^1 dy$$

$$= \int_0^1 4y[1/2(1-y)^2]dy = \int_0^1 2(y - 2y^2 + y^3)dy$$

$$= 2(y^2/2 - 2y^3/3 + y^4/4) = 1/6$$

$$\therefore P(X+Y<1) = 1/6$$

Cont…

e. Conditiona l pdf of X and Y

i. Conditiona l pdf of X

$$f_{X/Y}(x/y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{4xy}{2y} = 2x$$

$$\therefore f_{X/Y}(x/y) = \begin{cases} 2x, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

$e.$ Conditional pdf of X and Y

$ii.$ Conditional pdf of Y

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{4xy}{2x} = 2y$$

$$\therefore f_{Y/X}(y/x) = \begin{cases} 2y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

Example-2.29:

The joint pdf of two continuous random variables ***X*** and ***Y*** is given by:

$$f_X(x) = \begin{cases} k, & 0 < y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

a. Determine the value of k.
b. Find the marginal pdf of X and Y.
c. Are X and Y independen t?
d. Find $P(0 < X < 1/2)$
e. Find the conditiona l pdf of X and Y.

Cont…

Solution:

$$a. \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1 \Rightarrow \int_{0}^{1}\int_{y}^{1} k\,dx\,dy = 1$$

$$\Rightarrow k\int_{0}^{1}\left(x\right)\Big|_{y}^{1} = 1$$

$$\Rightarrow k\int_{0}^{1}(1-y)\,dy = k\left(y-\frac{y^2}{2}\right)\Bigg|_{0}^{1} = \frac{k}{2} = 1$$

$$\therefore k = 2$$

Cont…

b. Marginal pdf of X and Y

i. Marginal pdf of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^x 2dy$$

$$\Rightarrow f_X(x) = (2y)\Big|_0^x = 2x$$

$$\therefore f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

$b.$ Marginal pdf of X and Y

$ii.$ Marginal pdf of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_y^1 2dx$$

$$\Rightarrow f_Y(y) = (2x)\Big|_y^1 = 2(1-y)$$

$$\therefore f_Y(y) = \begin{cases} 2(1-y), & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

$$c.\quad f_{XY}(x,y) \neq f_X(x) f_Y(y)$$

$\therefore X$ and Y are not independen t

$$d.\quad P(0 < X < 1/2) = \int_0^{1/2}\int_0^{x} f_{XY}(x,y)\,dy\,dx$$

$$= \int_0^{1/2}\int_0^{x} 2\,dy\,dx = \int_0^{1/2} (2y)\Big|_0^{x} dx$$

$$= \int_0^{1/2} 2x\,dx = x^2\Big|_0^{1/2} = 1/4$$

$$\therefore P(0 < X < 1/2) = 1/4$$

Cont…

e. Conditiona l pdf of X and Y

i. Conditiona l pdf of X

$$f_{X/Y}(x/y)=\frac{f_{XY}(x,y)}{f_Y(y)}=\frac{2}{2(1-y)}=\frac{1}{(1-y)}$$

$$\therefore f_{X/Y}(x/y)=\begin{cases}\dfrac{1}{1-y}, & 0<y\le x<1\\ 0, & \text{otherwise}\end{cases}$$

Cont…

$e.\quad$ Conditional pdf of X and Y

$ii.\quad$ Conditional pdf of Y

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$$

$$\therefore f_{Y/X}(y/x) = \begin{cases} \dfrac{1}{x}, & 0 < y \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Cont…

Example 2.30 The joint CDF of two discrete random variables X and Y is given as follows:

$$F_{XY}(x,y) = \begin{cases} \frac{1}{8} & x=1, y=1 \\ \frac{5}{8} & x=1, y=2 \\ \frac{1}{4} & x=2, y=1 \\ 1 & x=2, y=2 \end{cases}$$

Determine the following:

a. Joint PMF of X and Y

b. Marginal PMF of X

c. Marginal PMF of Y.

Cont…

Solution The joint PMF is obtained from the relationship $F_{XY}(a,b) = \sum_{x \le a} \sum_{y \le b} p_{XY}(x,y)$. Thus,

$$F_{XY}(1,1) = 1/8 = p_{XY}(1,1)$$

$$F_{XY}(1,2) = p_{XY}(1,1) + p_{XY}(1,2) = 5/8 \Rightarrow p_{XY}(1,2) = 5/8 - 1/8 = 1/2$$

$$F_{XY}(2,1) = p_{XY}(1,1) + p_{XY}(2,1) = 1/4 \Rightarrow p_{XY}(2,1) = 1/4 - 1/8 = 1/8$$

$$F_{XY}(2,2) = p_{XY}(1,1) + p_{XY}(1,2) + p_{XY}(2,1) + p_{XY}(2,2) = 1 \Rightarrow p_{XY}(2,2) = 1/4$$

a, The joint PMF becomes

$$p_{XY}(x,y) = \begin{cases} \frac{1}{8} & x=1, y=1 \\ \frac{1}{2} & x=1, y=2 \\ \frac{1}{8} & x=2, y=1 \\ \frac{1}{4} & x=2, y=2 \end{cases}$$

Cont…

b, The marginal PMF of X is given by

$$p_X(x) = \begin{cases} p_{XY}(1,1) + p_{XY}(1,2) = 5/8 & x = 1 \\ p_{XY}(2,1) + p_{XY}(2,2) = 3/8 & x = 2 \end{cases}$$

c, The marginal PMF of Y is given by

$$p_Y(y) = \begin{cases} p_{XY}(1,1) + p_{XY}(2,1) = 1/4 & y = 1 \\ p_{XY}(1,2) + p_{XY}(2,2) = 3/4 & y = 2 \end{cases}$$

<u>Example 2.31</u> Two random variables X and Y have the following joint PDF:

$$f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)} & 0 \le x < \infty; 0 \le y < \infty \\ 0 & \text{otherwise} \end{cases}$$

- Determine the conditional PDF of X given Y and the conditional PDF of Y given X. **<u>Solution</u>**

Cont…

$$f_X(x) = \int_0^{\infty} f_{XY}(x, y)dy = \int_0^{\infty} xe^{-x(y+1)}dy = xe^{-x}\int_0^{\infty} e^{-xy}dy$$

$$= xe^{-x}\left[-\frac{e^{-xy}}{x}\right]_0^{\infty}$$

$$= e^{-x} \quad 0 \le x < \infty$$

$$f_Y(y) = \int_0^{\infty} f_{XY}(x, y)dx = \int_0^{\infty} xe^{-x(y+1)}dx$$

Let $u = x$, which means that $du = dx$; and let $dv = e^{-x(y+1)}dx$, which means that $v = -e^{-x(y+1)}/(y+1)$. Integrating by parts we obtain

$$f_Y(y) = \int_0^{\infty} xe^{-x(y+1)}dx = \left[-\frac{xe^{-x(y+1)}}{y+1}\right]_0^{\infty} + \frac{1}{y+1}\int_0^{\infty} e^{-x(y+1)}dx$$

$$= 0 - \frac{1}{(y+1)^2}\left[e^{-x(y+1)}\right]_0^{\infty}$$

$$= \frac{1}{(y+1)^2} \quad 0 \le y < \infty$$

Thus, the conditional PDFs are given by

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = xe^{-xy} \qquad 0 \le y < \infty$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{1/(y+1)^2} = x(y+1)^2 e^{-x(y+1)} \quad 0 \le x < \infty$$

THANK YOU!!!!!

After we will do some of these Examples on a board; Chapter Two be Ended!!!

If You have Any question??? Welcome!!!!!!