DEBRE MARKOS UNIVERSITY

DEBRE MARKOS INSTITUTE OF TECHNOLOGY

SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING

Probability and Random Process (ECEg2103)

For 2012 Second Year ECE Regular Students;

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Chapter Four

Stationery and Ergodicity

Outlines

- Introduction
- Stationary Stochastic Processes
- Ergodic Processes
- Power Spectral Density

4.1 Introduction

- As we recall, A stochastic process (random process) $X(t)$ is a mapping that assigns a time function $X(t,w)$ to every outcome w of points in the sample space S.
- Since $X(t1)$ and $X(t2)$ are random variables, various types of **joint moments** can be defined; as we were try to see in chapter three. To revise these;

1. **Mean:-** of $X(t)$ is a function of time called the **ensemble average** and is denoted by $\mu x(t)=E[X(t)]$
2. **Autocorrelation:**-this function provides a **measure of similarity between two observations** of the random process $X(t)$ at different points t1&t2.

Cont…

- ✓ The autocorrelation function (AC) RX(t1,t2) is defined as the expected value of the product X(t1) and X(t2):

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1\, dx_2$$

- ✓ By substituting t1=t2=t; in the above equation; we can obtain the second moment or the **average power** of the random process:

$$R_X(t) = E\left[X^2(t)\right] = \int_{-\infty}^{\infty} x^2 f_X(x; t) dx$$

Cont…

3.**Autocovariance**:- it is quantitative measure of the statistical **coupling** between X(t1) and X(t2).

- ✓ The autocovariance function (ACF) CX(t1,t2) is defined as the covariance between X(t1) and X(t2):

$$C_X(t_1,t_2) = E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2))]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X(t_1))(x_2 - \mu_X(t_2)) f_X(x_1, x_2; t_1, t_2) dx_2\, dx_1$$

$$= E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2)$$

- ✓ Thus, the interrelationships between AC and ACF are

$$C_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$$

$$R_X(t_1,t_2) = C_X(t_1,t_2) + \mu_X(t_1)\mu_X(t_2)$$

Cont…

4. **<u>Normalized Autocovariance</u>**:- The normalized autocovariance function (NACF) is the ACF normalized by the variance and is defined by

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}} = \frac{C_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)}$$

- The NACF finds wide applicability in many problems in random processes, particularly in time-series analysis.

Cont…

- The following definitions pertain to two different random processes X(t) and Y(t):

1. **Cross-Correlation**:- The cross-correlation function (CC) RXY(t1,t2) is defined as the expected value of the product X(t1) and Y (t2):

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f_{XY}(x_1, y_2; t_1, t_2) dy_2\, dx_1$$

- ✓ By substituting t1=t2=t in the above equation; we obtain the joint moment between the random processes X(t) and Y(t) as

$$R_{XY}(t) = E[X(t)Y(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x, y; t) dydx$$

Cont…

2. **Cross-Covariance**:-The cross-covariance function (CCF) CXY(t1,t2) is defined as the covariance between X(t1) and Y(t2):

$$C_{XY}(t_1,t_2) = E[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X(t_1))(y_2 - \mu_Y(t_2)) f_{XY}(x_1,y_2;t_1,t_2)dy_2\, dx_1$$

$$= E[X(t_1)Y(t_2)] - \mu_X(t_1)\mu_Y(t_2)$$

- Thus, the interrelationships between CC and CCF are

$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2)$$

$$R_{XY}(t_1,t_2) = C_{XY}(t_1,t_2) + \mu_X(t_1)\mu_Y(t_2)$$

Cont…

3. **<u>Normalized Cross-Covariance</u>**:- The normalized cross-covariance function (NCCF) is the CCF normalized by the variances and is defined by

$$\rho_{XY}(t_1,t_2)=\frac{C_{XY}(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_Y^2(t_2)}}=\frac{C_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}$$

<u>Some Properties of X(t) and Y(t)</u>

- Two random processes X(t) and Y(t) are **independent** if for all t1 and t2

$$F_{XY}(x,y;t_1,t_2)=F_X(x;t_1)F_Y(y;t_2)$$

Cont…

✓ They are **uncorrelated** if

$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2) = 0$$

Or

$$R_{XY}(t_1,t_2) = \mu_X(t_1)\mu_Y(t_2) \quad \text{for all } t_1 \text{ and } t_2$$

✓ They are **orthogonal** if for all t1 and t2

$$R_{XY}(t_1,t_2) = 0$$

Example 4.1:- A random process X(t) is given by X(t) $X(t) = A\sin(\omega t + \phi)$ where A is a uniformly distributed random variable; then find the mean, variance, autocorrelation, autocovariance, and normalized autocovariance of X(t).

Cont…

Solution

➢Review of Some Trigonometric Identities

1. $$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Adding (1); $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

2. $$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Adding(2);

$$\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

Subtracting(2);

$$\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

Cont…

- More common trigonometric identities

$$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$$

$$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$$

$$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$$

$$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$$

$$\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{\cos(\theta-\varphi) + \cos(\theta+\varphi)}$$

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = -\frac{\cos\alpha-\cos\beta}{\sin\alpha-\sin\beta}$$

$$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$$

$$\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

Cont…

- Now come to Example#1 solution

Mean:

$$E[X(t)] = \mu_X(t) = E[A\sin(\omega t + \phi)] = \mu_A \sin(\omega t + \phi)$$

Variance:

$$\begin{aligned}\operatorname{var}[X(t)] = \sigma_X^2(t) &= E[A^2 \sin^2(\omega t + \phi)] - \mu_A^2 \sin^2(\omega t + \phi) \\ &= \{E[A^2] - \mu_A^2\} \sin^2(\omega t + \phi) = \sigma_A^2 \sin^2(\omega t + \phi)\end{aligned}$$

Autocorrelation

$$\begin{aligned}R_X(t_1, t_2) = E[X(t_1)X(t_2)] &= E[A^2] \sin(\omega t_1 + \phi) \sin(\omega t_2 + \phi) \\ &= \frac{1}{2} E[A^2] \{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2\phi]\}\end{aligned}$$

Cont…

Autocovariance

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

$$= E[A^2]\sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi) - \mu_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)$$

$$= \sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)$$

$$= \frac{1}{2}\sigma_A^2\{\cos[\omega(t_1 - t_2)] - \cos[\omega(t_1 + t_2) + 2\phi]\}$$

Normalized Autocovariance

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \frac{\sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)}{\sigma_A^2 \sin(\omega t_1 + \phi)\sin(\omega t_2 + \phi)} = 1$$

Cont…

Example 4.2:- Two random processes X(t) and Y(t) are given by $X(t) = A\sin(\omega t + \phi_1); \quad Y(t) = B\sin(\omega t + \phi_2)$

where A and B are two random variables; then find the means and variances of X(t) and Y(t) and their crosscorrelation, cross-covariance, and normalized cross-covariance.

Solution

- Means:

$$\mu_X(t) = E[A\sin(\omega t + \phi_1)] = \mu_A \sin(\omega t + \phi_1)$$

$$\mu_Y(t) = E[B\cos(\omega t + \phi_2)] = \mu_B \cos(\omega t + \phi_2)$$

- Variances:

$$\sigma_X^2(t) = \sigma_A^2 \sin^2(\omega t + \phi_1)$$

$$\sigma_Y^2(t) = \sigma_B^2 \cos^2(\omega t + \phi_2)$$

Cont…

- Cross-Correlation:

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = E[AB]\sin(\omega t_1 + \phi_1)\cos(\omega t_2 + \phi_2)$$

$$= \frac{1}{2}E[AB]\{\sin[\omega(t_1+t_2)+\phi_1+\phi_2] + \sin[\omega(t_1-t_2)+\phi_1-\phi_2]\}$$

- Cross-Covariance:

$$C_{XY}(t_1,t_2) = R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2)$$

$$= E[AB]\sin(\omega t_1+\phi_1)\cos(\omega t_2+\phi_2) - \mu_A\mu_B\sin(\omega t_1+\phi_1)\cos(\omega t_2+\phi_2)$$

$$= \sigma_{AB}\sin(\omega t_1+\phi_1)\cos(\omega t_2+\phi_2)$$

$$= \frac{1}{2}\sigma_{AB}\{\sin[\omega(t_1+t_2)+\phi_1+\phi_2] + \sin[\omega(t_1-t_2)+\phi_1-\phi_2]\}$$

- Normalized Cross-Covariance:

$$\rho_{XY}(t_1,t_2) = \frac{C_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)} = \frac{\sigma_{AB}\sin(\omega t_1+\phi_1)\cos(\omega t_2+\phi_2)}{\sigma_A\sigma_B\sin(\omega t_1+\phi_1)\cos(\omega t_2+\phi_2)} = \frac{\sigma_{AB}}{\sigma_A\sigma_B} = \rho_{AB}$$

4.2 Stationary Stochastic Processes

- Stationary processes exhibit **statistical** properties that are **invariant** to **shift** in the **time index**.

✓ **First-order stationarity** implies that the statistical properties of $X(t_i)$ and $X(t_i + \tau)$ are the same for any τ

❖ A random process is first-order stationary if

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x)$$

$$f_X(x; t) = f_X(x; t + \tau) = f_X(x)$$

and the distribution and density functions are independent of time.

Cont…

- ✓ **Second-order stationarity** implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1+\tau), X(t_2+\tau)\}$ are the same for *any* τ.

❖ A random process is second-order stationary if

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_1+\tau, t_2+\tau) = F_X(x_1, x_2; \tau)$$

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\tau, t_2+\tau) = f_X(x_1, x_2; \tau)$$

- The distribution and density functions are dependent not on two time instants t1 and t2 but on the time difference τ =t1–t2 only.
- **Second-order stationary** processes are also called **wide-sense stationary** or **weakly stationary.**

Cont…

- Random processes in which the **mean** and **autocorrelation** function **do not depend on absolute time** are called **wide-sense stationary (WSS) processes**.
- If a random process is wide-sense stationary, then it is **necessary** and **sufficient** that the following two conditions be satisfied:

1. The expected value is a constant, $E[X(t)] = \mu_X$.
2. The autocorrelation function R_X is a function of the time difference $t_2 - t_1 = \tau$ and not individual times, $R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$.

Cont…

- ✓ **nth-order stationary** processes- the probabilities of the samples of a random process X(t) at times t1,...,tn will not differ from those at times t1+τ ,...,tn+τ .

i.e

$$F_X(x_1,\ldots,x_n;t_1,\ldots,t_n)=F_X(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau)$$

$$f_X(x_1,\ldots,x_n;t_1,\ldots,t_n)\equiv f_X(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau)$$

- **Astrict-sense** or **strongly stationary** process is a random process that satisfies the above equestions for all n, and any τ.

- ❖ nth-order stationarity implies lower-order stationarities.

- ➢ Strict-sense stationarity implies wide-sense stationarity; but the converse is not necessarily true; except Gaussian.

Cont…

Some Properties of Correlation Functions of WSS

1. $R_X(0) = E[X^2(t)] =$ average power ≥ 0
2. $R_X(\tau) = R_X(-\tau)$. Or, $R_X(\tau)$ is an even function.

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t+\tau)X(t)] = R_X(-\tau)$$

3. $|R_X(\tau)| \leq R_X(0)$.
4. If a constant $T > 0$ exists such that $R_X(T) = R_X(0)$, then the $R_X(\tau)$ is periodic
5. $E[X(t+\phi)X(t+\tau+\phi) = R_X(\tau) = E[X(t)X(t+\tau)]$

$$E[X(t+\phi)Y(t+\tau+\phi) = R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

6. If $E[X(t)] = \mu_X$ and $Y(t) = a + X(t)$, where a is constant, then $E[Y(t)] = a + \mu_X$

and

$$\begin{aligned} R_Y(\tau) &= E\{[a + X(t)][a + X(t+\tau)]\} \\ &= a^2 + aE[X(t+\tau)] + aE[X(t)] + E[X(t)X(t+\tau)] \\ &= a^2 + 2a\mu_X + R_X(\tau) = a^2 + 2a\mu_X + \mu_X^2 + C_X(\tau) \\ &= (a + \mu_X)^2 + C_X(\tau) \end{aligned}$$

Cont…

- In a similar manner, two processes X(t) and Y(t) are **jointly stationary** if for all n

$$F_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1,\ldots,t_n)$$
$$= F_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau)$$

or

$$f_{XY}(x_1,\ldots,x_n;y_1,\ldots,y_n;t_1,\ldots,t_n)$$
$$= f_{XY}(x_1,\ldots,x_n;\ y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau)$$

and they are **jointly wide-sense** stationary if

$$F_{XY}(x_1,y_2;t_1,t_2)$$
$$= F_{XY}(x_1,y_2;t_1+\tau,\ t_2+\tau) = F_{XY}(x_1,y_2;\tau)$$

and

$$f_{XY}(x_1,y_2;t_1,t_2) = f_{XY}(x_1,y_2;t_1+\tau,t_2+\tau) = f_{XY}(x_1,y_2;\tau)$$

Cont…

- The cross-moments of two jointly stationary processes X(t)and Y(t) are defined below:

1. Cross-Correlation:

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 f(x_1, y_2; \tau)\,dy_2\,dx_1$$

2. Cross-Covariance

$$\begin{aligned} C_{XY}(\tau) &= E\{[X(t) - \mu_X][Y(t+\tau) - \mu_Y]\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1 - \mu_X)(y_2 - \mu_Y) f(x_1, y_2; \tau)\,dy_2\,dx_1 \\ &= R_{XY}(\tau) - \mu_X\mu_Y \end{aligned}$$

3. Normalized Cross-Covariance:

$$\rho_{XY}(\tau) = \frac{C_{XY}(\tau)}{\sigma_X \sigma_Y}$$

Cont…

- If a stationary random process X(t) is passed through a **linear** system with impulse response h(t); the input–output relationship will be given by the **convolution integral:**

$$Y(t) = \int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha = \int_{-\infty}^{\infty} X(t)h(t-\alpha)d\alpha$$

- The cross-correlation function RXY(t) between the input and the output can be found as follows:

$$E[X(t)Y(t+\tau)] = E\int_{-\infty}^{\infty} X(t)X(t+\tau-\alpha)h(\alpha)d\alpha$$

or

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} R_X(\tau-\alpha)h(\alpha)d\alpha$$

- Since the cross-correlation function depends only on τ , the output Y(t) will also be a stationary random process.

Cont…

<u>Example 4.3</u>:- A random process X(t) is defined by $X(t)=A\cos t + B\sin t\ ;\ -\infty< t <\infty$

where A and B are independent random variables each of which has a value −2 with probability 1/3 and a value 1 with probability 2/3. Show that X(t) is a wide-sense stationary process.

<u>Solution</u>

$$E[A]=E[B]=\frac{1}{3}(-2)+\frac{2}{3}(1)=0$$

$$E[A^2]=E[B^2]=\frac{1}{3}(-2)^2+\frac{2}{3}(1)^2=2$$

- $E[X(t)]=0$

- Since A and B are independent, $E[AB]=E[A]E[B]=0$.

Cont…

- Thus; let; t1=t & t- τ =t2=s

$$R_{XX}(t,s) = E[X(t)X(s)] = E[\{A\cos(t) + B\sin(t)\}\{A\cos(s) + B\sin(s)\}]$$

$$= \left[A^2\cos(t)\cos(s) + AB\cos(t)\sin(s) + AB\sin(t)\cos(s)\right.$$

$$\left. + B^2\sin(t)\sin(s)\right]$$

$$= E\left[A^2\right]\cos(t)\cos(s) + E[AB]\{\cos(t)\sin(s) + \sin(t)\cos(s)\}$$

$$+ E\left[B^2\right]\sin(t)\sin(s)$$

$$= 2\{\cos(t)\cos(s) + \sin(t)\sin(s)\}$$

$$= 2\cos(t - s)$$

- Since the mean is constant and the autocorrelation function is a function of the difference between the two times, we conclude that the random process X(t) is wide-sense stationary.

Cont…

<u>Example 4.4</u>:-Find the conditions necessary for the random process $X(t) = A\sin(\omega t + \Phi)$ to be WSS where;

a, A&w are constants and ф is a random variable,

b, w is constants and ф&A are random variables.

<u>Solution</u>

a; 1. Mean: $E[X(t)] = AE[\sin(\omega t + \Phi)] = A\int_{-a}^{b} \sin(\omega t + \phi) f_{\Phi}(\phi) d\phi$

One of the ways the integral will be independent of t is for ф to be uniformly distributed in (0,2π), in which case we have

$$E[X(t)] = \frac{1}{2\pi}\int_0^{2\pi} \sin(\omega t + \phi) d\phi = \frac{-1}{2\pi}\cos(\omega t + \phi)\Big|_0^{2\pi} = 0$$

Cont…

2. Autocorrelation:

$$E[X(t_1)X(t_2)] = A^2E[\sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)]$$

$$= A^2E\{\cos[\omega(t_2 - t_1)] - \cos[\omega(t_2 + t_1) + 2\Phi]\}$$

$$= A^2\cos[\omega(t_2 - t_1)] - A^2E\{\cos[\omega(t_2 + t_1) + 2\Phi]\}$$

$$E\{\cos[\omega(t_2 + t_1) + 2\Phi]\} = \frac{1}{2\pi}\int_0^{2\pi}\cos[\omega(t_2 + t_1) + 2\phi]\,d\phi$$

$$= \frac{1}{2\pi}\frac{1}{2}\sin[\omega(t_2 + t_1) + 2\phi]\Big|_0^{2\pi} = 0$$

Hence, $R_X(t_1, t_2) = A^2\cos[\omega(t_2 - t_1)] = A^2\cos[\omega\tau]$

- X(t) is stationary if ϕ is uniformly distributed in (0,2π). Since RX(τ) is periodic, X(t) is a periodic WSS process.

Cont…

b; 1. <u>Mean</u>: $E[X(t)] = E[A\sin(\omega t + \Phi)] = \int\int a\sin(\omega t + \phi) f_{A\Phi}(a, \phi) d\phi\, da$

- The first condition for the double integral to be independent of t is for A and ϕ to be statistically independent, in which case we have

$$E[A\sin(\omega t + \Phi)] = \int\int a\sin(\omega t + \phi) f_A(a) f_\Phi d\phi\, da$$

- and the second condition is for ϕ to be uniformly distributed in (0,2π), in which case we have

$$(1/2\pi)\int_0^{2\pi} \sin(\omega t + \phi)\, d\phi = 0 \text{ and } E[X(t)] = 0.$$

2. <u>Autocorrelation</u>:

$$E[X(t_1)X(t_2)] = E[A^2 \sin(\omega t_1 + \Phi)\sin(\omega t_2 + \Phi)]$$

- Since A and ϕ are independent, we have

Cont…

$$E[X(t_1)X(t_2)] = E[A^2]E\{\cos[\omega(t_2 - t_1) - \cos[\omega(t_2 + t_1) + 2\Phi]\}$$

$$= E[A^2]\cos[\omega(t_2 - t_1)] - E[A^2]E\{\cos[\omega(t_2 + t_1) + 2\Phi\}$$

$$E\{\cos[\omega(t_2 + t_1) + 2\Phi]\} = 0.$$

- Hence,

$$R_X(t_1, t_2) = \quad E[A^2]\cos[\omega(t_2 - t_1)] = E[A^2]\cos[\omega\tau]$$

- X(t) is stationary if A and ϕ are independent and if ϕ is uniformly distributed in (0,2π).
- Since RX(τ) is periodic, X(t) a periodic WSS process.

Cont…

<u>Example 4.5</u>:- If X(t)= Acos(wt)+Bsin(wt), where A and B are random variables with density functions fA(a) and fB(b); and w is constant. Then find the conditions under which X(t) will be WSS.

<u>Solution</u>

1. Mean: $E[X(t)] = E[A\cos(\omega t) + B\sin(\omega t)] = \cos(\omega t)E[A] + \sin(\omega t)E[B]$

- If E[X(t)] is to be independent of t, then E[A]=E[B]=0, in which case E[X(t)]=0.

2. Autocorrelation:

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = E\{[A\cos(\omega t_1) + B\sin(\omega t_1)][A\cos(\omega t_2) + B\sin(\omega t_2)]\}$$

$$= E\{A^2\cos(\omega t_1)\cos(\omega t_2) + B^2\sin(\omega t_1)\sin(\omega t_2) + AB[\sin(\omega t_1)\cos(\omega t_2) + \cos(\omega t_1)\sin(\omega t_2)]\}$$

Cont…

$$= \frac{1}{2}E[A^2][\cos(\omega(t_2 - t_1)) + \cos(\omega(t_2 + t_1))]$$

$$+ \frac{1}{2}E[B^2][\cos(\omega(t_2 - t_1)) - \cos(\omega(t_2 + t_1))]$$

$$+ E[AB]\sin(\omega(t_2 + t_1))$$

- The conditions under which this equation will be dependent only on (t2-t1) are $E[A^2] = E[B^2]$ and $E[AB] = 0$. In this case $R_X(t_1,t_2) = E[A^2][\cos(\omega(t_2 - t_1))] = E[A^2][\cos(\omega\tau)]$
- We can now summarize the conditions for WSS:

(a) $E[A] = E[B] = 0$

(b) $E[A^2] = E[B^2]$

(c) $E[AB] = 0$

4.3. Ergodic Processes

- **Ergodicity** refers to **certain time averages** of random processes **converging** to their respective **statistical averages**.
- An ergodic random process is a stationary process in which **every member** of the **ensemble** exhibits the **same statistical behavior** as the ensemble.
- This implies that it is possible to determine the statistical behavior of the ensemble by examining only one typical sample function.
- Thus, for an ergodic random process, the **mean values and moments** can be determined by time averages as well as by ensemble averages (or expected values), which are equal. i.e;

$$E[X^n] = \overline{X^n} = \int_{-\infty}^{\infty} x^n f_X(x)dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X^n(t)dt$$

Cont…

❖ **Mean-Ergodic**

- A stationary random process X(t) is called mean-ergodic if the **ensemble average** is equal to the time **average of the sample** function x(t).

$X(t)$ is stationary, implying that it has a constant mean μ_X, and the autocorrelation function $R_X(t, t+\tau)$ is a function of τ only, or $R_X(t, t+\tau) = R_X(\tau)$.

$R_X(0) = E[X^2(t)]$ is bounded, or $R_X(0) < \infty$. Hence $C_X(0) = R_X(0) - \mu_X^2 < \infty$.

- A necessary and sufficient condition for a random process to be mean-ergodic is $\int_{-\infty}^{\infty} |C_X(\tau)| d\tau < \infty$

Cont…

- Even though we have derived necessary and sufficient condition for a mean-ergodic process, we cannot use it to test the ergodicity of any process since it **involves prior knowledge** of the autocovariance function CX (τ).
- However, if a partial knowledge of the ACF such as |CX(τ)| goes to 0 as τ goes to infinity, then we can conclude that the process is ergodic.
- Although **ergodic processes must be stationary**, stationary processes need not be ergodic.

Cont…

❖ Correlation-Ergodic

- A stationary random process X(t) is correlation-ergodic if time autocorrelation is equal to ensemble autocorrelation.

i.e $$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\lambda)dt\right] = R_X(\lambda)$$ for all λ.

Where; λ is time shit,

$\lim_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\lambda)dt\right]$ is time autocorrelation,

$R_X(\lambda)$ is ensemble autocorrelation,

Cont…

Example 4.6:- A random process has sample functions of the form $X(t)=A\cos(wt+\Theta)$

where w is constant, A is a random variable that has a magnitude of $+1$ and -1 with equal probability, and Θ is a random variable that is uniformly distributed between 0 and 2π. Assume that the random variables A and Θ are independent.

(a) Is $X(t)$ a wide-sense stationary process?

(b) Is $X(t)$ a mean-ergodic process?

Cont…

Solution

$$E[A] = 0$$

$$\sigma_A^2 = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1 = E[A^2]$$

$$E[\Theta] = \pi$$

$$\sigma_\Theta^2 = \frac{(2\pi)^2}{12} = \frac{\pi^2}{3}$$

(a) Since A and Ɵ are independent, E[X(t)]=E[A]E[cos(wt+ Ɵ)]=0, which is a constant. Also, the autocorrelation function of X(t) is given by

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[A\cos(wt+\Theta)A\cos(wt+w\tau+\Theta)] \\ &= E[A^2]E[\cos(wt+\Theta)\cos(wt+w\tau+\Theta)] \\ &= \frac{1}{2}E[\cos(-w\tau)+\cos(2wt+w\tau+2\Theta)] \end{aligned}$$

Cont…

$$= \frac{1}{2}E[\cos(-w\tau)] + \frac{1}{2}E[\cos(2wt + w\tau + 2\Theta)]$$

$$= \frac{1}{2}\cos(w\tau) + \frac{1}{2}E[\cos(2wt + w\tau + 2\Theta)]$$

$$= \frac{1}{2}\cos(w\tau) + \frac{1}{2}\int_0^{2\pi} \frac{\cos(2wt + w\tau + 2\theta)}{2\pi}\,d\theta$$

$$= \frac{1}{2}\cos(w\tau) + \frac{1}{8\pi}[\sin(2wt + w\tau + 2\theta)]_0^{2\pi}$$

$$= \frac{1}{2}\cos(w\tau) + \frac{1}{8\pi}\{\sin(2wt + w\tau + 4\pi) - \sin(2wt + w\tau)\}$$

$$= \frac{1}{2}\cos(w\tau)$$

- Since the mean is constant and the autocorrelation function depends only on the difference between the two times and not on t, we conclude that the process is wide-sense stationary.

Cont…

(b)

Time average is;

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)dt=\lim_{T\to\infty}\frac{1}{2T}\int_{0}^{2\pi}A\cos(wt+\Theta)dt=\lim_{T\to\infty}\frac{A}{2Tw}[\sin(wt+\Theta)]_0^{2\pi}$$

$$=\lim_{T\to\infty}\frac{A}{2Tw}[\sin(2\pi w+\Theta)-\sin\Theta]=0$$

- time average=E[X(t)]=0.
- Thus, X(t) is mean-ergodic.

4.4 Power Spectral Density

- The power spectral density (psd) function of a real **stationary** random process X(t) is defined as the **Fourier transform** of the **autocorrelation** function:

➤ **For Continuous Time;**

$$S_X(\omega) = \mathrm{FT}[R_X(\tau)] = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau$$

- It represents the average power per hertz, and hence the term power spectral density.
- From the Fourier inversion theorem we can obtain the autocorrelation function from the power spectral density:

$$R_X(\tau) = \mathrm{IFT}[S_X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega$$

Cont…

- The mean-square value of the random process , which is also called the average power, is given by

$$E[X^2(t)] = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)dw$$

- Other properties of the power spectral density;

1. $S_{XX}(w) \geq 0$, which means that $S_{XX}(w)$ is a nonnegative function
2. $S_{XX}(-w) = S_{XX}(w)$, which means that $S_{XX}(w)$ is an even function
3. The power spectral density is a real function if $X(t)$ is real because we have that

$$S_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-jw\tau}d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\{\cos(w\tau) - j\sin(w\tau)\}d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau - j\int_{-\infty}^{\infty} R_{XX}(\tau)\sin(w\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)\cos(w\tau)d\tau$$

Cont…

- The **cross-spectral density** (csd) SXY(w) of two real stationary random processes X(τ) and Y(τ) is defined as the Fourier transform of the cross-correlation function, RXY(τ)

$$S_{XY}(\omega) = \mathrm{FT}[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau$$

and the inverse Fourier transform of SXY(w) gives the cross-correlation function

$$R_{XY}(\tau) = \mathrm{IFT}[S_{XY}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}d\omega$$

- The **cross-spectral density** SXY(w) **will be complex**, in general, even when the random processes X(t) and Y(t) are real.

Cont…

Some Common Fourier Transform Pairs

$x(\tau)$	$X(w)$
$e^{-a\|\tau\|}, a > 0$	$\dfrac{2a}{a^2 + w^2}$
$e^{-a\tau}, a > 0, \tau \geq 0$	$\dfrac{1}{a + jw}$
$e^{b\tau}, b > 0, \tau < 0$	$\dfrac{1}{b - jw}$
$\tau e^{-a\tau}, a > 0, \tau \geq 0$	$\dfrac{1}{(a + jw)^2}$
1	$2\pi\delta(w)$
$\delta(\tau)$	1
$e^{jw_0\tau}$	$2\pi\delta(w - w_0)$
$\begin{cases} 1 & -T/2 < \tau < T/2 \\ 0 & \text{otherwise} \end{cases}$	$T\dfrac{\sin(wT/2)}{(wT/2)}$
$\begin{cases} 1 - \|\tau\|/T & \|\tau\| < T \\ 0 & \text{otherwise} \end{cases}$	$T\left[\dfrac{\sin(wT/2)}{(wT/2)}\right]^2$
$\cos(w_0\tau)$	$\pi\delta(w - w_0) + \pi\delta(w + w_0)$
$\sin(w_0\tau)$	$-j\pi[\delta(w - w_0) - \delta(w + w_0)]$
$e^{-a\|\tau\|}\cos(w_0\tau)$	$\dfrac{a}{a^2 + (w - w_0)^2} + \dfrac{a}{a^2 + (w + w_0)^2}$

Cont…

➢ **For discrete-time:**

- The power spectral density of X[n]is given by the following discrete-time Fourier transform of its autocorrelation function:

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m}$$

Note that $e^{-j\Omega n}$ is periodic with period 2π. That is, $e^{-j(\Omega+2\pi)n} = e^{-j\Omega n}e^{-j2\pi n} = e^{-j\Omega n}$ because $e^{-j2\pi n} = 1$. Thus, $S_{XX}(\Omega)$ is periodic with period 2π, and it is sufficient to define $S_{XX}(\Omega)$ only in the range $(-\pi, \pi)$. This means that the autocorrelation function is given by

$$R_{XX}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)e^{j\Omega m}\,d\Omega$$

Cont…

- The properties of SXX(Ω) include the following:

1. $S_{XX}(\Omega + 2\pi) = S_{XX}(\Omega)$, which means that $S_{XX}(\Omega)$ is periodic with period 2π as stated earlier.

2. $S_{XX}(-\Omega) = S_{XX}(\Omega)$, which means that $S_{XX}(\Omega)$ is an even function.

3. $S_{XX}(\Omega)$ is real, which means that $S_{XX}(\Omega) \geq 0$.

4. $E[X^2[n]] = R_{XX}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\Omega)d\Omega$, which is the average power of the process.

Cont…

- **<u>White-Noise Process:-</u>** is a **zero mean** stationary random process X(t) whose a**utocovarianc**e or **autocorrelation** is given by $C_X(\tau) = R_X(\tau) = \sigma_X^2 \delta(\tau)$
- The energy of a white-noise process is infinite.
- White noise is the term used to define a random function whose **power spectral density is constant** for all frequencies.
- Thus, if N(t) denotes white noise, $S_{NN}(w) = N_0/2$

  where N0 is a real positive constant.
- The **inverse Fourier transform** of SNN(w) gives the **autocorrelation** function of N(t), $R_{NN}(\tau) = (N_0/2)\delta(\tau)$

where δ(τ) is the impulse function.

Cont…

Example 4.7:- Determine the autocorrelation function of the random process with the power spectral density given by $S_{XX}(w) = \begin{cases} S_0 & |w| < w_0 \\ 0 & \text{otherwise} \end{cases}$

Solution

- Just find the inverse Fourier transform

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(w)e^{jw\tau}dw$$

$$= \frac{1}{2\pi}\int_{-w_0}^{w_0} S_0 e^{jw\tau}dw = \frac{S_0}{2j\pi\tau}\left[e^{jw\tau}\right]_{-w_0}^{w_0}$$

$$= \frac{S_0}{2j\pi\tau}\left[e^{jw_0\tau} - e^{-jw_0\tau}\right] = \frac{S_0}{\pi\tau}\left(\frac{e^{jw_0\tau} - e^{-jw_0\tau}}{2j}\right)$$

$$= \frac{S_0}{\pi\tau}\sin(w_0\tau)$$

Cont…

<u>Example 4.8:</u>- Let Y(t)=X(t)+N(t) be a wide-sense stationary process where X(t) is the actual signal and N(t) is a zero-mean noise process with variancσ_N^2 and independent of X(t). Find the power spectral density of Y(t).

<u>Solution</u>

- Since X(t) and N(t) are independent random processes, the autocorrelation function of Y(t) is given by

$$
\begin{aligned}
R_{YY}(\tau) &= E[Y(t)Y(t+\tau)] = E[\{X(t)+N(t)\}\{X(t+\tau)+N(t+\tau)\}] \\
&= E[X(t)X(t+\tau) + X(t)N(t+\tau) + N(t)X(t+\tau) + N(t)N(t+\tau)] \\
&= E[X(t)X(t+\tau)] + E[X(t)]E[N(t+\tau)] + E[N(t)]E[X(t+\tau)] \\
&\quad + E[N(t)N(t+\tau)]
\end{aligned}
$$

Cont…

$$= R_{XX}(\tau) + R_{NN}(\tau) = R_{XX}(\tau) + \sigma_N^2 \delta(\tau)$$

- The power spectral density of Y(t) is the inverse Fourier transform its autocorrelation function.

➢ Thus, the power spectral density of Y(t) is given by

$$S_{YY}(w) = S_{XX}(w) + \sigma_N^2$$

<u>Example 4.9:</u>- Assume that X[n] is areal process, which means that RXX[−m]= RXX[m]. Find the power spectral density SXX(Ω).

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m} = \sum_{m=-\infty}^{-1} R_{XX}[m]e^{-j\Omega m} + \sum_{m=0}^{\infty} R_{XX}[m]e^{-j\Omega m}$$

$$= \sum_{k=1}^{\infty} R_{XX}[-k]e^{j\Omega k} + \sum_{m=0}^{\infty} R_{XX}[m]e^{-j\Omega m}$$

Cont…

$$= R_{XX}[0] + 2\sum_{m=1}^{\infty} R_{XX}[m]\frac{\{e^{j\Omega m} + e^{-j\Omega m}\}}{2}$$

$$= R_{XX}[0] + 2\sum_{m=1}^{\infty} R_{XX}[m]\cos(m\Omega)$$

<u>Example 4.10:</u>- Find the power spectral density of a random sequence X[n] whose autocorrelation function is given by $R_{XX}[m] = a^{|m|}$.

<u>Solution</u>

The power spectral density is given by

$$S_{XX}(\Omega) = \sum_{m=-\infty}^{\infty} R_{XX}[m]e^{-j\Omega m} = \sum_{m=-\infty}^{\infty} a^{|m|}e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{-1} a^{-m}e^{-j\Omega m} + \sum_{m=0}^{\infty} a^{m}e^{-j\Omega m}$$

Cont...

$$= \sum_{k=1}^{\infty} a^k e^{j\Omega k} + \sum_{m=0}^{\infty} a^m e^{-j\Omega m} = \sum_{k=1}^{\infty} \{ae^{j\Omega}\}^k + \sum_{m=0}^{\infty} \{ae^{-j\Omega}\}^m$$

$$= \frac{1}{1 - ae^{j\Omega}} - 1 + \frac{1}{1 - ae^{-j\Omega}}$$

$$= \frac{1 - a^2}{1 + a^2 - 2\cos(\Omega)}$$

THANK YOU!!!!!

After we will do these and some other Examples on a board; Chapter Four be Ended!!!

If You have Any question??? Welcome!!!!!!