Applied Mathematics I

Handout

DMU

CHAPTER ONE

1. VECTOR AND VECTOR SPACE

Vectors can be used by air-traffic controllers when tracking planes, by meteorologists to describe wind conditions, and also it helps to computer programmers to design virtual world. In this chapter, applications of vectors which are commonly used in the study of physics: work, torque and magnetic force will be presented along with the concept of vector and vector space.

1.1 Scalar and Vectors in R^2 and R^3

Definition 1.1.1 A Physical quantities that is described by its magnitude only is called scalar.

Definition 1.1.2 A physical quantities that is described using both magnitude and direction is called vector

Example 1.1 Temperature, Mass, area, density, volume, etc, are examples of scalars because they are completely described by a number that tells "How Much" like 10^oC and length of 5 m whereas force, displacement, velocity, acceleration, etc are examples of vectors.

Definition 1.1.3 Vectors in R2 and R3

A vector in the plane R^2 can be described as $v = (v_1, v_2)$ or $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where $v_1, v_2 \in R$.

Similarly, a vector in the space R^3 can be described as a triple of numbers w = (w_1, w_2, w_3) or

$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, where $w_1, w_2, w_3 \in$ R.

Definition 1.1.4. A number x can be used to represent a point on a line. A pair of numbers or a couple of numbers(x, y) can be used to represent appoint in the plane. A triple of numbers (x, y, z) can be used to represent a point in space.

We can say that a single number represents a point in1-space or on a line, a couple of numbers represents a point in 2-space or on a plane and a triple of numbers represents a point in3-space.

Although we cannot draw a picture to go further, a quadruple of numbers(x, y, z, w) or (x_1, x_2, x_3, x_4) represent a point in 4-space.

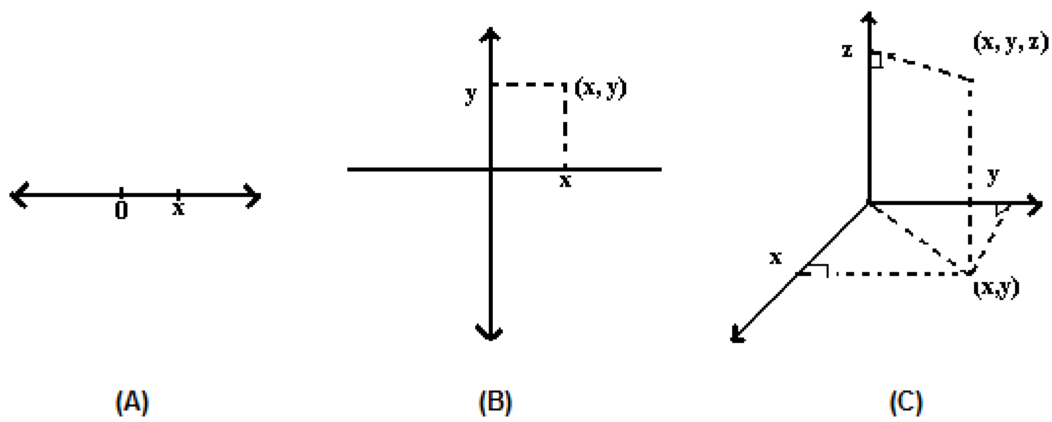


Figure 1.1: Representation of a point on a line, plane and on 3- space

Definition 1.1.5 Vectors in n–space,

Every pair of distinct points P and Q in R^n determines a directed line segment with initial point at P and terminal point at Q. We call such a directed line segment a vector and denote it by $\overrightarrow{PQ}$. The length of the line segment is the magnitude of the vector. Although $\overrightarrow{PP}$ has zero length, and strictly speaking, no direction, it is convenient to view it as a vector. It is called a zero or a null vector. It is often denoted by $\vec{0}$

Definition 1.1.6 A position vector is a vector whose initial point is at the origin otherwise it is a located vector.

Definition 1.1.7 Two non–zero vectors v and w of the same dimension are said to be parallel if they are scalar multiples of one another. In other words, the two vectors v and w are said to be parallel, denoted by v//w if there is a scalar k such that v = kw and if k > 0, then they have the same direction and if k < 0, then they are in the opposite direction.

The vector $\vec{0}$ is parallel to every vector v in the same dimension, since it can be expressed as the scalar multiple 0 = 0v. Although, zero vectors has no natural direction, it can be assigned any direction that is convenient for the problem at hand.

Example 1.2 Consider P_1 = (3,7), P_2 = (5,1), Q_1 = (−4,2) and Q_2 = (−16,−14) are points on a plane. Then $\overrightarrow{P_1Q_1}=Q_1 - P_1$ = (−7,−5) and $\overrightarrow{P_2Q_2}=Q_2 - P_2$ = (−21,−15) = 3(−7,−5).

Therefore, $\overrightarrow{P_1Q_1}$ and $\overrightarrow{P_2Q_2}$ are parallel and have the same direction, since 3 > 0.

Exercise 1.1 Show that vector v = (1,2,3) and w = (−2,−4,−6) are parallel vectors and determine the direction of the vectors?

Definition 1.1.8 Two vectors v and w will be considered to be equal (or equivalence), v = w, if they have the same magnitude and direction even though they may be located in different position.

That is, if v = (v_1,v_2) and w = (w_1,w_2) in R^2, v = w if and only if v_1 = w_1 and v_2= w_2.

Figure 1.2: Equal Vectors

The definition of equality of two vectors does not require that the vectors have the same initial and terminal points. Rather it suggests that we can move vectors freely provided we make no change in magnitude and direction.

1.2 Vector Addition and Scalar Multiplication

Definition 1.2.1 If v and w are any two vectors, then the sum $v + w$ is the vector determined as follows; position the vector w so that its initial point coincides with the terminal point of v. The vector $v + w$ is represented by the arrow from the initial point of v to the terminal point of w.

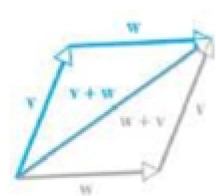


Figure 1.3: The sum of vector v and w

More than two vectors can also be added by joining the terminal point of the first to the initial point of the second and so on, finally the result will be a vector from the initial point of the first to the terminal point of the last vector.

Definition 1.2.2 If v is a non-zero vector and k a non-zero real numbers(scalar), then the product kv is defined to be the vector whose length is |k| times the length of v and whose direction is the same as that of v if k > 0 and opposite to that of v if k < 0. We define kv = 0 if k = 0 or v = 0.

Note that the vector (−1)v has the same length as v but is oppositely direction. Thus (−1)v is just the negative of v.

Figure 1.4: -v is in the opposite direction of v

Vectors in Coordinate System

Let v be any vectors in the plane, that v has been positioned. So, its initial point is at the origin of a rectangular coordinates system. The coordinates (v_1, v_2) of the terminal point of v are called the components of v and we write $v = (v_1, v_2)$. An order pair consists of two terms the abscissa (horizontal, usually x) and the ordinate (vertical, usually y) which define the location of a point in two-dimensional rectangular space.

The operation of vector addition in terms of components for $v = (v_1, v_2)$ and w = (w_1, w_2), then

Figure 1.5: The location of a point in two dimensional rectangular space

$$v + w = (v_1 + w_1, v_2 + w_2)$$

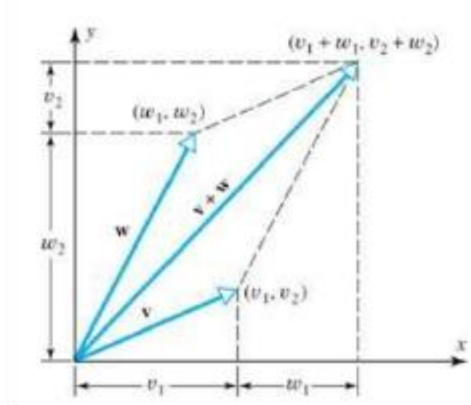


Figure 1.6: The sum of vector v and w component wise

Definition 1.2.3 If $v = (v_1, v_2)$ and k is any number or scalar. Then kv is a vector and defined as $kv = (kv_1, kv_2)$.

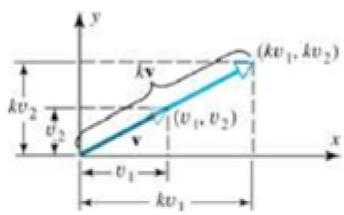


Figure 1.7: Scalar multiple of a vector

Example 1.4 If $u = (4,3,2)$ andα = 2, then $\alpha u = 2(4,3,2) = (8,6,4)$.

If a vector v in 3-space is positioned. So its initial point is at the origin of rectangular coordinate system, the coordinates of the terminal point are called the components of v, and we write $v = (v_1, v_2, v_2)$.

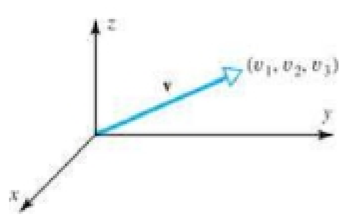


Figure 1.8: Position Vector

Definition 1.2.4 If v and w are any two vectors, then the difference of w from v is defined by

$$v - w = v + (-w)$$

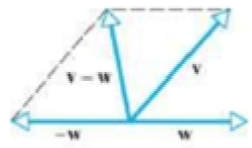


Figure 1.9: The difference of vector v and w

Example 1.5 Consider $v = (1,2,4)$ and $w = (3,1,2)$. $Find$ $v + w, 2v$ and $v - 2w$.

Solution: From the definition of vector addition and scalar multiplication

$$v + w = (1,2,4) + (3,1,2) = (4,3,6)$$

$$2v = 2(1,2,4) = (2,4,8)$$

$$v - 2w = (1,2,4) - 2(3,1,2) = (-5,0,0)$$

Properties of Vector addition and Scalar Multiplication

Let u,v and w be vectors in R^2 and α and β are scalars. Then

1. $v + w \in R2(R3)$

2. $v + w = w + v$

3. u+$\vec{0} = \vec{0} + u = u$, where $\vec{0}$ = (0,0) ∈ R^2

4. There exist w∈R^2, such that $u + w = \vec{0}$ for every $u \in R^2$

5. $(u + v) + w = u + (v + w)$

6. $\alpha(\beta u) = (\alpha\beta)u$

7. $(\alpha + \beta)u = \alpha u + \alpha u$

8. $1 \cdot u = u$

The properties described above also hold true for every vectors in R^3, where $\vec{0} = (0,0,0) \in R^3$ and generally is also true in R^n, where $\vec{0} = (0,0,0,\cdots,0) \in R^n$.

1.3 Norm of vector and Scalar Product, Orthogonal Projection, and Direction Cosines

1.3.1 Norm of a Vector

Definition 1.3.1 Let v = (v_1, v_2) be a vector in R^2. Then the norm or magnitude of v, denoted by $\|v\|$

is defined by $\|v\| = \sqrt{{v_1}^2 + {v_2}^2}$

Similarly, for a vector w = (w_1, w_2, w_3) be a vector in R^3. Then the magnitude of w, denoted by $\|w\|$

is defined by $\|w\| = \sqrt{{w_1}^2 + {w_2}^2 + {w_3}^2}$

Example 1.6 Find the norm of a vector u = (2,3,5,4**).**

Solution: From the definition of norm $\|u\| = \sqrt{2^2 + 3^2 + 5^2 + 4^2} = \sqrt{54}$

Example 1.7 If $\|v\|$ = 6, find x such that $v = (-1, x, 5)$.

Solution: From the definition of norm $\|v\| = \sqrt{(-1)^2 + (x)^2 + (5)^2}$

$$36 = 1 + x^2 + 25$$

$$x = \pm\sqrt{10}$$

Remark 1.3.1 i. $\|v\| \neq 0$ if $v \neq 0$

ii. $\|v\| = \|-v\|$

Theorem 1.3.2 if $k \in R$, then $\|kv\| = |k|\|v\|$

Proof: suppose that $v \in R^n$, then

$$\|v\| = \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2}$$

$$= \sqrt{k^2({v_1}^2 + {v_2}^2 + \cdots + {v_n}^2)}$$
$$= \sqrt{k^2}\sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$$
$$= |k|\|v\|$$

Example 1.8 Let $v = (1,3,5)$. Then find the norm or magnitude of the vector $-3v$.

Solution: From the definition and properties of the norm $\|-3v\| = |-3|\|v\|$

$$= 3\sqrt{(1)^2 + (3)^2 + (5)^2}$$

$$= 3\sqrt{35}$$

Definition 1.3.2 A vector u satisfying $\|u\| = 1$ is called a unit vector.

NB: $\|u\| = 0 \Leftrightarrow u = 0$

Example 1.9 The vector $(0,1), (-1,0), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (1,0,0)$ are examples of unit vectors

Example 1.10 Find a unit vector in the same direction as w = (3,–4).

Solution: First, note that $\|w\| = \|3,-4\| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$

A unit vector in the same direction as w is then u = $\frac{1}{\|w\|}w = \frac{1}{5}(3,-4) = (\frac{3}{5}, -\frac{4}{5})$

Example 1.11 Find a unit vector in the same direction as (1,–2,3) and write (1,–2,3) as the product of its magnitude and a unit vector.

Solution: First, we find the magnitude of the vector $\|1,-2,3\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

The unit vector having the same direction as (1,–2,3) is given by

$$\frac{1}{\sqrt{14}}(1,-2,3) = (\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$$

Furthermore, $(1,-2,3) = \sqrt{14}(\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

Remark 1.3.3 • All unit vectors in R^2 are of the form $(cos\theta, sin\theta)$, where θ ∈[0,2π].

• For any non-zero vector w, the unit vector u corresponding to w in the direction of w

can be obtained as u = $\frac{1}{\|w\|}w$

• For two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ on the plane R^3,

we calculate the distance d(P_1, P_2) between the two points as

$$d(P_1, P_2)= \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

where $\overrightarrow{P_1P_2}$ is the vector with initial point P_1 and terminal point P_2.

That is $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

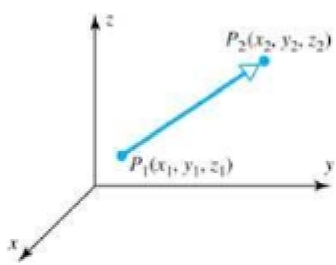


Figure 1.10: Vector

1.3.2 Scalar Product

Definition 1.3.3 Suppose v and w be two vectors in R^2 or R^3 and θ ∈ [0,π] represents the angle between them. Then scalar product of v and w is the number defined by

$$v.w = \begin{cases} \|v\|\|w\| \cos\theta \text{ if } v \neq 0 \text{ and } w \neq 0 \\ 0 \text{ if } v = 0 \text{ or } w = 0 \end{cases}$$

The scalar product of the two vectors is a scalar quantity and its value is maximum when θ = 0^0 and minimum if θ = 180^0 and the scalar product is also called a dot product or inner product and its value are scalar.

Let v= (v_1, v_2) and w= (w_1, w_2) be two non -zero vectors. If θ is the angle between v and w, then the law of cosines yields

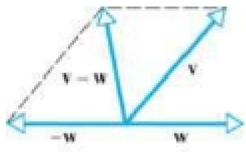


Figure 1.11: The dot product of two vectors

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$$

$$\Rightarrow 2\|v\|\|w\|\cos\theta = \|v\|^2 + \|w\|^2 - \|v-w\|^2$$

$$\Rightarrow \|v\|\|w\|\cos\theta = \frac{1}{2}[\|v\|^2 + \|w\|^2 - \|v-w\|^2]$$

$$\Rightarrow v.w = \frac{1}{2}[\|v\|^2 + \|w\|^2 - \|v-w\|^2]$$

$$= \frac{1}{2}[{v_1}^2 + {v_2}^2 + {w_1}^2 + {w_2}^2 + 2v_1w_1 + 2v_2w_2 - {v_1}^2 - {v_2}^2 - {w_1}^2 - {w_2}^2]$$

$$= \frac{1}{2}[2v_1w_1 + 2v_2w_2] = v_1w_1 + v_2w_2$$

There fore $v \cdot w = v_1w_1 + v_2w_2$

Similarly, if $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are non zero vectors in R^3, then the dot product can be given by $v \cdot w = v_1w_1 + v_2w_2 + v_3w_3$

Properties of Scalar Product

If u, v *and* w are vectors in the same dimension and α ∈R, then

1. $u \cdot u = \|u\|^2$

2. $v \cdot w = w \cdot v$

3. $u \cdot (v + w) = u \cdot v + u \cdot w$

4. $0 \cdot u = 0$

5. $(\alpha v) \cdot w = \alpha(v \cdot w) = w \cdot (\alpha v)$

6. $u \cdot u > 0$ if $u \neq 0$ and $u \cdot u = 0$ if and only if $u = 0$

Example 1.12 If v = (1,-2,3) and w = (0,1,-5), then find $v \cdot v, v \cdot w \ and \ (v + w) \cdot v$.

Solution: From the definition of dot product and properties of dot product

$$v \cdot v = (1, -2,3) \cdot (1, -2,3) = 14$$

$$v \cdot w = (1, -2,3) \cdot (0,1, -5) = -17$$

$$(v + w) \cdot v = ((1, -2,3) + (0,1, -5)) \cdot (0,1, -5) = 9$$

1.3.3 Angle between two vectors

If θ is the angle between two vectors v and w, then the angle between the two vectors can be obtained by

$$\cos\theta = \frac{v.w}{\|v\|\|w\|} \Rightarrow \theta = \cos^{-1}(\frac{v.w}{\|v\|\|w\|}) \text{ where } \theta \in [0,\pi].$$

Example 1.13 Find the angle between the vectors v = (2,0,−2) and w = (2,2,0).

Solution: Let θ be the angle between the two vectors, then

$$\cos\theta = \frac{v.w}{\|v\|\|w\|} \Rightarrow \theta = \cos^{-1}(\frac{v.w}{\|v\|\|w\|}) = \cos^{-1}\left(\frac{4}{\sqrt{8}\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Hence, $\theta = \frac{\pi}{3} = 60^0$

Definition 1.3.4 Two non-zero vectors v and w are said to be orthogonal (perpendicular) if and only if $v \cdot w = 0$; that is, $\theta = \frac{\pi}{2}$.

Example 1.14 Find the value(s) of x such that the vectors $v = (1,4,3)$and $w = (x, -1,2)$ are orthogonal.

Solution: From the definition of orthogonality, $v \cdot w = 0$

$$\Rightarrow (1,4,3) \cdot (x, -1,2) = 0$$

$$\Rightarrow x - 4 + 6 = 0$$

$$\Rightarrow x = -2$$

Remark 1.3.4 If v is orthogonal to w, then it is also orthogonal to any scalar multiple of w.

Definition 1.3.5 If P and Q are points in 2 or 3 space, the distance between P and Q, by using dot product, denoted by $\|P - Q\|$ is given by

$$\|P - Q\| = \sqrt{(P - Q).(P - Q)}$$

Theorem 1.3.5 Given two vectors v and w in space, $\|v + w\| = \|v - w\|$ if and only if v and w are orthogonal vectors.

Proof: (⇒) Given $\|v + w\| = \|v - w\|$, we want to show v and w are orthogonal

$$\|v + w\|^2 = (v + w).(v + w)$$
$$= \|v\|^2 + 2v.w + \|w\|^2$$
$$\|v - w\|^2 = (v - w).(v - w)$$
$$= \|v\|^2 - 2v.w + \|w\|^2$$

By hypothesis $\|v + w\| = \|v - w\|$ implies that

$$\|v\|^2 + 2v.w + \|w\|^2 = \|v\|^2 - 2v.w + \|w\|^2$$
$$\Rightarrow 4v.w = 0$$

Therefore, v and w are orthogonal.

Proof: (⇐) Given v and w are orthogonal We want to show $\|v + w\| = \|v - w\|$

$$\|v + w\|^2 = \|v\|^2 + 2v.w + \|w\|^2$$
$$\|v - w\|^2 = \|v\|^2 - 2v.w + \|w\|^2$$

Since, v and w are orthogonal, $v.w = 0$

$$\therefore \|v + w\| = \|v - w\|$$

Theorem 1.3.6 Pythagoras Theorem If v and w are orthogonal vectors, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof: $\|v + w\|^2$ can be written as

$$\|v + w\|^2 = (v + w).(v + w)$$

$$= \|v\|^2 + 2v.w + \|w\|^2$$
$$= \|v\|^2 + \|w\|^2, \text{ since } v.w = 0$$

Example 1.15 Find any unit vectors that are orthogonal to the vector $v = (6,8)$.

Solution: Let $w = (a, b)$be a unit vector orthogonal to v, then $\|w\| = 1 = \sqrt{a^2 + b^2}$ and

$$v.w = 6a + 8b = 0$$

By using simultaneous equation, $w = \left(\frac{-4}{5}, \frac{3}{5}\right) \, or \, w = \left(\frac{4}{5}, \frac{-3}{5}\right)$

Example 1.16 If the angle between the vector v and w is $\theta = \frac{\pi}{6}$ with each other and $\|v\| = \sqrt{3}$ and $\|w\|$= 1, then calculate the cosine of the angle between the vectors $v + w \, and \, v - w$.

Solution: Let $A = v + w \, and \, B = v - w$. Now we need to find the angle between A and B. If Φ is the angle between A and B, then

$$\cos \Phi = \frac{A.B}{\|A\|\|B\|} = \frac{(v+w).(v-w)}{\|v+w\|\|v-w\|} = \frac{1}{\sqrt{7}}$$

$$\Rightarrow \Phi = \cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$$

Since, $\|v + w\|^2 = \|v\|^2 + 2v.w + \|w\|^2$

$$= 3 + 2(\sqrt{3})(1) \cos \frac{\pi}{6} + 1$$

$$= 7$$

$\|v + w\| = \sqrt{7}$ and $\|v + w\| = 1$

Hence, $(v + w).(v - w) = \|v\|^2 - \|w\|^2 = 1$

Exercise 1.17 Let v and w be a pair orthogonal vectors such that$\|v\|$ =t and $\|w\|$ = r. Find the angle between the vector p = $\frac{tw+rv}{t+r}$ and the vector v.

1.3.4 Orthogonal Projection

Definition 1.3.6 Suppose S is the foot of the perpendicular from R to the line containing $\overrightarrow{PR}$, then the magnitude of the vector with representation $\overrightarrow{PS}$ is called the component of w along v and is denoted by $comp_v^w$; that is

$$comp_v^w = \|w\| \cos \theta, \text{ where } \theta \text{ is the angle between v \& w.}$$

$$= \|w\| \frac{v.w}{\|v\|\|w\|}, \text{ since } \cos \theta = \frac{v.w}{\|v\|\|w\|}$$

$$= \frac{v.w}{\|v\|}$$

Figure 1.12: Component of w along v

Definition 1.3.7 The projection of w on to v is defined to be the vector w in the direction of vector v, which is denoted by $proj_v^w$; that is

$$proj_v^w = (\frac{v.w}{||v||})\frac{v}{||v||}$$

$$= \frac{v.w}{||v||^2}v$$

$$proj_v^w = (\frac{v.w}{||v||})\frac{v}{||v||} = \frac{v.w}{||v||^2}v$$

Note: 1. $proj_v^w$ is **parallel** to A. That is $proj_v^w = tA$, for some $t \in \Re$

2. $w - proj_v^w$ is **orthogonal (perpendicular) to** A.

Example 1.18 Find the component of v along w and the projection of v on to w,

Where $v = (1,2)\ and\ w = (3,4)$.

Solution: Since, $comp_w^v = \frac{v.w}{||w||} = \frac{1x3+2x4}{\sqrt{25}} = \frac{11}{5}$

Similarly, $proj_w^v = \frac{v.w}{||w||^2}w = \left(\frac{11}{5}\right)(3,4) = (\frac{33}{5}, \frac{44}{5})$

Remark 1.3.7 $comp_v^w \neq comp_w^v$ and $proj_v^w \neq proj_w^v$

Theorem 1.3.8 Let u be a non-zero vector, then for any other vector w

$v = w - \frac{w.u}{||u||^2}.u$ is orthogonal to u.

Proof: $v.u = \left(w - \frac{w.u}{||u||^2}.u\right).u$

$= w.u - \frac{w.u}{||u||^2}||u||^2$

$= w.u - w.u = 0$

Example 1.19 Find an orthogonal vector to $u = (0,2,0,2,1)$.

Solution: Let $w = (0, -1,0, -1,0)$, then

$$v = w - \frac{w.u}{\|u\|^2}.u$$

$$= \frac{1}{9}(0, -1,0, -1,0)$$ is orthogonal (perpendicular) to u.

Theorem 1.3.9 Cauchy-Schwarz inequality For any two vectors v and w, $v.w = \|v\|\|w\|$

Equality holds if and only if either v is a scalar multiple of w or one of v or w is 0.

Proof: Let p is the end point of $proj_w^v$; that is, $p = proj_w^v$ and let d is the distance from the terminal point of v to the terminal point of the vector $proj_w^v$ from the figure below,

$$d = \left\| v - \frac{v.w}{\|w\|^2}.w \right\|$$

So, from the above assumption, the square of the distance from the line to the origin to be

$$\left\| v - \frac{v.w}{\|w\|^2} w \right\|^2 = v.w - 2\frac{(v.w)^2}{w.w} + \frac{(v.w)^2}{w.w}$$

$$= \|v\|^2 - \frac{(v.w)^2}{\|w\|^2}$$

$$= \frac{1}{\|w\|^2}(\|v\|^2\|w\|^2 - (v.w)^2)$$

Since, the number is square, it cannot be negative.

Hence, $(v.w)^2 \le \|v\|^2\|w\|^2$

$$\Rightarrow v.w \le \|v\|\|w\|$$

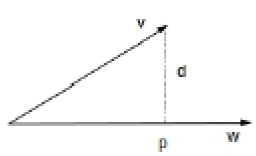


Figure 1.13: The distance from a point on vector v to a point p on vector w

Theorem 1.3.10 Triangular Inequality For vectors v and w in space $\|v + w\| \le \|v\|\|w\|$

Proof: $\|v + w\|^2 = (v + w).(v + w)$

$$= \|v\|^2 + 2v.w + \|w\|^2$$

By the cauchy-schwarz inequality, we have

$$\|v + w\|^2 = \|v\|^2 + 2v.w + \|w\|^2$$

$$\leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$$
$$\leq (\|v\| + \|w\|)^2$$
Hence, $\|v + w\| \leq \|v\| + \|w\|$

1.3.5 Directional angles and cosines

Let $u = u_1 i + u_2 j + u_3 k$ be a vector positioned at the origin in R^3, making an angle of α,β and γ with the positive x, y and z −axis respectively. Then the angles α, β and γ are called the directional angles of u, and the quantities $cos\alpha, cos\beta$ and $cos\gamma$ are called the directional cosines of u, which can be computed as

- $cos\alpha = \frac{u_1}{\|u\|}, \alpha \in [0, \pi]$
- $cos\beta = \frac{u_2}{\|u\|}, \beta \in [0, \pi]$
- $cos\gamma = \frac{u_3}{\|u\|}, \gamma \in [0, \pi]$

Remark 1.3.11 $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

Example 1.20: Let u = (1, -2, 3). Find the direction cosines of u.

Solution: Since $\|u\| = \sqrt{u.u} = \sqrt{(1,-2,3).(1,-2,3)} = \sqrt{1+4+9} = \sqrt{14}$

Thus the direction cosines are: $Cos\alpha = \frac{u_1}{\|u\|} = \frac{1}{\sqrt{14}}$, $Cos\alpha = \frac{u_2}{\|u\|} = \frac{-2}{\sqrt{14}}$, and $Cos\gamma = \frac{u_3}{\|u\|} = \frac{3}{\sqrt{14}}$

1.4 The Vector product

The second type of product of two vectors is the cross product. Unlike the dot product, the cross product of two vectors is a vector.

Definition 1.4.1 The cross product (or vector product) A x B of two vectors $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ is defined by

$$\mathbf{A \times B = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)}$$

$$\mathbf{AxB} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

So using the concept of determinants we can compute the cross product of vectors

Note that the cross product is defined in $\mathfrak{R}^3$

Example 1.4.1 Let A = (2, -1, 3) and B = (-1, -2, 4)

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ -1 & -2 & 4 \end{vmatrix} = i\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - j\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} + k\begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = 2i-11j-5k$$

=(2, -11, -5)

Activity 1.4.1: Find B x A.

Properties of cross product

Theorem 1.4.1 Let A and B be non zero vectors in R^3,then the length of AxB is by $\|AxB\|=\|A\|\|B\||sin\theta|$ where θ is the between A and B

Proof: Suppose that A = (a_1, a_2, a_3) and B = (b_1, b_2, b_3), then

A x B = ($a_2 b_3 - a_3 b_2$, $a_3 b_1 - a_1 b_3$, $a_1 b_2 - a_2 b_1$)

Hence $\|AxB\|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

$=({a_1}^2 + {a_2}^2 + {a_3}^2)({b_1}^2 + {b_2}^2 + {b_3}^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$

$=\|A\|^2\|B\|^2 - (A.B)^2$ but we A.B=$\|A\|\|B\|COS\theta$

$\Longrightarrow \|AxB\|^2 = \|A\|^2\|B\|^2 - (\|A\|\|B\|COS\theta)^2$

$=(\|A\|^2\|B\|^2)(1 - cos^2\theta)$

$=(\|A\|^2\|B\|^2)(sin^2\theta)$

$\|AxB\| =\|A\|\|B\||sin\theta|$

Theorem 1.4.2 For vectors A, B and C,

1) A x B = -(B x A)
2) A x A = O
3) tA x B = t(A x B) = A x (tB), $t \in \mathfrak{R}$
4) $\|\mathbf{AxB}\|^2 = \|\mathbf{A}\|^2\|\mathbf{B}\|^2 - (\mathbf{A.B})^2$
5) C. (A x B) = B. (C x A) = A. (B x C)
6) (A + B) x C = (A x C) + (B x C)
7) C x (A + B) = (C x A) + (C x B)
8) A . (A x B) = 0 and B. (A x B) = 0 (that is, AxB is perpendicular to both A and B.)
9) (AxB) x C = (A. C) B – (B.C)A

Proof: The following is the proof for 1, 2 and 8. The rest are left as an exercise

1) From the definition of cross product,

$A \times B = (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

For B x A, interchange A and B to obtain

$B \times A = (b_2 a_3 - b_3 a_2,\ b_3 a_1 - b_1 a_3,\ b_1 a_2 - b_2 a_1)$

$= (a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$

$= -(A \times B)$

2) $A \times A = (a_2 a_3 - a_3 a_2,\ a_3 a_1 - a_1 a_3,\ a_1 a_2 - a_2 a_1)$

$= (0, 0, 0)$

8) Setting C = A in 5) yields

$A \cdot (A \times B) = B \cdot (A \times A)$

$= B \cdot 0$ (why?)

$= 0$

By setting C = B in 5),

$B \cdot (A \times B) = A \cdot (B \times B)$

$= A \cdot 0 = 0$

This shows that for non zero vectors A and B, the cross product A x B is orthogonal to both A and B.

Activity 1.4.2: Are the usual commutative and associative laws valid?i.e. for any vectors A, B and C in $\Re^3$, is A x B = B x A? Is A x (B x C) = (A x B) x C?

Exercise 1.4.1: Let A = (2,1,0), B = (2,-1,1) and C = (0,1,1). Find

a. AxB b. (AxB)xC c. (A.C)B – (B.C)A d. BxC e. Ax(BxC)

From 4) of theorem 1.4.1, we derive an important formula for the norm of the cross product.

$$\|\mathbf{AxB}\|^2 = \|\mathbf{A}\|^2 \|\mathbf{B}\|^2 - (\mathbf{A.B})^2$$

$$= \|A\|^2 \|B\|^2 - \|A\|^2 \|B\|^2 \cos^2\theta \quad (\theta \text{ is the angle between A and B})$$

$$= \|\mathbf{A}\|^2 \|\mathbf{B}\|^2 (1 - \cos^2\theta)$$

$$= \|\mathbf{A}\|^2 \|\mathbf{B}\|^2 \sin^2\theta$$

$$\Rightarrow \|\mathbf{AxB}\| = \|\mathbf{A}\| \|\mathbf{B}\| \sin\theta \quad (\text{For } 0 \le \theta \le \pi, \sin\theta \text{ is non- negative})$$

Activity 1.4.3:

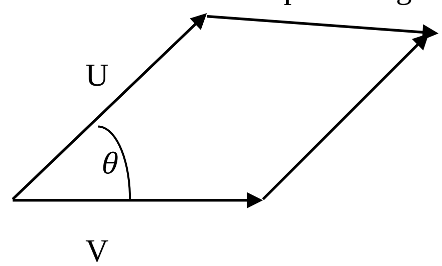
- For the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, find $\mathbf{i} \times \mathbf{j}$, $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$. What is $\mathbf{j} \times \mathbf{i}$?
- If A and B are parallel, what is A ×B?
- If A and B are orthogonal, What is $\|\mathbf{A} \times \mathbf{B}\|$?

Example

1. Find a unit vector perpendicular to both A = (2,-3,1) and B = (1,2,-4).

   Solution: AxB=(2-3,1)x(1,2,4)=(10,9,7) is orthogonal to both A and B
2. **Activity** Prove that (A – B)x(A + B) = 2 (AxB).

Let **u** and **v** be vectors and consider the parallelogram that the two vectors make.



Then Area $=\|U\|\|V\|sin\theta$ $= \|u \times v\|$

$\|u \times v\|$ **= Area of the Parallelogram**

The direction of **uxv** is a right angle to the parallelogram that follows the right hand rule.

To find the volume of the parallelepiped spanned by three vectors **u, v,** and **w**, we find the triple product:

$|u.(vxw)|$ **= Volume**

This can be found by computing the determinant of the three vectors:

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = u_1(v_2w_3 - v_3w_2) - v_1(u_2w_3 - u_3w_2) + w_1(u_2v_3 - u_3v_2)$$

Example 1.5.1: 1. Find the area of the parallelogram which is formed by the two vectors u= (1, 3, 2) and v= (-2, 1, 3).

2. Find the volume of the parallelepiped spanned by the vectors

u = (3, -2, -1), v = (1, 3, 2), and w = (-2, 1, 3).

Solution: 1. The area of the parallelogram is given by:

$$\|u \times v\| = \|(1,3,2) \times (-2,1,3)\| = \|(7,-7,7)\| = \sqrt{147}$$

2. The volume of the parallelepiped spanned by the three vectors is:

$$|u.(vxw)| = |(3,-2,-1).((1,3,2) \times (-2,1,3))| = |(3,-2,-1).(7,-7,7)| = 28$$

Exercise: Find the area of the triangle having vertices at u = (3, -2, -1),

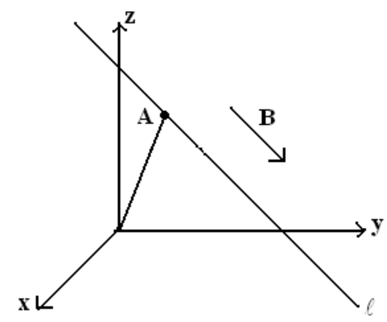
v = (1, 3, 2), and w = (-2, 1, 3).

1.5 Lines and planes

Vector equations of lines

Definition 1.5.1: A **line** L is any set of the form $\{\mathbf{p} : \mathbf{p} = \mathbf{A} + \mathbf{tB},\ \mathbf{t} \in \Re\}$ where **B** is assumed to be a **non-zero vector** and A is a **fixed point** on the line.

Note that if (x, y, z) is on line L and if A = (a_1, a_2, a_3) a point and B = (b_1, b_2, b_3) be a vector then (x, y, z) = (a_1, a_2, a_3) + t(b_1, b_2, b_3) for some real number t.



Activity 1.5.1: Is point A on L? Is B parallel to a vector formed by any points of L?P = A + tB is a **vector equation** of a line through A

Example 1.5.1 Find equation of a line through P_1 = (0, 1, 2) and P_2 = (-1, 1, 1).

Solution: We need a point A on the line and a vector B parallel to the vector formed by two point of the line.

Take A = P_1 and B = $P_2 - P_1$. Then

A + t B = (0, 1, 2) + t (-1, 0, -1)

(x, y, z) = (0, 1, 2) + t (-1, 0, -1) is equation of the line. By giving distinct values for t we will obtain distinct points on the line. Find some of the points.

Note: The equation of a line passing through points A and B is given by:

$$P = A + t(B - A) \qquad \text{or} \qquad P = (1 - t)A + B,\ t \in \Re$$

Exercise 1.5.1: Let the line L_1 passes through the points (5,1,7) and (6,0,8) and the line L_2 passes through the points (3,1,3) and $(-1,3,\alpha)$. Find the value of α for which the two lines intersect.

Suppose P = (x, y, z) is a point on line ℓ through A = (a_1, a_2 a_3) in the direction of B = (b_1, b_2, b_3). Then p = A + tB $\Rightarrow$ (x, y, z) = (a_1, a_2, a_3) + t(b_1, b_2, b_3) or equivalently

$$x = a_1 + b_1t$$

$$y = a_2 + b_2t$$

$$z = a_3 + b_3t$$

These equations are **parametric equation** of a line and t is called a **parameter**.

Activity 1.5.2: 1) Find the parametric equation of a line that contains (2, -1, 1) and is parallel to the vector $(3, \frac{1}{2}, 0)$.

2) From the parametric equation of a line in $\Re^3$, derive the equation

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

It is called **standard form** of equation of a line. If the line is on a plane show that the standard form reduces to an equation of the form y = mx + c.

Parallel and Perpendicular lines

Two lines L and m given by $A_1 + tB_1$ and $A_2 + tB_2$ are said to be **parallel** if B_1 and B_2 are parallel (their directional vectors are Parallel. Two lines are said to be perpendicular if their directional vectors are perpendicular That is the vectors $P_1 - Q_1$ and $P_2 - Q_2$ are parallel for any two points P_1, Q_1 of L and P_2, Q_2 of m.

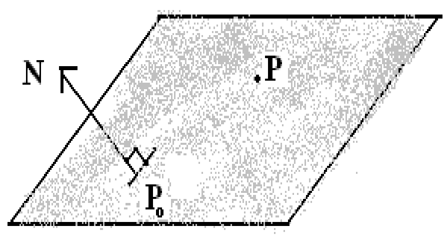
Let L be a line through A in the direction of B $(B \neq 0)$. Consider the distance between L and the origin. This distance is the minimum of the lengths of all vectors with initial point the origin and terminal point on L. That is, minimum of $\|A + tB\|$ for any real number t.

Planes and their equation

Now put $f(t) = \|A + tB\|^2$

This is a quadratic function whose graph opens upward: $f(t) = \|A\|^2 + 2t(A.B) + t^2\|B\|^2$

So it has minimum at: $t = \frac{-2A.B}{2\|B\|^2} = \frac{-A.B}{\|B\|^2}$



Let P_o be a point and N be a non zero vector. We define the **plane** passing through P_o to perpendicular to N to be the collection of all points P such that the vector $\overrightarrow{\mathbf{p_o p}}$ is perpendicular N. According to our definition, if **P** is any point on the plane through P_0 and perpendicular to N, then $\mathbf{N} \cdot \overrightarrow{\mathbf{p_o p}} = 0$ or $N \cdot (p - p_0) = 0$

Activity 1.5.3: Starting from the equation $\mathbf{N} \cdot \overrightarrow{\mathbf{p_o p}} = 0$, show that equation of a plane through point $P_o = (x_o, y_o, z_o)$ perpendicular to N = (a, b, c) is

ax + by + cz = d where $\mathbf{d = ax_o + by_o + cz_o}$.

This equation can be written as $N.P = d$. The vector N is said to be **normal** to the plane.

Hence a plane is any set of the form **{P: N.P = d}.** Where N is a given non-zero vector and d is a given number.

Example 1.5.2: Find an Equation of the plane that contains point p_0 (-2, 4, 5) and that is normal to N(7, 0, -6).

Solution: Let P=(x,y,z) be any point on the plane then the equation of the plane given by

N.p=N.p_0=(7,0,6).(x,y,z)=(7,0,6)(-2,4,5)=7x+6z=16

Does this plane intersect the y-axis?

Remark:. Two planes in 3 spaces are said to be **parallel** if their normal vectors are parallel. They are said to be **perpendicular** if their normal vectors are perpendicular. The **angle** between two planes is defined to be the angle between their normal vectors.

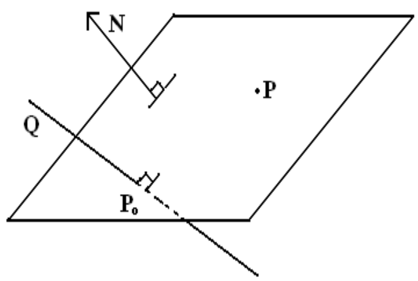
Activity 1.5.4:

1. A plane passes through (-1, 2, 3) and is perpendicular to the y-axis. What is the equation?
2. Consider the planes x + 2y - 3z = 2 and 15x - 9y – z = 2. Are they parallel or perpendicular planes? Or neither parallel nor perpendicular?

Exercise 1.5.2: Find the equation of the plane passing through the three points

$P_1 = (2,1,1)$, $P_2 = (3,-1,1)$, $P_3 = (4,1,-1)$.

Let Q be a point outside a plane normal to N. We define the distance from point Q to the plane as fo
Let P_o be the point of intersection of the line through Q, in the direction of N, and the plane through
The distance d from Q to the plane is the distance between Q and P_o.



Now we find a formula for this distance. Clearly $d = \left\|Proj_{\overrightarrow{QP}}\overrightarrow{QP}\right\| = \left\|\mathbf{Proj}_{\mathbf{N}}\overrightarrow{\mathbf{QP}}\right\|$

However, $\mathbf{Proj}_{\mathbf{N}}\overrightarrow{\mathbf{QP}} = \left(\frac{\mathbf{N}.\overrightarrow{\mathbf{QP}}}{\|\mathbf{N}\|^2}\right)\mathbf{N}$

Hence $\mathbf{d} = \left\|\left(\frac{\mathbf{N}.\overrightarrow{\mathbf{QP}}}{\|\mathbf{N}\|^2}\right)\mathbf{N}\right\| = \frac{\left|\mathbf{N}.\overrightarrow{\mathbf{QP}}\right|}{\|\mathbf{N}\|^2}\|\mathbf{N}\| = \frac{\left|\mathbf{N}.\overrightarrow{\mathbf{QP}}\right|}{\|\mathbf{N}\|}$

Therefore the **distance d** of a point Q from a plane through **P** which is normal to **N** is given by:

$$\mathbf{d} = \frac{\left|\mathbf{N}.\overrightarrow{\mathbf{QP}}\right|}{\|\mathbf{N}\|}$$

1.6 VECTOR SPACES

1.6.1 The axioms of a vector space

Definition 1.6.1: A set F having at least 2 elements is called a **field** if two operations called addition (+) and multiplication (·) are defined in F and satisfy the following two axioms:

a) If x is an element of K, then –x is also an element of K. Furthermore, if x ≠ 0, then x^{-1} is also an element of K.

b) 0 and 1 are elements of K.

Definition: The set of complex numbers is C = {(a, b) | a, b ∈ ℜ}. Define addition on C as (a, b) + (c, d) = (a + c, b + d) and multiplication on C as (a, b) · (c, d) =(ac − bd, bc + ad).

Remark: The order pair (a,b) refers a+bi

Example 1.6.1: The set of all complex numbers ℂ are fields.

Solution:

a). Let $u = a + bi$, and $v = c + di \in \mathbb{C}$

$$u + v = (a + bi) + (c + di) = (a + c) + (b + d)i$$

$$u + v \in \mathbb{C} \quad (\text{since } (a + c) \in \Re \text{ and } (b + d) \in \Re$$

ℂ is closed under addation.

Similarly, $uv = (a + bi)(c + di)$ = (ac-bd)+(ad+bc)i

$$uv \in \mathbb{C} \quad (\text{ since } (ac - bd) \in \Re \text{ and } (ad + bc) \in \Re$$

ℂ is closed under addation.

b). Let $u = a + bi$, and $-1 \in \Re$

(i). (-1) $u = (-1)(a + bi) = (-a - bi) \in \mathbb{C}$ ……. (Note: $-a \in \Re$ and $-b \in \Re$)

$$-u \in \mathbb{C} \quad \ldots (-u \text{ is an addative inverse of } u)$$

ii. suppose $u = (a + bi) \neq 0$

Then the multiplicative inverse of u is $u^{-1} = \frac{1}{u} = \frac{1}{a+bi}$

$$u^{-1} = \frac{a+bi}{a^2+b^2} = \frac{a}{a^2+b^2} + (\frac{b}{a^2+b^2})i$$

$$\therefore, \quad u^{-1} \in \mathbb{C}$$

C) $0 = (0,0) = 0 + 0i \in \mathbb{C} \quad (\because 0 \in \Re\ )$

$1 = (1,0) = 1 + 0i \in \mathbb{C}$

Hence, The set ℂ is is Field.

Activity 1.6.1: Are ℤ (The set of all integers) and Q (the set of all rational numbers fields?

Definition 1.6.2: A vector space V over a field K is a set of objects which can be added and can be multiplied by elements of K. It satisfies the following properties.

V_1) For any u, v $\in$ V and a $\in$ K, we have

$$u + v \in V \quad \text{and} \quad a\,u \in V$$

V_2) For any u, v, w $\in$ V,

$$(u + v) + w = u + (v + w)$$

V_3) There is an element of V, denoted by O (called the zero element), such that

$0 + u = u + 0 = u$ for all elements u of V.

V_4) For u $\in$ V, there exists –u $\in$ V such that

$$u + (-u) = 0$$

V_5) For u, v $\in$ V, we have

$$u + v = v + u$$

V_6) For u, v $\in$ V and a $\in$ k,

$$a(u + v) = au + av$$

V_7) For u $\in$ V and a, b $\in$ k, $(a + b)\,u = au + bu$ and $(ab)\,u = a(bu)$

V_8) For u $\in$ v,

$$1u = u$$

Activity 1.6.2: What is the name given for each of the above properties?

Other properties of a vector space can be deduced from the above eight properties. For example, the property $0u = O$ can be proved as :

$0u + u = 0u + 1.u$ (by V_8)

$= (0 + 1)\,u$ (by V_7)

= 1. u

= u

By adding –u to both sides of ou + u = u, we have 0u = O.

1.6.2 Examples of different models of a vector space

Example 1.6.2. (n-tuples space)

$(u_1, u_2, \dots u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$, Vector addition $k((u_1, u_2, \dots u_n) = (ku_1, ku_2, \dots, ku_n)$, scalar multiplication and 0=(0,0,0,…0) with these operations $\Re^n$ is a vector space over filed $\Re$.

The other 5 properties can be easily verified. Hence $\Re^2$ is a vector space over$\Re$.

Example 1.6.3.

Let V = $\Re^2$ and K = $\mathbb{C}$

For any u, v $\in \Re^2$, we have u + v $\in \Re^2$.

But for a $\in \mathbb{C}$, au is not always in $\Re^2$.

For example for a = 3i and $u = (1, -2), au = (3i, -2i) \notin \Re 2$.

Hence $\Re^2$ is not vector space over $\mathbb{C}$.

Thus when dealing with vector spaces, we shall always specify the field over which we take the vector space.

Example 1.6.4.

Let F be the set of all functions form $\Re$ to $\Re$, for any f and g in F, $f + g$ is a function from $\Re$ to $\Re$ defined by $(f + g)(x) = f(x) + g(x)$.

$$For\ a \in \Re, af = af(x)\ is\ in\ F.$$

The zero element O of F is the zero function f$(x) = 0\ for\ all\ x \in \Re$.

By verifying the other properties, we can see that F is a vector space over $\Re$.

Example 1.6.5.

$\mathbf{v_3} =$ the set containing of all polynomials of degree 3 or less in the set of real numbers together with standard polynomial addation and scalar multiplication.

Is v_3 a vector space.

Solution*: we need to examine whether conditions*

$Let \quad u = a_2x^2 + a_1x + a_0$

$v = b_2x^2 + b_1x + b_0$

$w = c_2x^2 + c_1x + c_0$ *and c and d be scalars. Then*

1). $U+V = (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0)$

$= (a_2+b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0) \in V_3$

$U+V \in V_3$

2). $u+v = (a_2x^2 + a_1x + a_0) + (b_2x^2 + b_1x + b_0)$

$= (a_2+b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$

$= (b_2+a_2)x^2 + (b_1+ a_1)x + (b_0+ a_0)$

$= (b_2x^2 + b_1x + b_0) + (a_2x^2 + a_1x + a_0) = v+u$

3). $U+(v+w) = (a_2x^2 + a_1x + a_0) + [(b_2x^2 + b_1x + b_0) + (c_2x^2 + c_1x + c_0)]$

$=(a_2+b_2+c_2)\, x^2 + (a_1 + b_1+c_1)x + (a_0 + b_0 + c_0)$

$=[(a_2+b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)] +(c_2x^2 + c_1x + c_0)$

$= (u+v)+w$

4). Let $0 = 0x^2 + 0\,x + 0.$ then

U+0 = $(a_2 + 0)x^2 + (a_1 + 0)x + (a_0 + 0) = (0 + a_2)x^2 + (0 + a_1)x + (0 + a_0)$

U+0=0+u= u

5). Let -u= $(-a_2)x^2 + (-a_1)x + (-a_0)$. Then

u+ - u = $[a_2 + (-a_2)]x^2 + [a_1 + (-(a_1)]x + [((a_0) + -a_0)] = 0x^2 + 0\,x + 0$=0

6). Cu = $(ca_2)x + (ca_1)x + (ca_0) \in v_3$

Since cu is a polynomial of degree 2 or less.

7). C(u+v) = $[c(a_2+b_2)]\,x^2 + [c(\,a_1 + b_1)]x + [c(a_0 + b_0)]$

$= (ca_2+cb_2)\,x^2 + (\,ca_1 + cb_1)x + (ca_0 + cb_0)$ = cu + cv

8). (c+d)u = $[(c+d)\,a_2]\,x^2 + [(c + d)a_1\,]x + [(c + d)a_0]$ = cu +dv

9). C(du) = $c[(da_2)\,x^2 + (da_1\,)x + (da_0)$

$= (cd)\,a_2x^2 + (cd)a_1\,x + (cd)a_0 = (cd)u$

10). $1u = 1.\,a_2x^2 + 1.\,a_1x + 1.\,a_0 = u$

Hence, the given space is vector space.

The algebraic properties of elements of an arbitrary vector space are very similar to those of elements of $\Re^2$, $\Re^3$, or $\Re^n$. Consequently, we call elements of a vector space as **vectors**

Activity 1.6.1: Which of the following are vector spaces?

a) C on $\Re^2$
b) C^n over $\mathbb{C}$
c) Q^n over Q
d) $\Re^n$ over $\mathbb{C}$

1.6.3 Subspaces, Linear Combinations and generators

Definition 1.6.3.1: Suppose V is a vector space over k and W is a subset of V. If, under the addition and scalar multiplication that is defined on V, W is also a vector space then we call W a **subspace** of V.

Using this definition and the axioms of a vector space, we can easily prove the following:

A subset W of a vector space V is called a **subspace** of V if:

i) **W** is closed under addition. That is, if $u, w \in W$, then $u + w \in W$

ii) **W** is closed under scalar multiplication. That is, if $u \in W$ and $a \in k$, then $au \in W$.

iii) **W** contains the additive identity 0.

Then as $W \subseteq V$, properties $V_1 - V_8$ are satisfied for the elements of W.

Hence W itself is a vector space over k. We call W a **subspace** of V.

Example Consider $H = \{(x, y): x, y \in \Re \text{ and } x + 4y = 0\}$.

H is a subset of the vector space $\Re^2$ over $\Re$. To show that H is a subspace of V, it is enough to show the above three properties hold in H.

Let $u = (x_1, y_1)$ and $w = (x_2, y_2)$ be in H. Then $x_1 + 4y_1 = 0$ and $x_2 + 4y_2 = 0$

$u + w = (x_1 + x_2, y_1 + y_2)$ and $(x_1 + x_2) + 4(y_1 + y_2) = x_1 + 4y + x_2 + 4y_2 = 0 + 0 = 0$

Which shows $\boldsymbol{u + w \in H}$.

For $\boldsymbol{a} \in \Re$, $\boldsymbol{au} = (\boldsymbol{ax_1, ax_2})$ and $\boldsymbol{(ax_1) + 4(ax_2) = a(x_1 + 4x_2)} = 0\boldsymbol{a} = 0$.

Hence $au \in H$. Now, the element O of $\Re^2$ is (0, 0). 0 + 4(0) = 0. Hence O = (0, 0) is in H.

$\therefore$ H is a subspace of $\Re^2$

Activity: Take any vector A in $\Re^3$. Let W be the set of all vectors B in $\Re^3$ where B.A = 0. Discuss whether W is a subspace of $\Re^3$ or not.

1.7 Linear dependence and independence of vectors

Definition 1.7.1: Let **V** be a vector space over k. Elements $v_1, v_2, \ldots, v_n$ of **V** are said to be **linearly independent** if and only if the following condition is satisfied:

whenever $a_1, a_2, \ldots, a_n$ are in k such that $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$, then $a_i = 0$ for all

$i = 1, 2, \ldots, n$.

If the above condition does not hold, the vectors are called **linearly dependent**. In other words $v_1, v_2, \ldots, v_n$ are linearly dependent if and only if there are numbers $a_1, a_2, \ldots, a_n$ where $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ for at least one non-zero a_i.

Example 1.7.1: Consider $v_1 = (1, -1, 1)$, $v_2 = (2, 0, -1)$ and $v_3 = (2, -2, 2)$

i) $a_1v_1 + a_2v_2 = a_1 (1, -1, 1) + a_2 (2, 0, -1) = (a_1 + 2a_2, -a_1, a_1 - a_2)$

$$a_1v_1 + a_2v_2 = 0 \Rightarrow a_1 + 2a_2 = 0, -a_1 = 0 \text{ and } a_1 - a_2 = 0$$

$$\Rightarrow a_1 = 0 \text{ and } a_2 = 0$$

Hence v_1 & v_2 are linearly independent.

ii) $a_1v_1 + a_2v_3 = a_1 (1, -1, 1) + a_2 (2, -2, 2)$

$$= (a_1 + 2a_2, -a_1 - 2a_2, a_1 + 2a_2)$$

$$a_1v_1 + a_2v_3 = 0 \Rightarrow a_1 + 2a_2 = 0, -a_1 - 2a_2 = 0 \text{ and } a_1 + 2a_2 = 0$$

$$\Rightarrow a_1 = -2a_2$$

Take $a_1 = 2$ and $a_2 = -1$, we get $2(1, -1, 1) + (-1)(2, -2, 2) = 0$.

As the constants are not all equal to zero, v_1 and v_3 are linearly dependent.

Activity 1.7.1: Show that v_1, v_2 and v_3 are also linearly dependent.

Remark: If vectors are linearly dependent, at least one of them can be written as a linear combination of the others.

Activity 1.7.2: Show that (1, 0, 0, …,0), (0, 1,0,…)…, (0,0,0, …, 1) are linearly independent vectors in $\Re^n$.

1.8 Bases and dimension of a vector space

Definition 1.8.1: If elements $e_1, e_2, \dots, e_n$ of a vector space **V** are linearly independent and generate **V**, then the set $B = \{e_1, e_2, \dots, e_n\}$ is called a **basis** of **V**. we shall also say that the elements $e_1, e_2, \dots, e_n$ **constitute** or **form** a basis of **V**.

Example 1.8.1:

1) Show that $e_1 = (0, -1)$ and $e_2 = (2, 1)$ form a basis of $\Re^2$.

Solution: we have to show that

i) e_1 and e_2 are linearly independent

ii) They generate $\Re^2$ i.e every element (x,y) of $\Re^2$ can be written as a linear combination of $e_1$and e_2.

i) $a_1 e_1 + a_2 e_2 = O \Rightarrow a_1(0, -1) + a_2(2,1) = (0, 0)$

$$\Rightarrow 2a_2 = 0 \text{ and } -a_1 + a_2 = 0$$

$$\Rightarrow a_2 = 0 \text{ and } a_1 = 0$$

$\therefore$ e_1 and e_2 are linearly independent

ii) $(x, y) = a_1e_2 + a_2 e_2 \Rightarrow (x, y) = (0, -a_1) + (2a_2, a_2)$

$$\Rightarrow x = 2a_2 \text{ and } y = -a_1 + a_2$$

$$\Rightarrow a_2 = \frac{x}{2} \text{ and } a_1 = a_2 - y \qquad \dots (*)$$

$$= \frac{x-2y}{2}$$

Therefore, given any (x, y), we can find a_1 and a_2 given by (*) and (x, y) can be written as a linear combination of e_1 and e_2 as

$$(x, y) = \left(\frac{x - 2y}{2}\right)(0, -1) + \left(\frac{x}{2}\right)(2, 1)$$

For example, $(4, 3) = \left(\frac{4-6}{2}\right)(0,-1) + \left(\frac{4}{2}\right)(2,1)$

Or $(4, 3) = -(0, -1) + 2(2, 1)$

Note that {(1, 0), (0, 1)} is also a basis of $\Re^2$. Hence a vector space can have two or more basis. Find other bases of $\Re^2$.

2) Show that $e_1 = (2, 1, 0)$ and $e_2 = (1, 1, 0)$ form a basis of $\Re^3$.

Solution: $e_1 = (2, 1, 0)$ and $e_2 = (1, 1, 0)$ are linearly independent but they do not generate $\Re^3$. There are no numbers a_1 and a_2 for which

$$(3, 4, 2) = a_1 (2, 1, 0) + a_2 (1, 1, 0).$$

Hence {(2, 1, 0), (1,1,0)} is not a basis of $\Re^3$.

The vectors $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$, $E_3 = (0, 0, 1)$ are linearly independent and every element (x, y, z) of $\Re^3$ can be written as

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z (0, 0, 1)$$

$$= xE_1 + yE_2 + zE_3$$

Hence $\{E_1, E_2, E_3\}$ is a basis of $\Re^3$.

Note that the set of elements $E_1 = (1, 0, 0,\ldots,0)$, $E_2 = (0, 1, 0, \ldots 0),\ldots,E_n = (0, 0, 0, \ldots,1)$ is a basis of $\Re^n$. It is called a **standard basis**.

Let $B = \{e_1 , e_2, \ldots, e_n\}$ be a basis of **V**. since B generates **V**, any u in **V** can be represented as $u = a_1e_1 + a_2 e_2 + \ldots + a_n e_n$. Since the e_i are linearly independent, such a representation is unique. We call $(a_1, a_2, \ldots, a_n)$ the **coordinate vector** of u with respect to the basis B, and we call a_i the **i – th coordinate.**

Example 1.8.2

1) In 1) of example 3.3.1 The coordinate vector of (4,3) with respect to the basis {(0, -1), (2,1)} is (-1, 2). But with respect to the standard basis it is (4, 3).

   Find coordinates of (4,3) in some other basis of $\Re^2$.

2) Consider the set **V** of all polynomial functions f: $\Re \rightarrow \Re$ which are of degree less than or equal to 2.

   Every element of **V** has the form $f(x) = bx^2 + cx + d$, where $b, c, d \in \Re$

   V is a vector space over $\Re$ (show).

   Clearly, $e_1 = x^2$, $e_2 = x$ and $e_3 = 1$ are in **V** and $a_1e_1 + a_2 e_2 + a_3e_3 = O$

   (0 is the zero function)

   $\Rightarrow a_1x^2 + a_2e_2 + a_3 e_3 = 0$ for all x

   $\Rightarrow a_1 = a_2 = a_3 = 0.$

   Which shows e_1, e_2 and e_3 are linearly independent

   $bx^2 + cx + d = a_1e_1 + a_2e_2 + a_3e_3$ for all x

   $\Rightarrow bx^2 + cx + d = a_1x^2 + a_2x + a_3$

   $\Rightarrow b = a_1$, $c = a_2$ and $d = a_3$

   Thus e_1, e_2 and e_3 generate **V**.

   $\therefore$ $\{x^2, x_1\ 1\}$ is a basis of **V** and the coordinate vector of an element

   $f(x) = bx^2 + cx + d$ is (b, c, d)

   The coordinate vector of $x^2 - 3x + 5$ is (1, -3, 5)

Activity 1.8.1: Show that the polynomials

$E_1 = (x - 1)^2 = x^2 - 2x + 1$

$E_2 = x - 1$

and $E_3 = 1$

form a basis of a vector space **V** defined in **2)** of **example 1.8.2**. What is the coordinate of **f(x) = $2x^2 - 5x + 6$** with respect to the basis **{E_1, E_2, E_3}**?

E = {(1, 0, 0), (0,1,0), (0,0,1)} and B = {(-1,1,0), (-2, 0, 2), (1, 1, 1)} are bases of $\Re^3$ and each has three elements. Can you find a basis of $\Re^3$ having two elements? four elements?

The main result of this section is that any two bases of a vector space have the same number of elements. To prove this, we use the following theorem.

Theorem 1.8.1: Let V be a vector space over the field K. Let {v_1, v_2,…,v_n} be a basis of V. If ***w_1, w_2,…,w_m*** are elements of V, where m > n, then w_1, w_2, …, w_m are linearly dependent.

Proof (reading assignment)

Theorem 1.8.2: Let V be a vector space and suppose that one basis B has n elements, and another basis W has m elements. Them m = n.

Proof: As B is a basis, m > n is impossible. Otherwise by theorem 3.4.1, W will be a linearly dependent set. Which contradicts the fact that W is a basis. Similarly, as W is a basis, n > m is also impossible. Hence n = m.

Definition 1.8.2: Let V be a vector space having a basis consisting of n elements. We shall say that n is the **dimension** of V. It is denoted by **dim V**.

Remarks: 1. If V = {0}, then V doesn't have a basis, and we shall say that dim v is zero.

2. The zero vector space or a vector space which has a basis consisting of a finite number of elements, is called **finite dimensional**. Other vector

spaces are called **infinite dimensional**.

Example 1.8.3:

1) $\Re^3$ over $\Re$ has dimension 3. In general $\Re^n$ over $\Re$ has dimension n.

2) $\Re$ over $\Re$ has dimension 1. In fact, {1} is a basis of $\Re$, because

$\boldsymbol{a}.1 = 0 \Rightarrow \boldsymbol{a} = 0$ and

any number $x \in \Re$ has a unique expression $x = x.1$.

Definition 1.8.3: The set of elements $\{v_1, v_2, \ldots, v_n\}$ of a vector space V is said to be a **maximal set of linearly independent elements** if $v_1, v_2, \ldots, v_n$ are linearly independent and if given any element w of V, the elements $w, v_1, v_2, \ldots, v_n$ are linearly dependent.

Example 1.8.4: In $\Re^3$ {(1, 0, 0), (0, 1, 1), (0, 2, 1)} is a maximal set of linearly independent elements.

We now give criteria which allow us to tell when elements of a vector space constitute a basis.

Theorem 1.8.3: Let **V** be a vector space and $\{v_1, v_2, \ldots, v_n\}$ be a maximal set of linearly independent elements of **V**. Then $\{v_1, v_2, \ldots, v_n\}$ is a basis of **V**.

Proof: It suffices to show that $v_1, v_2, \ldots, v_n$ generate **V**. (Why?)

Let $\boldsymbol{w} \in \boldsymbol{v}$.

Then $w, v_1, v_2, \ldots, v_n$ are linearly dependent (why?).

Hence there exist numbers $a_o, a_1, a_2, \ldots, a_n$ not all 0 such that

$a_o w + a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = O$

In particular $\boldsymbol{a_o} \neq 0$ (why?

Therefore, by solving for w,

$$w = \frac{-a_1}{a_o}V_1 - \frac{a_2}{a_o}V_2 \;...\; \frac{-a_n}{a_o}V_n$$

This proves that w is a linear combination of $v_1, v_2, \ldots, v_n$.

Theorem 1.8.4: Let dim **V** = n, and let $v_1, v_2, \ldots, v_n$ be linearly independent elements of v. Then

$\{v_1, v_2, \ldots, v_n\}$ is a basis of v.

Proof: According to theorem 3.4.1, $\{v_1, v_2, \ldots, v_n\}$ is a maximum set of linearly independent elements of **V**.

Hence it is a basis by theorem 2.5.3

Corollary 1.8.1: Let W be a subspace of V. If dim W = dimV, then V = W

Proof: Exercise

1.9 Direct sum and direct product of subspaces

Let **V** be a vector space over the field K. Let **U, W** be subspaces of **V**. We define the **sum** of **U** and **W** to be the subset of **V** consisting of all sums **u + w** with $u \in U$ and $w \in W$. We denote this sum by **U +W** and it is a subspace of **V**. Indeed, if $u_1, u_2 \in U$ and $w_1, w_2 \in W$ then

$$(u_1 + w_1) + (u_2 + w_2) = u_1 + u_2 + w_1 + w_2 \in U + W$$

If $c \in K$, then

$$c(u_1 + w_1) = cu_1 + cw_1 \in U + W$$

Finally, $0 + 0 \in U + W$. This proves that U + W is a subspace.

Definition 1.9.1: A vector space V is a **direct sum** of U and W if for every element v in V there exist unique elements $u \in U$ and $w \in W$ such that $v = u + w$.

Theorem 1.9.1: Let V be a vector space over the field K, and let U, W be subspaces. If U + W = V, and if $U \cap W = \{0\}$, then V is the direct sum of U and W.

Proof: Exercise

Note: When V is the direct sum of subspaces U, W we write:

$$V = U \oplus W$$

Theorem 1.9.2: Let V be a finite dimensional vector space over the field K. Let W be a subspace. Then there exists a subspace U such that V is the direct sum of W and U.

Proof: Exercise

Theorem 1.9.3: If **V** is a finite dimensional vector space over the field K, and is the direct sum of subspaces U, W then

dim V = dim U + dim W

Proof: Exercise

Remark: We can also define V as a direct sum of more than two subspaces. Let W_1, W_2, …., W_r be subspaces of V. We shall say that V is their **direct sum** if every element of can be expressed in a unique way as a sum

$$v = w_1 + w_2 + ... + w_r$$

With w_i in W_i.

Suppose now that U, W are arbitrarily vector spaces over the field K(i.e. not necessarily subspaces of some vector space). We let UXW be the set of all pairs (u, w) whose first component is an element u of U and whose second component is an element w of W. We define the addition of such pairs component wise, namely, if $(u_1, w_1) \in UXW$ and $(u_2, w_2) \in UXW$ we define

$$(u_1, w_1) + (u_2, w_2) = u_1 + u_2, w_1 + w_2)$$

If $c \in K$, we define the product $c(u_1, w_1)$ by

$$c(u_1, w_1) = (cu_1, cw_1)$$

It is then immediately verified that UXW is a vector space, called the **direct product** of U and W.

Note: If n is a positive integer, written as a sum of two positive integers, $n = r + s$, then we see that K^n is the direct product $K^r X K^s$ and $\dim(UXW) = \dim U + \dim W$.

Example 1.9.1: Let, $V = R^3$, $U = \{(0,0,x_3), x_3 \in \Re\}$, and

$W = \{(x_1, x_2, 0), x_1, x_2 \in \Re\}$. Show that V is the direct sum of W and U.

Solution: Since V, U and W are vector spaces, and in addition to that U and W are subspaces of V. The sum of U and W is:

U + W = $\{(x_1, x_2, x_3), x_1, x_2, x_3 \in \Re\} = R^3 = V$

Thus; V = U + W

The intersection of U and W is: $U \cap W = \{0\}$

Therefore, V is the direct sum of W and U.

Activity 1.9.1: 1. Let, $V = R^3, U = \{(x_1, 0, x_3), x_1, x_3 \in \Re\}$, and $W = \{(0, x_2, 0), x_2 \in \Re\}$.

Show that V is the direct sum of W and U.

2. Let, $V = R^3, U = \{(x_1, x_2, 0), x_1, x_2 \in \Re\}$, and $W = \{(0,0,x_3), x_3 \in \Re\}$.

Show that V is the direct sum of W and U.

Exercise:

1. Let, $V = R^2, U = \left\{(\frac{a+b}{2}, \frac{a+b}{2}), a, b \in \Re\right\}$, and $W = \left\{(\frac{a-b}{2}, \frac{b-a}{2}), a, b \in \Re\right\}$.

Show whether V is the direct sum of W and U or not.

2. Let, $V = R^3, U = \{(x_1, x_2, 0), x_1, x_2 \in \Re\}$, and $W = \{(0, x_2, x_3), x_2, x_3 \in \Re\}$ Show whether V is the direct sum of W and U or not.

Exercise 2.1

1. Let k be the set of all numbers which can be written in the form $\mathbf{a}+\mathbf{b}\sqrt{2}$, where a, b are rational numbers. Show that k is a field.
2. Show that the following sets form subspaces
   a. The set of all (x, y) in $\Re^2$ such that x = y
   b. The set of all (x, y) in $\Re^2$ such that x – y = 0
   c. The set of all (x, y, z) in $\Re^3$ such that x + y = 3z
   d. The set of all (x, y, z) in $\Re^3$ such that x = y and z = 2y
3. If U and W are subspaces of a vector space V, show that $\mathbf{U}\cap\mathbf{W}$ and $\mathbf{U}+\mathbf{W}$ are subspaces.
4. Decide whether the following vectors are linearly independent or not (on $\Re$)

   a) (π, 0) and (0, 1)

   b) (-1, 1, 0) and (0, 1, 2)

   c) (0, 1, 1), (0, 2, 1), and (1, 5, 3)

5. Find the coordinates of X with respect to the vectors A, B and C
   a. X = (1, 0, 0), A = (1, 1, 1), B = (-1, 1, 0), C = (1, 0, -1)
   b. X = (1, 1, 1) , A = (0, 1, -1), B = (1, 1, 0), C = (1, 0, 2)
6. Prove: The vectors (a, b) and (c, d) in the plane are linearly dependent if and only if ad – bc = 0
7. Find a basis and the dimension of the subspace of $\Re^4$ generated by
   {(1, -4, -2, 1), (1, -3, -1, 2), (3, -8, -2, 7)}.

8. Let W be the space generated by the polynomials $x^3 + 3x^2 – x + 4$, and
   $2x^3 + x^2 – 7x – 7$. Find a basis and the dimension of W.

9. Let V = {(a, b, c, d) $\in \Re^4$: b – 2c + d = 0}

   W = {(a, b, c, d) $\in \Re^4$: a = d, b = 2c}

   Find a basis and dimension of

   a) V   b) W   c) $\mathbf{V}\cap\mathbf{W}$

10. What is the dimension of the space of 2 x 2 matrices? Give a basis for this space. Answer the same question for the space of n x m matrices.

CHAPTER TWO

2 MATRICES AND DETERMINANTS

2.1 Definition of matrix and basic operations

The concept of matrices has had its origin in various types of linear problems, the most important of which concerns the nature of solutions of any given system of linear equations. Matrices are also useful in organizing and manipulating large amounts of data. Today, the subject of matrices is one of the most important and powerful tools in mathematics which has found applications to a very large number of disciplines such as engineering, business and economics, statistics etc.

Definition and Examples of Matrices

Definition: A matrix is a rectangular table of form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A matrix is said to be of dimension $m \times n$ when it has m rows and n columns. This method of describing the size of a matrix is necessary in order to avoid all confusion between two matrices containing the same amount of entries. For example, a matrix of dimension 3×4 has 3 rows and 4 columns. It would be distinct from a matrix 4×3, that has 4 rows and 3 columns, even if it also has 12 entries. The elements are matrix entries a_{ij}, that are identified by their position. The element a_{32} would be the entry located on the third row and the second column of matrix A. This notation is essential in order to distinguish the elements of the matrix. The element a_{23}, distinct from a_{32}, is situated on the second row and the third column of the matrix A.

Remark: By the size of a matrix or the dimension of a matrix we mean the order of the matrix.

Example: Let $A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution: Since A has 2 rows and 3 columns, we say A has order 2×3, where the number of rows is specified first. The element 6 is in the position a_{23} (read a two three) because it is in row 2 and column 3.

Example: What is the value of a_{23} and a_{32} in $A = \begin{bmatrix} -1 & 4 & 7 \\ 2 & 3 & 1 \\ 5 & 7 & 8 \end{bmatrix}$?

Solution: a_{23}, the element in the second row and third column, is 1 and a_{32}, the element in the third row and second column, is 7. What is the size of this matrix?

Activity: 1. Suppose A is a 5×7 matrix, then

a) A has 7 rows. (True/False)

b) a_{ij} is an element of A for $i = 6$ and $j = 4$.(True/False)

c) For what values of i and j, a_{ij} is an element of A?

2. Suppose A $= \begin{bmatrix} 4 & -7 & 5 \\ 8 & 1 & 6 \end{bmatrix}$ and B $= \begin{bmatrix} 4 & 8 \\ -7 & 1 \\ 5 & 6 \end{bmatrix}$

a) What is the order of A and B?

b) Find a_{22}, a_{13}, b_{13} and b_{31}.

It is customary to abbreviate the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

by the symbol $(a_{ij})_{m\times n}$ or more simply (a_{ij}). This notation merely indicates what type of symbols we are using to denote the general entry.

Example: Form a 4 by 5 matrix, B, such that $b_{ij} = i + j$.

Solution: Since the number of rows is specified first, this matrix has four rows and five columns.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}.$$

Activity: Form a 4 by 3 matrix, B, such that

a) $b_{ij} = i \times j$

b) $b_{ij} = (-1)^{i+j}$

Definition: Two matrices A and B are said to be **equal**, writtenA = B, if they are of the same order and if all corresponding entries are equal.

Example: Given the matrix equation$\begin{bmatrix} x+y & 6 \\ x-y & 8 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 3 & 8 \end{bmatrix}$. Find x and$y$.

Solution: By the definition of equality of matrices, $\begin{matrix} x+y=1 \\ x-y=3 \end{matrix}$ solving gives $x = 2$ and $y = -1$.

Activity: Find the values of x, y, z and w which satisfy the matrix equation

$$\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2w \end{bmatrix}$$

Types of matrices

Row Matrix: A matrix that has exactly one row. For example, the matrix $A = [5 \quad 2 \quad -1 \quad 4]$ is a row matrix of order 1×4.

Column Matrix: A matrix consisting of a single column. For example, the matrix $B = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$ is a 3×1 column matrix.

Zero or Null Matrix: A matrix whose entries are all 0 is called a zero or null matrix. It is usually denoted by $0_{m\times n}$ or more simply by 0. For example, $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a 2×4 zero matrix.

Square Matrix: An $m \times n$ matrix is said to be a square matrix of order n if $m = n$. That is, if it has the same number of columns as rows.

For example, $\begin{bmatrix} -3 & 4 & 6 \\ 2 & 1 & 3 \\ 5 & 2 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}$ are square matrices of order 3 and 2 respectively.

In a square matrix $A = (a_{ij})$ of order n, the entries a_{11}, a_{22}, $\cdots$, a_{nn} which lie on the diagonal extending from the left upper corner to the lower right corner are called the main diagonal entries, or more simply the main diagonal. Thus, in the matrix $C = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 6 & 0 \\ 5 & 1 & 8 \end{bmatrix}$ the entries $C_{11} = 3$, $C_{22} = 6$ and $C_{33} = 8$ constitute the main diagonal.

Note: The sum of the entries on the main diagonal of a square matrix A of order n is called the trace of A. That is, Trace of $A = \sum_{i=1}^{n} a_{ii}$.

Activity: Find the trace of C in the above example.

Triangular Matrix: A square matrix is said to be an upper (lower) triangular matrix if all entries below (above) the main diagonal are zeros.

For example, $\begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ and $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 6 & 1 & 2 & 0 \\ -2 & -4 & 8 & 6 \end{bmatrix}$ are upper and lower triangular matrices, respectively.

Diagonal Matrix: A square matrix is said to be diagonal if each of the entries not falling on the main diagonal is zero. Thus a square matrix $A = (a_{ij})$ is diagonal if $a_{ij} = 0$ for $i \neq j$.

Activity: What about for $i = j$?

For example, $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

Notation: A diagonal matrix A of order n with diagonal elements $a_{11}, a_{22}, \cdots , a_{nn}$ is denoted by $\mathrm{A} = \mathrm{diag}\,(a_{11}, a_{22}, \cdots , a_{nn})$.

Scalar matrix: A diagonal matrix whose all the diagonal elements are equal is called a scalar matrix.

For example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix.

Note: Let $\mathrm{A} = (a_{ij})$ be a square matrix. A is a scalar matrix if and only if $a_{ij} = \begin{cases} k, & if \quad i = j \\ 0, & if \quad i \neq j \end{cases}$.

Identity Matrix or Unit Matrix: A square matrix is said to be identity matrix or unit matrix if all its main diagonal entries are 1's and all other entries are 0's. In other words, a diagonal matrix whose all main diagonal elements are equal to 1 is called an identity or unit matrix. An identity matrix of order n is denoted by I_n or more simply by I.

For example, $\mathrm{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is identity matrix of order 3. $\mathrm{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is identity matrix of order 2.

Note: Let $\mathrm{A} = (a_{ij})$ be a square matrix. A is an identity matrix if and only if $a_{ij} = \begin{cases} 1, & if \quad i = j \\ 0, & if \quad i \neq j \end{cases}$.

Algebra of matrices

Activity: $\begin{bmatrix} 30 & 18 \\ 20 & 12 \\ 16 & 10 \end{bmatrix} + \begin{bmatrix} 36 & 20 \\ 24 & 18 \\ 20 & 12 \end{bmatrix} = \begin{bmatrix} 66 & ? \\ ? & 30 \\ ? & ? \end{bmatrix}$. Can you guess what number should appear in the entries marked by question mark?

Addition of matrices

Let A and B be two matrices of the same order. Then the addition of A and B, denoted by A + B, is the matrix obtained by adding corresponding entries of A and B. Thus, if

$\mathrm{A} = (a_{ij})_{m\times n}$ and $\mathrm{B} = (b_{ij})_{m\times n}$ then $\mathrm{A} + \mathrm{B} = (a_{ij} + b_{ij})_{m\times n}$.

Remark: Notice that we can add two matrices if and only if they are of the same order. If they are, we say they are conformable for addition. Also, the order of the sum of two matrices is same as that of the two original matrices.

Activity: Given the matrices A, B, and C below

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \quad .$$

Find, if possible. a) $A + B$ b) $B + C$

If A is any matrix, the negative of A, denoted by $-A$, is the matrix obtained by replacing each entry in A by its negative. For example, if

$A = \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ -6 & 0 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -5 & -4 \\ 6 & 0 \end{bmatrix}$.

Properties of Addition of Matrices

1. *Matrix addition is commutative*. That is, if A and B are two matrices of the same order, then $A + B = B + A$.
2. *Matrix addition is associative*. That is, if A, B and C are three matrices of the same order, then $(A + B) + C = A + (B + C)$.
3. *Existence of additive identity*. That is, if 0 is the zero matrix of the same order as that of the matrix A, then $A + 0 = A = 0 + A$.
4. *Existence of additive inverse*. That is, if A is any matrix, then $A + (-A) = 0 = (-A) + A$.

Note: The zero matrices play the same role in matrix addition as the number zero does in addition of numbers.

Subtraction of Matrices

Let A and B be two matrices of the same order. Then by $A - B$, we mean $A + (-B)$. In other words, to find $A - B$ we subtract each entry of B from the corresponding entry of A.

Example: Let $A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \\ 5 & -7 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 5 & -2 \\ 6 & 1 \end{bmatrix}$. Then $A - B = \begin{bmatrix} 4-0 & -1-2 \\ 2-5 & 3-(-2) \\ 5-6 & -7-1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ -1 & -8 \end{bmatrix}$.

Multiplication of a Matrix by a Scalar

Let A be an $m \times n$ matrix and k be a real number (called a scalar). Then the multiplication of A by k, denoted by kA, is the $m \times n$ matrix obtained by multiplying each entry of A by k. This operation is called scalar multiplication.

Example: If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$. Find 2A + 3B.

Solution: $2A = 2\begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix}$ and $3B = 3\begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix}$

$$2A + 3B = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix} = \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$$

Example: Express the matrix equation $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} - y\begin{bmatrix} 3 \\ 5 \end{bmatrix} = 2\begin{bmatrix} 8 \\ 11 \end{bmatrix}$ as a system of equations and solve.

Solution: The given matrix equation gives

$$\begin{bmatrix} 2x \\ x \end{bmatrix} - \begin{bmatrix} 3y \\ 5y \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x - 3y \\ x - 5y \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

By equality of matrices we have

$$2x - 3y = 16$$
$$x - 5y = 22$$

Solving gives $x = 2$, $y = -4$.

Properties of scalar multiplications

1. If A and B are two matrices of the same order and if k is a scalar, then $k(A + B) = kA + kB$.
2. If k_1 and k_2 are two scalars and if A is a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.
3. If k_1 and k_2 are two scalars and if A is a matrix, then $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$.

Product of Matrices and some algebraic properties

While the operations of matrix addition and scalar multiplication are fairly straightforward, the product AB of matrices A and B can be defined under the condition that the number of columns of A must be equal to the number of rows of B. If the number of columns in the matrix A equals the number of rows in the matrix B, we say that the matrices are conformable for the product AB. Because of wide use of matrix multiplication in application problems, it is important that we learn it well. Therefore, we will try to learn the process in a step by step manner. We first begin by finding a product of a row matrix and a column matrix.

Example: Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, find the product AB.

Solution: The product is a 1×1 matrix whose entry is obtained by multiplying the corresponding entries and then forming the sum.

$$AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [(2a + 3b + 4c)]$$

Note that AB is a 1×1 matrix, and its only entry is $2a + 3b + 4c$.

Example: Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$, find the product AB.

Solution: $AB = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = [10 + 18 + 28] = [56]$

Note: In order for a product of a row matrix and a column matrix to exist, the number of entries in the row matrix must be the same as the number of entries in the column matrix.

Example: Here is an application: Suppose you sell 3 T-shirts at $10 each, 4 hats at $15 each, and 1 pair of shorts at $20. Then your total revenue is

$$\underbrace{\begin{pmatrix} 10 & 15 & 20 \end{pmatrix}}_{\text{Price}} \underbrace{\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}_{\text{Quantity}} = \underbrace{\begin{pmatrix} (10 \times 3) & (15 \times 4) & (20 \times 1) \end{pmatrix}}_{\text{Revenue}} = (110).$$

Example: Given $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 6 & 4 \\ 7 & 5 \end{bmatrix}$, find the product AB.

Solution: We already know how to multiply a row matrix by a column matrix. To find the product AB, in this example, we will be multiplying the row matrix A to both the first and second columns of matrix B, resulting in a 1×2 matrix.

$$AB = \begin{bmatrix} 2 \times 5 + 3 \times 6 + 4 \times 7 & 2 \times 3 + 3 \times 4 + 4 \times 5 \end{bmatrix} = \begin{bmatrix} 56 & 38 \end{bmatrix}$$

We have just multiplied a 1×3 matrix by a matrix whose order is 3×2. So unlike addition and subtraction, it is possible to multiply two matrices with different dimensions as long as the number of entries in the rows of the first matrix is the same as the number of entries in columns of the second matrix.

Activity: Given the matrices E, F, G and H, below

$E = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 3 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $G = [4 \quad 1]$ and $H = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

Find, if possible. a) GH b) FH c) EF d) FE

We summarizes matrix multiplication as follows: In order for product AB to exist, the number of columns of A must equal to the number of rows of B. If matrix A is of dimension $m \times n$ and B of dimension $n \times p$, the product AB will have the dimension $m \times p$. Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{jk})$ be an $n \times p$ matrix. Then the product AB is the $m \times p$ matrix defined by $AB = (c_{ik})$, where

$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$, $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

Thus, the product AB is the $m \times p$ matrix, where each entry c_{ik} of AB is obtained by multiplying corresponding entries of the i^{th} row of A by those of the k^{th} column of B and then finding the sum of the results.

Remark: The definition refers to the product AB, in that order, A is the left factor called pre factor and B is the right factor called post factor.

Example: Find the product AB if $A = \begin{bmatrix} 1 & -4 \\ 5 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 1 & 6 \\ 2 & 7 & 3 & 8 \end{bmatrix}$.

Solution: Since the number of columns of A is equal to the number of rows of B, the product $AB = C$ is defined. Since A is 3×2 and B is 2×4, the product AB will be 3×4

$$AB = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

The entry C_{11} is obtained by summing the products of each entry in row 1 of A by the corresponding entry in column 1 of B, that is. $C_{11} = (1)(-2) + (-4)(2) = -10$. Similarly, for C_{21} we use the entries in row 2 of A and those in column 1 of B, that is $C_{21} = (5)(-2) + (3)(2) = -4$.

Also, $C_{12} = (1)(4) + (-4)(7) = -24$

$C_{13} = (1)(1) + (-4)(3) = -11$

$C_{14} = (1)(6) + (-4)(8) = -26$

$C_{22} = (5)(4) + (3)(7) = 41$

$C_{23} = (5)(1) + (3)(3) = 14$

$C_{24} = (5)(6) + (3)(8) = 54$

$C_{31} = (0)(-2) + (2)(2) = 4$

$C_{32} = (0)(4) + (2)(7) = 14$

$C_{33} = (0)(1) + (2)(3) = 6$

$C_{34} = (0)(6) + (2)(8) = 16$

Thus $AB = \begin{bmatrix} -10 & -24 & -11 & -26 \\ -4 & 41 & 14 & 54 \\ 4 & 14 & 6 & 16 \end{bmatrix}$

Observe that the product BA is not defined since the number of columns of B is not equal to the number of rows of A. This shows that matrix multiplication is not commutative. That is, for any two matrices A and B, it is usually the case that $AB \neq BA$ (even if both products are defined).

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Thus, $AB \neq BA$.

Activity

1. Which of the following are defined?

   i) $(5 \quad 2 \quad 10)\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

   ii) $\begin{pmatrix} a \\ b \end{pmatrix}(c \quad d)$

   iii) $(a \quad b)(c \quad d)$

   iv)

Note: 1. $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$.

2. $AB = AC$ does not necessarily imply $B = C$.
3. If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$, find a and b such that $AB = BA$.

Example

1) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2) Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$, then

   $AB = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix} = AC$. But $B \neq C$.

Properties of Matrix multiplication

If A, B and C are any matrices, and if I is an identity matrix, then the following hold, whenever the dimensions of the matrices are such that the products are defined.

$A(BC) = (AB)C$	Associative Law
$A \cdot I = I \cdot A = A$	Multiplicative Identity Law
(The order of I and A is the same)	
$A(B + C) = AB + AC$	Left Distributive Law
$(A + B)C = AC + BC$	Right Distributive Law
$A \cdot 0 = 0 \cdot A = 0$	Multiplication by Zero

Remark: For real numbers, a multiplied by itself n times can be written as a^n. Similarly, a square matrix A multiplied by itself n times can be written as A^n. Therefore, A^2 means AA, A^3 means AAA and so on.

Exercise

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{bmatrix}$, then find each of the following

   (i) $A + B$ (iii) $A + B - C$ (v) $2A - C$

   (ii) $2B - 3C$ (iv) $A - 2B + 3C$

2. Let $A = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 \\ 7 & 2 \end{bmatrix}$, then find the following:

   (i) AB (ii) BC (iii) (AB)C (iv) A(BC)

3. If $A = \begin{bmatrix} 4 & -1 & -4 \\ 4 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$ compute A^2. Is it equal to I_3, where I_3 is the identity matrix of order 3?

2.2 Transpose of a matrix

Definition: Let A be an $m \times n$ matrix. The transpose of A, denoted by A' or A^t, is the $n \times m$ matrix obtained from A by interchanging the rows and columns of A. Thus the first row of A is the first column of A^t, the second row of A is the second column of A^t and so on.

Example: If $A = \begin{pmatrix} 2 & -4 & 6 \\ 3 & 1 & 4 \end{pmatrix}$, then $A^t = \begin{pmatrix} 2 & 3 \\ -4 & 1 \\ 6 & 4 \end{pmatrix}$

Activity: Find a 3×3 matrix A for which $A = A^t$.

Properties of matrix transpose

a) If A and B have the same order, $(A \pm B)^t = A^t \pm B^t$.

b) For a scalar k, $(kA)^t = kA^t$.

c) If A is $m \times n$ and B is $n \times p$, then $(AB)^t = B^t A^t$.

d) $(A^t)^t = A$.

Definition: A square matrix A is said to be orthogonal if $AA^t = A^t A = I$.

Example: $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is orthogonal.

Definition: A square matrix $A = (a_{ij})$ is said to be symmetric if $A^t = A$, or equivalently, if $a_{ij} = a_{ji}$ for each i and j.

Example: $A = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{pmatrix}$ is symmetric.

Activity

1. For $A = \begin{pmatrix} a & 3 & 4 & 8 \\ b & c & -3 & 9 \\ d & e & f & 10 \\ g & h & i & j \end{pmatrix}$ is to be a symmetric matrix, what numbers should the letters a to j represent?

2. a) Does a symmetric matrix have to be square?

c) Are all square matrices symmetric?

Definition: A square matrix $A = (a_{ij})$ is said to be skew-symmetric if $A^t = -A$, or equivalently, if $a_{ij} = -a_{ji}$ for each i and j.

Remark: $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$ or $a_{ii} = 0$. Hence elements of main diagonal of a skew-symmetric matrix are all zero.

Example: For $A = \begin{pmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 0 & -5 & -7 \\ 5 & 0 & -3 \\ 7 & 3 & 0 \end{pmatrix} = -A$. So A is skew-symmetric.

Properties of symmetric and skew-symmetric matrices

1. For any square matrix A, $A + A^t$ is symmetric and $A - A^t$ is skew- symmetric. That is,

   $(A + A^t)^t = A^t + (A^t)^t = A^t + A = A + A^t$

   $(A - A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$

2. If A and B are two symmetric (or skew symmetric) matrices of the same order, then so is $A + B$. That is,

   (i) Suppose A and B are symmetric

$(A + B)^t = A^t + B^t = A + B$

(ii) Suppose A and B are skew symmetric

$(A + B)^t = A^t + B^t = -A - B = -(A + B)$

3. If A is symmetric or skew symmetric, then so is kA. That is,

(i) Suppose A is symmetric

$(kA)^t = kA^t = kA$

(ii) Suppose A is skew symmetric

$(kA)^t = kA^t = -kA$

4. Let A and B be symmetric matrices of the same order. Then the product AB is symmetric if and only if AB = BA. That is,

($\Longrightarrow$) AB is symmetric $\Rightarrow AB = (AB)^t = B^t A^t = BA$ ($\because$ A and B be symmetric)

$\therefore\ AB = BA$

($\Longleftarrow$) Suppose AB = BA. Then $AB = BA = B^t A^t = (AB)^t$

$\therefore$ AB is symmetric

Exercise

1. Form a 4 by 5 matrix, **B**, such that $\boldsymbol{b_{ij} = i * j}$, where $*$ represents **multiplication**.

   a) What is $\mathbf{B^t}$? b) Is **B** symmetric? Why or why not?

2. Given $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 4 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 3 \\ 5 & 1 & 7 \\ 2 & 3 & 8 \end{bmatrix}$. Verify that

   i) $(A \pm B)^t = A^t \pm B^t$ ii) $(AB)^t = B^t A^t$ iii) $(2A)^t = 2A^t$

3. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then show that $A^t A$ is symmetric.

2.3 Elementary operations and its properties

Elementary row operations

1. (Replacement) Replace one row (say R_i) by the sum of itself and a multiple of another row

   (say R_j). This is abbreviated as $R_i \to kR_j + R_i$.

2. (Interchange) Interchange two rows (say R_i and R_j). This is abbreviated as $R_i \leftrightarrow R_j$.

3. (Scaling) Multiply all entries in a row (say R_i) by a nonzero constant (scalar) k. This is abbreviated as $R_i \to kR_i$.

For elementary column operations "row" by "column" in (1), (2) and (3) above.

We say that two matrices are row equivalent if one is obtained from the other by a finite sequence of elementary row operations.

It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a non-zero constant C, then multiplying the new row by $\frac{1}{C}$ produces the original row. Finally, consider a replacement operation involving two rows, say rows i and j, and suppose c times row i is added to row j to produce a new row j. To "reverse" this operation, add $-C$ times row i to the new row j and obtain the original row j.

Example: Find the elementary row operation that transforms the first matrix in to the second, and then find the reverse row operation that transforms the second matrix in to the first.

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & -4 \\ 0 & -3 & 4 \end{bmatrix} \xleftarrow{2R_2 \leftarrow R_2} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ 0 & -3 & 4 \end{bmatrix}$

Activity: Find the elementary row operation that transforms the first matrix in to the second, and then find the reverse row operation that transforms the second matrix in to the first.

a) $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 1 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -2 \\ 0 & 5 & -3 \\ 2 & 1 & 8 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 2 & -5 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 3 & -5 \end{bmatrix}$

In the definition that follows, a non-zero row (or column) in a matrix means a row (or column) that contains at least one non-zero entry; a leading entry of a row refers to the left most non-zero entry (in a non zero row).

Definition: A matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All non-zero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

If a matrix in echelon form satisfies the following additional condition, then it is in reduced echelon form (or row reduced echelon form)

4. The leading entry in each non-zero row is 1
5. Each leading 1 is the only non-zero entry in its column.

Example: The following matrices are in row echelon form; in fact the second matrix is in row reduced echelon form

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Definition: (i) A matrix which is in row echelon form is called an echelon matrix.

(ii) A matrix which is in row reduced echelon form is called a reduced echelon matrix.

Note: 1) Each matrix is row equivalent to one and only one row reduced echelon matrix.

2) But a matrix can be row equivalent to more than one echelon matrices.

If matrix A is row equivalent to an echelon matrix U, we call U an echelon form of A. If U is in reduced echelon form, we call U the reduced echelon form of A.

Activity: Determine which of the following matrices are in row reduced echelon form and which others are in row echelon form (but not in reduced echelon form)

i) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ii) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

iii) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix}$

v) $\begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

2.4 Determinants of a matrix and its properties

Definition of a Determinant

If A is a square matrix, then the determinant function associates with A exactly one numerical value called the **determinant of A,** that gives us valuable information about the matrix. By denoting the determinant of A by $|\boldsymbol{A}|$ or **det A** we can think of the *determinant function* as correspondence:

$$\underset{\textit{square matrix}}{A} \rightarrow \underset{\textit{determinant of } A}{|A|}$$

In this case, the straight bars do NOT mean absolute value; they represent the determinant of the matrix. Let's find out how to compute the determinant of a square matrix .

Definition: (Determinant of order 1): Let $A=[a_{11}]$ be a square matrix of order 1. Then determinant of A is defined as the number $\boldsymbol{a}_{11}$ itself. That is, $|\boldsymbol{a}_{11}| = \boldsymbol{a}_{11}$.

Example $|3| = 3,$ $|-5| = -5$ and $|0| = 0$

Definition: (Determinant of order 2): Let $\begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} \end{bmatrix}$ be a 2×2 matrix, then

$$\text{detA}=|A|=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22} - a_{12}a_{21}$$

That is, the determinant of a 2×2 matrix is obtained by taking the product of the entries in the main diagonal and subtracting from it the product of the entries in the other diagonal.

To define the determinant of a square matrix A of order n(n > 2), we need the concepts of the minor and the cofactor of an element.

Let $|\boldsymbol{A}| = |\boldsymbol{a}_{ij}|$ be a determinant of order n. The **minor of** $\boldsymbol{a}_{ij}$, is the determinant that is left by deleting the *i*th row and the *j*th column. It is denoted by $\boldsymbol{M}_{ij}$**.**

For example, given the 3 x 3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$. The minor of a_{11} is

$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, the minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, and so on.

Let $|A| = |a_{ij}|$ be a determinant of order n. The **cofactor of a*ij*** denoted **C*ij* or A*ij*,** is defined as $(-1)^{i+j} M_{ij}$, where $i + j$ is the sum of the row number i and column number j in which the entry lies. Thus $C_{ij} = \begin{cases} M_{ij}, & if \ i+j \ is \ even \\ -M_{ij}, & if \ i+j \ is \ odd \end{cases}$. For example, the cofactor of a_{12} in the 3 x 3 determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is

$$C_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Example: Evaluate the cofactor of each of the entries of the matrix: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$

Solution: $C_{11} = -1$, $C_{21} = 1$, $C_{31} = -1$, $C_{12} = 8$, $C_{13} = -5$, $C_{22} = -6$, $C_{32} = 2$, $C_{23} = 3$, $C_{33} = -1$

Activity: Evaluate the cofactor of each of the entries of the given matrices:

a. $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ **b.** $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

Definition: (Determinant of order n): If A is a square matrix of order n (n >2), then its determinant may be calculated by multiplying the entries of any row (or column) by their cofactors and summing the resulting products. That is,

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in} \quad \textbf{Or} \quad \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$

Remark: It is a fact that determinant of a matrix is unique and does not depend on the row or column chosen for its evaluation.

Example: Find the value of $\begin{vmatrix} 1 & -3 & 4 \\ 0 & 2 & 5 \\ -2 & 6 & 3 \end{vmatrix}$

Solution: Choose a given row or column. Let us arbitrarily select the first row. Then

$$\begin{vmatrix} 1 & -3 & 4 \\ 0 & 2 & 5 \\ -2 & 6 & 3 \end{vmatrix} = (1)\begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} + (-3)(-1)\begin{vmatrix} 0 & 5 \\ -2 & 3 \end{vmatrix} + 4\begin{vmatrix} 0 & 2 \\ -2 & 6 \end{vmatrix} = 1(6-30) + 3(0+10) + 4(0+4) \quad = 22$$

If we had expanded along the first column, then

$$\begin{vmatrix} 1 & -3 & 4 \\ 0 & 2 & 5 \\ -2 & 6 & 3 \end{vmatrix} = (1)\begin{vmatrix} 2 & 5 \\ 6 & 3 \end{vmatrix} + 0 \; + (-2)\begin{vmatrix} -3 & 4 \\ 2 & 5 \end{vmatrix} = 1(6-30) - 2(-15-8) = 22, \text{ as before}$$

Example: Find the value of $|A| = \begin{vmatrix} 1 & 2 & 0 & -1 \\ 3 & -1 & 4 & 1 \\ -2 & 0 & -3 & 3 \\ 4 & 3 & 1 & 2 \end{vmatrix}$

Solution: Expanding along first row, we have

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - a_{14}M_{14}$$

$$= (1)\begin{vmatrix} -1 & 4 & 1 \\ 0 & -3 & 3 \\ 3 & 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 & 1 \\ -2 & -3 & 3 \\ 4 & 1 & 2 \end{vmatrix} + 0 + (1)\begin{vmatrix} 3 & -1 & 4 \\ -2 & 0 & -3 \\ 4 & 3 & 1 \end{vmatrix} = \quad 54 - 94 + 13 = -27$$

Activity: Compute the determinant of A if:

a. $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix}$ c. $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

b. $A = \begin{pmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{pmatrix}$ d. $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ -8 & 3 & 1 & 0 \\ 4 & -7 & 5 & 2 \end{pmatrix}$

Note: 1. $\det I_n = 1$**,** where I_n is an identity matrix of order n.

2. $\det A =$ The product of the diagonal elements, if A is a diagonal matrix or lower triangular matrix or upper triangular matrix.

Exercise:

1) Evaluate the following determinants: a) $\begin{vmatrix} 3 & 2 \\ -5 & -4 \end{vmatrix}$ b) $\begin{vmatrix} -2 & -\boldsymbol{a} \\ -\boldsymbol{a} & 2 \end{vmatrix}$ c) $\begin{vmatrix} 2 & 1 & 5 \\ -3 & 4 & -1 \\ 0 & 6 & -1 \end{vmatrix}$

2) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Determine each of the following

a) the minor of a_{21} b) the minor of a_{22} c) the cofactor of a_{22}

d) the cofactor of a_{23} e) the cofactor of a_{32}.

Properties of Determinants

We now state some useful properties of determinants. These properties help a good deal in the evaluation of determinants. We use the notations R_i and C_j to denote respectively the *i-th* row and the j-*th* column of a determinant.

Property 1: The value of a determinant remains unchanged if rows are changed into columns and columns into rows. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \quad \text{or} \quad \det A = \det A^t$$

Property 2: If any two rows (or columns) of a determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}.$$

Remark: The notation $R_i \leftrightarrow R_j (C_i \leftrightarrow C_j)$ is used to represent interchange of *ith* and *jth* row (column).

Property 3: If any two rows (or columns) of a determinant are identical, the value of the determinant is zero. That is, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$. R_1 and R_2 are identical

Property 4: If each element of a row (or column) of a determinant is multiplied by a constant k, the value of the determinant so obtained is k times the value of the original determinant. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Remark: 1. The notation $R_i \to k\,R_i (C_i \to kC_i)$ is used to represent multiplication of each element of ith row (column) by the constant k.

2. $\det(kA) = k^n \det A$**,** where k is any real number and A is an nxn matrix**.**

Property 5: If to the elements of a row (or column) of a determinant are added k times the elements of another row (or column), the value of the determinant so obtained is equal to the value of the original determinant. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+ka_{31} & a_{12}+ka_{32} & a_{13}+ka_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Property 6: If each element of a row (or column) of a determinant is the sum of two elements, the determinant can be expressed as the sum of two determinants. That is,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31}+b_1 & a_{32}+b_2 & a_{33}+b_3 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Property 7: If a row or a column of a square matrix is zero then the determinant is zero.

Property 8: The determinant of a triangular matrix is the product of the diagonal elements.

Example: Find the value of the determinant $|A| = \begin{vmatrix} 1 & 18 & 72 \\ 2 & 40 & 148 \\ 3 & 45 & 150 \end{vmatrix}$

Solution: By applying various properties of determinants, we make maximum number of zeros in a row or a column. We shall make maximum number of zeros in C_1. Performing the operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 2R_1$ (property 5),

$$|A| = \begin{vmatrix} 1 & 18 & 72 \\ 0 & 4 & 4 \\ 0 & 9 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 9 & 6 \end{vmatrix} = 24 - 36 = -12$$

Activity: Evaluate the following determinants by using the properties listed above:

a) $\begin{vmatrix} 3 & 1 & 43 \\ 2 & 7 & 35 \\ 1 & 3 & 17 \end{vmatrix}$ b) $\begin{vmatrix} 2 & 4 & 6 \\ 7 & 9 & 11 \\ 8 & 10 & 12 \end{vmatrix}$ c) $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$

Product of two determinants

Theorem: The determinant of the product of two matrices of order n is the product of their determinants. That is, $|\boldsymbol{AB}| = |\boldsymbol{A}|\ |\boldsymbol{B}|$.

Example: Let $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$, then $|AB| = |A||B| = \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = (3)(3) = 9$

Example: Let A and B be 3x3 matrix with det A = 2 and det B = -3.

Find det ($2AB^t$).

Solution: $\det(2AB^t) = 2^3 \det A \det B^t = 8(2)(-3) = -48$**,** since det B = det B^t.

2.5 Inverse of a matrix and its properties

Adjoint of a matrix

Definition: Let A = (a_{ij}) be a square matrix of order n and let C_{ij} be the cofactor of a_{ij}. Then the **ad joint of A**, denoted by **adj A**, is defined as the transpose of the cofactor matrix (C_{ij}).

Let C=$\left(c_{ij}\right)_{nxn}$ be cofactor matrix then a**djA**=$C^+ = \left(c_{ij}\right)_{nxn}^{+} = \left(c_{ji}\right)_{nxn}$

Example: Find adj A, if $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 3 & 2 \end{bmatrix}$

Solution: We have C_{11}=-3, C_{12}=6, C_{13}= -3, C_{21}=5, C_{22}=-10, C_{23}=5, C_{31}=2,C_{32}=-4, C_{33}=2.

Thus, $adj\ A = \begin{bmatrix} -3 & 5 & 2 \\ 6 & -10 & -4 \\ -3 & 5 & 2 \end{bmatrix}$.

Activity: Find adj A if $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$

Properties of the Adjoint of a matrix

1. If A is a square matrix of order n, then

   A(adj A) = |A| $\mathbf{I_n}$ = (adj A)A, where $\mathbf{I_n}$ is an identity matrix of order n.

2. If A is a square matrix of order n, then ***adj*** $(A') = (adjA)'$
3. If A and B are two square matrices of the same order, then

adj(AB) = adj(B) adj (A).

Example: If $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & 1 \\ -4 & 5 & 6 \end{bmatrix}$, verify that A(adjA) = |A| I_3 = (adjA)A

Solution: We have |A| = 2(-5)-1(12+4)+3(10) = -10 – 16 + 30 = 4

Now C_{11}=-5, C_{12}=-16, C_{13}=10, C_{21} = 9, C_{22}=24, C_{23}=-14, C_{31}=1, C_{32}=4, C_{33}=-2.

$$Therefore, \qquad adj\ A = \begin{bmatrix} -5 & 9 & 1 \\ -16 & 24 & 4 \\ 10 & -14 & -2 \end{bmatrix}$$

Hence $A(adjA) = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 0 & 1 \\ -4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -5 & 9 & 1 \\ -16 & 24 & 4 \\ 10 & -14 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3$

Similarly, it can be proved that $(adjA)A = |A| I_3$

Definition: Let A be a square matrix of order n. Then a square matrices B of order n, if it exists, is called an **inverse of A** if **AB = BA = I_n**. A matrix A having an inverse is called **an invertible (non-singular) matrix**. It may easily be seen that if a matrix A is invertible, its inverse is **unique**. The inverse of an invertible matrix A is denoted by $\mathbf{A^{-1}}$.

Does every square matrix possess an inverse? The answer is No

Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. If B is any square matrix of order 2, we find that **AB = BA = 0.**

We thus see that there cannot be any matrix **B** for which **AB** and **BA** both are equal to $\mathbf{I_2}$. Therefore A is not invertible. Hence, we conclude that a square matrix may not to have an inverse. However, if A is a square matrix such that $|A| \neq 0$, then A is invertible and

$A^{-1} = \frac{1}{|A|} adj\ A$. For, we know that $A(adjA) = (adiA)A = |A|\ I_n$

$A\left(\frac{1}{|A|} adjA\right) = \left(\frac{1}{|A|} adjA\right)A = I_n$. Thus A is invertible and $A^{-1} = \frac{1}{|A|} adjA$.

A square matrix A is said to be **singular (not invertible)** if $|A| = 0$, and it is called **non-singular (invertible)** if $|A| \neq 0$.

Example: Find λ if the matrix $A = \begin{bmatrix} 6 & 7 & -1 \\ 3 & \lambda & 5 \\ 9 & 11 & \lambda \end{bmatrix}$ has no inverse.

Solution: $6(\lambda^2 - 55) - 7(3\lambda - 45) - (33 - 9\lambda) = 0$

- ➢ $\lambda^2 - 2\lambda - 8 = 0$
- ➢ $(\lambda - 2)(\lambda - 4) = 0$
- ➢ $\lambda = 2$ or $\lambda = 4$

Example: If $\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, then $|\boldsymbol{A}| = 3(-3+2) - 1(2+1) + 2(4+3) = 8$

Since $|\boldsymbol{A}| \neq 0, \boldsymbol{A}$ is non-singular or invertible.

Activity: Find $\mathbf{A^{-1}}$.

Further, if $\boldsymbol{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 4 \\ 5 & -2 & 8 \end{bmatrix}$ then $|B| = 0$ and it is singular.

Note: If A is an invertible nxn matrix, then $AA^{-1} = I_n$ and det $A^{-1} = \frac{1}{\det A}$, where $\det A \neq 0$.

Properties of the inverse of a matrix

1. A square matrix is invertible if and only if it is non-singular.
2. The inverse of the inverse is the original matrix itself, i.e. $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
3. The inverse of the transpose of a matrix is the transpose of its inverse, i.e.,

   $$(A^t)^{-1} = (A^{-1})^t$$

4. If A and B are two invertible matrices of the same order, then AB is also invertible and moreover, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

Example: Find the inverse of the matrix A=$\begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 3 \\ 2 & 0 & 1 \end{bmatrix}$.

Solution: $|A| = (4)(3) - (-2)(1) + 1(-6) = 8 \neq 0$. Thus A^{-1} exists and is given by $A^{-1} = \frac{1}{|A|} adjA$. To find *adjA*, let C_{ij} denote the cofactor of a_{ij}, the element in the *ith* row and *jth* column of |A|. Thus $C_{11}=3$, $C_{12} = -1$, $C_{13}=-6$, $C_{21} = 2$, $C_{22}=2$, $C_{23}=-4$, $C_{31}=-9$ $C_{32}=-5$ and $C_{33} =26$.

$adj\ A = \begin{bmatrix} 3 & 2 & -9 \\ -1 & 2 & -5 \\ -6 & -4 & 16 \end{bmatrix}$. Hence $A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{8}\begin{bmatrix} 3 & 2 & -9 \\ -1 & 2 & -5 \\ -6 & -4 & 16 \end{bmatrix}$

Activity: 1. Find the inverse of A, if i) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ii) $A = \begin{bmatrix} 1 & 4 & 0 \\ -1 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

2. Find matrix A such that $A\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 9 & 4 \end{bmatrix}$.

3. If $AX = b$ then $X = A^{-1}b$. (True/False)

2.6 System of Linear equations

Definition: A **linear equation** in the variables $x_1, x_2,..., x_n$ over the real field $\Re$ is an equation that can be written in the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ ……….(1),where b and the coefficients $a_1, a_2,..., a_n$ are given real numbers.

Definition: A **system of linear equations** (or **a linear system**) is a collection of one or more linear equations involving the same variables, say $x_1, x_2,..., x_n$.

Now consider a system of m linear equations in n-unknowns $x_1, x_2,..., x_n$:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{array} \qquad (2)$$

If $\boldsymbol{b}_1 = \boldsymbol{b}_2 = ... = \boldsymbol{b}_m = 0$ then we say that the system is **homogeneous**. If $\boldsymbol{b}_i \neq 0$ for some

$i \in \{1,2,3, . . ., m\}$ then the system is called **non homogeneous**.

Matrix Form of a Linear System

In matrix notation, the linear system (2) can be written as $\boldsymbol{AX} = \boldsymbol{B}$ where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \dots & \boldsymbol{a}_{1n} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \dots & \boldsymbol{a}_{2n} \\ . & & & \\ . & & & \\ . & & & \\ \boldsymbol{a}_{m1} & \boldsymbol{a}_{m2} & \dots & \boldsymbol{a}_{mn} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ . \\ . \\ . \\ \boldsymbol{x}_n \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ . \\ . \\ . \\ \boldsymbol{b}_m \end{bmatrix}$$

We call **A** the **coefficient matrix** of the system (2).

Observe that entries of the k-th column of **A** are the coefficients of the variable $\mathbf{x_k}$ in (2).

The $\mathbf{m \times (n+1)}$ matrix whose first **n** columns are the columns of A (the coefficient matrix) and whose last column is B is called the **augmented matrix** of the system. We denote it by **[A|B]**. The augmented matrix determines the system (2) completely because it contains all the coefficients and the constants to the right side of each equation in the system.

For example for the non homogeneous linear system

$$\begin{aligned} \boldsymbol{x}_1 + 3\boldsymbol{x}_2 - \boldsymbol{x}_3 &= 2 \\ \boldsymbol{x}_2 - 2\boldsymbol{x}_3 &= 4 \\ -2\boldsymbol{x}_1 - 3\boldsymbol{x}_2 - 3\boldsymbol{x}_3 &= 5 \end{aligned} \qquad (3)$$

The matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -2 \\ -2 & -3 & -3 \end{bmatrix}$ is the coefficient matrix and $\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ -2 & -3 & -3 & 5 \end{bmatrix}$ is the augmented matrix.

Definition: A **solution** of a linear system in n-unknowns $x_1, x_2, ..., x_n$ is an n-tuple $(\boldsymbol{s}_1, \boldsymbol{s}_2, ..., \boldsymbol{s}_n)$ of real numbers that makes each of the equations in the system a true statement when s_i is substituted for x_i, i = 1,2, . . ., n. The set of all possible solutions is called the **solution set** of the linear system. We say that two linear systems are **equivalent** if they have the same solution set.

Note: A system of linear equations has either

1. ***no solution***,this case happnes if rank(A)$< rank([A|B])$.
2. ***exactly one solution***,if rank(A)$= rank([A|B]) =$number of unknowns or
3. ***Infinitely many solutions*** if rank(A)$= rank([A|B]) <$number of unknowns.

We say that a linear system is **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Methods for solving a linear system

This is the process of finding the solutions of a linear system. We first see the technique of elimination (Gaussian elimination method) and then we add two more techniques, matrix inversion method and Cramer's rule.

Gaussian Elimination Method

The Gaussian elimination method is a standard method for solving linear systems. It applies to any system, no matter whether $m < n$, $m = n$ or $m > n$ (where m and n are number of equations and variables respectively). We know that equivalent linear systems have the same solutions. Thus the basic strategy in this method is to replace a given system with an equivalent system, which is easier to solve.

The basic operations that are used to produce an equivalent system of linear equations are the following:

1. Replace one equation by the sum of itself and a multiple of another equation.
2. Interchange two equations
3. Multiply all the terms in an equation by a non zero constant.

Example: Using Gaussian elimination method, solve the system of equations

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ -2x_1 - 3x_2 - 3x_3 &= 5 \end{aligned}$$

Solution: We perform the elimination procedure with and without matrix notation of the system. For each step we put the resulting system and its augmented matrix side by side for comparison:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ -2x_1 - 3x_2 - 3x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ -2 & -3 & -3 & 5 \end{bmatrix}$$

We keep x_1 in the first equation and eliminate it from the other equations. For this replace the third equation by the sum of itself and two times equation 1.

$$\begin{array}{ll} 2.[eq.1]: & 2x_1 + 6x_2 - 2x_3 = 4 \\ +[eq.3]: & -2x_1 - 3x_2 - 3x_3 = 5 \\ \hline [New\ eq.3] & 3x_2 - 5x_3 = 9 \end{array}$$

We write the new equation in place of the original third equation:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ 3x_2 - 5x_3 &= 9 \end{aligned} \qquad \xrightarrow{R_3 \rightarrow R_3 + 2R_1} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 3 & -5 & 9 \end{bmatrix}$$

Next use the x_2 in equation 2 to eliminate $3x_2$ in equation 3.

$$\begin{array}{ll} -3.[eq.2]: & -3x_2 + 6x_3 = -12 \\ +[eq.3]: & 3x_2 - 5x_3 = 5 \\ \hline [New\ eq.3] & x_3 = -3 \end{array}$$

The resulting equivalent system is:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ x_3 &= -3 \end{aligned} \qquad \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Now we eliminate the $-2x_3$ term form equation 2. For this we use x_3 in equation 3.

$$\begin{array}{ll} 2.[eq.3]: & 2x_3 = -6 \\ +[eq.2]: & x_2 - 2x_3 = 4 \\ \hline [New\ eq.3] & x_3 \quad = -3 \end{array}$$

From this we get

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 \quad &= -2 \\ x_3 &= -3 \end{aligned} \quad \xrightarrow{R_2 \rightarrow R_2 + 2R_3} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Again by using the x_3 term in equation 3, we eliminate the $-x_3$ term in equation 1.

$$\begin{array}{ll} 1.[eq.3]: & x_3 = -3 \\ +[eq.1]: & x_1 + 3x_2 - x_3 = 2 \\ \hline [New\,eq.1] & x_1 + 3x_2 \quad = -1 \end{array}$$

Thus we get the system

$$\begin{aligned} x_1 + 3x_2 &= -1 \\ x_2 &= -2 \\ x_3 &= -3 \end{aligned} \quad \xrightarrow{R_1 \rightarrow R_1 + R_3} \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Finally, we eliminate the $3x_2$ term in equation 1. We use the x_2 term in equation 2 to eliminate the $3x_2$ term above it.

$$\begin{array}{ll} -3.[eq.2]: & -3x_2 = 6 \\ +[eq.1]: & x_1 + 3x_2 = 2 \\ \hline [New\,eq.1] & x_1 \quad = 5 \end{array}$$

So we have an equivalent system (to the original system) that is easier to solve.

$$\begin{aligned} x_1 \quad &= 5 \\ x_2 \quad &= -2 \\ x_3 &= -3 \end{aligned} \quad \xrightarrow{R_1 \rightarrow R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Thus the system has only one solution, namely (5, -2, -3) or $\mathbf{x}_1 = 5, \mathbf{x}_2 = -2, \mathbf{x}_3 = -3$. To verify that (5, -2, -3) is a solution, substitute these values in to the left side of the original system, and compute:

5 + 3(-2) - (-3) = 5 – 6 + 3 = 2

$$-2 - 2(-3) = -2 + 6 = 4$$

$$-2(5) - 3(-2) - 3(-3) = -10 + 6 + 9 = 5$$

It is a solution, as it satisfies all the equation in the given system (3).

The example above illustrates how operations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the three elementary row operations on the augmented matrix.

Let us see how elementary row operations on the augmented matrix of a given linear system can be used to determine a solution of the system. Suppose a system of linear equations is changed to a new one via row operations on its augmented matrix. By considering each type of row operation it is easy to see that any solution of the original system remains a solution of the new system.

Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. From this we have the following important property.

- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Thus to solve a linear system by elimination we first perform appropriate row operations on the augmented matrix of the system to obtain the augmented matrix of an equivalent linear system which is easier to solve and use back substitution on the resulting new system. This method can also be used to answer questions about existence and uniqueness of a solution whenever there is no need to solve the system completely.

In Gaussian elimination method we either transform the augmented matrix to an echelon matrix or a reduced echelon matrix. That is we either find an echelon form or the reduced echelon form of the augmented matrix of the system. An echelon form of the augmented matrix enables us to answer the following two fundamental questions about solutions of a linear system. These are:

1. Is the system consistent; that is, does at least one solution exists?
2. If a solution exists, is it the only one; that is, is the solution unique?

Example: Determine if the following system is consistent. If so how many solutions does it have?

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ x_1 + 5x_2 - 5x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is $A = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix}$

Let us perform a finite sequence of elementary row operations on the augmented matrix.

$$[A|B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to -2R_1 + R_3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 3 & -3 & -5 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{2}R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 0 & 0 & -\frac{9}{2} \end{bmatrix}$$

The corresponding linear system of the last matrix is

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 6x_2 - 6x_3 &= -1 \qquad (*) \\ 0 &= \frac{-9}{2} \end{aligned}$$

But the last equation $0.x_1 + 0.x_2 + 0.x_3 = \frac{-9}{2}$ is never true. That is there are no values x_1, x_2, x_3 that satisfy the new system (*). Since (*) and the original linear system have the same solution set, the original system is inconsistent (has no solution)

Example: Use Gaussian elimination to solve the linear system

$$\begin{aligned} 2\mathbf{x} - \mathbf{y} + \mathbf{z} &= 2 \\ -2\mathbf{x} + \mathbf{y} + \mathbf{z} &= 4 \\ 6\mathbf{x} - 3\mathbf{y} - 2\mathbf{z} &= -9 \end{aligned}$$

Solution: The augmented matrix of the given system is

$$[\mathbf{A}|\mathbf{B}] = \begin{bmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & 1 & 4 \\ 6 & -3 & -2 & -9 \end{bmatrix}$$

Let us find an echelon form of the augmented matrix first. From this we can determine whether the system is consistent or not. If it is consistent we go ahead to obtain the reduced echelon form of **[A|B] ,** which enable us to describe explicitly all the solutions.

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & 1 & 4 \\ 6 & -3 & -2 & -9 \end{bmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 6 & -3 & -2 & -9 \end{bmatrix} \xrightarrow{R_3 \to -3R_1 + R_3}$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_3 \to \frac{5}{2}R_2 + R_3} \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2}$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_2 + R_1} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1}$$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The associated linear system to the reduced echelon form of **[A|B]** is

$$\begin{aligned} \mathbf{x} - \tfrac{1}{2}\mathbf{y} &= \tfrac{-1}{2} \\ \mathbf{z} &= 3 \\ 0 &= 0 \end{aligned}$$

The third equation is $0\mathbf{x} + 0\mathbf{y} + 0\mathbf{z} = 0$. It is not an inconsistency, it is always true whatever values we take for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

The system is consistent and if we assign any value λ for $\boldsymbol{y}$ in the first equation, we get $\mathbf{x} = \frac{-1}{2} + \frac{1}{2}\lambda$. From the second we have z = 3. Thus $\mathbf{x} = \frac{-1}{2} + \frac{1}{2}\lambda$, $\boldsymbol{y} = \lambda$ and $\boldsymbol{z} = 3$ is the solution of the given system, where λ is any real number. There are an infinite number of solutions, for example,

$$\boldsymbol{x} = \tfrac{-1}{2}, \boldsymbol{y} = 0, \boldsymbol{z} = 3$$

$$\boldsymbol{x} = 0, \boldsymbol{y} = 1, \boldsymbol{z} = 3 \quad \text{and so on.}$$

In vector form the general solution of the given system is of the form $(\frac{-1}{2} + \frac{1}{2}\lambda, \lambda, 3)$, where $\lambda \in \Re$.

Cramer's rule

Suppose we have to solve a system of n linear equations in n unknowns $\mathbf{Ax} = \mathbf{b}$. Let $\mathbf{A_i(b)}$ be the matrix obtained from **A** by replacing column **i** by the vector **b** and $\mathbf{A^k}$ be the **k**-th column vector of matrix **A**.

$$A_i(b) = [\mathbf{A}^1 \ \ \mathbf{A}^2 \ \dots \ \underset{\substack{\uparrow \\ \text{column } i}}{\mathbf{b}} \ \dots \ \mathbf{A}^n]$$

Now let $\mathbf{e_1}, \mathbf{e_2}, \dots \mathbf{e_n}$ be columns of the n × n identity matrix **I** and $\mathbf{I_i(x)}$ be the matrix obtained from **I** by replacing column **i** by **x.**

If $\mathbf{Ax} = \mathbf{b}$ then by using matrix multiplication we have

$$\mathbf{AI_i(x) = A[e_1 \ \dots \ x \ \dots \ e_n] = [Ae_1 \ \dots \ Ax \ \dots \ Ae_n]}$$

$$= [\mathbf{A}^1 \ \dots \ \mathbf{b} \ \dots \ \mathbf{A}^n] = \mathbf{A_i(b)}$$

By the multiplicative property of determinants, (det**A**)(det$\mathbf{I_i(x)}$) = det $\mathbf{A_i(b)}$

The second determinant on the left is x_i. (Make a cofactor expansion along the ith row.) Hence (det **A**). x_i = det $\mathbf{A_i(b)}$. Therefore if det**A** ≠ **0** then we have $\mathbf{x_i} = \dfrac{\det \mathbf{A_i(b)}}{\det \mathbf{A}}$.

This method for finding the solutions of n linear equations in n unknowns is known as Cramer's Rule.

Example: Solve the following system of linear equations by Cramer's Rule.

$$\begin{aligned} 2\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 &= 6 \\ \mathbf{x}_1 + 4\mathbf{x}_2 - 2\mathbf{x}_3 &= -4 \\ 3\mathbf{x}_1 \quad + \mathbf{x}_3 &= 7 \end{aligned}$$

Solution: Matrix form of the given system is $\mathbf{Ax} = \mathbf{b}$

where $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$ **and** $\mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$

By Cramer's Rule, $\mathbf{x_i} = \dfrac{\det \mathbf{A_i(b)}}{\det \mathbf{A}}$ $(\mathbf{i} = 1, 2, 3)$

$$\det A = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 3$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 6 & -1 & 1 \\ -4 & 4 & -2 \\ 7 & 0 & 1 \end{vmatrix}}{2} = \frac{6}{3} = 2 \quad , \quad x_2 = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 2 & 6 & 1 \\ 1 & -4 & -2 \\ 3 & 7 & 1 \end{vmatrix}}{2} = \frac{-3}{3} = -1 \quad \text{and}$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{\begin{vmatrix} 2 & -1 & 6 \\ 1 & 4 & -4 \\ 3 & 0 & 7 \end{vmatrix}}{2} = \frac{3}{3} = 1.$$

Example: Solve the following system of linear equations by Cramer's Rule.

$$\begin{aligned} 2x_1 + x_2 &= 7 \\ -3x_1 + 2x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

Solution: Matrix form of the given system is $\mathbf{Ax} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad and \quad b = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}$$

By Cramer's Rule, $\mathbf{x_i} = \dfrac{\mathbf{det\,A_i(b)}}{\mathbf{det\,A}}$ $(\mathbf{i} = 1, 2, 3)$

$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix}}{4} = \frac{6}{4} = \frac{3}{2}, \quad x_2 = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix}}{4} = \frac{16}{4} = 4 \quad \text{and}$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{\begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix}}{4} = \frac{-14}{4} = \frac{-7}{2}.$$

Remark: For the non homogeneous system $\mathbf{Ax} = \mathbf{b}$, if $\det A = 0$, then the Cramer's rule does not give any information whether or not the system has a solution.

Inverse matrix method

Consider the following linear system with n-**equations** in n-**unknowns** $x_1, x_2, x_3, \ldots, x_n$;

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

.... *(*)*

$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$.

The matrix notation of the given linear system (*) is $AX = B \ldots \ldots \ldots \ldots \ldots .. \ldots . (**)$, where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_n & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Now, to solve equation (*) the coefficient matrix (A) must be invertible and we multiply eqn(**) both sides by A^{-1} to get

$$X = A^{-1}B \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. \ldots (***)$$

which is the required solution of (*).

Example: Solve the given system of equations using the inverse of a matrix.

$$3x + 8y = 5$$
$$4x + 11y = 7$$

SOLUTION

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A = \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.

$$\begin{aligned} &\left(A^{-1}\right) AX = \left(A^{-1}\right) B \\ &\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11\,(5) + (-8)\,7 \\ -4\,(5) + 3\,(7) \end{bmatrix} \\ &\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

The solution is $(-1, 1)$.

Example:

Use matrix inversion to solve the following linear system.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + 2x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

Solution: The coefficient matrix A, the column vector b and the inverse A^{-1}, respectively, are given by

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, the unique solution of the given linear system from eqn(***) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

2.7 Eigenvalues and Eigenvectors of a matrix

Definition: Let A be an nxn matrix. A scalar λ in is said to be an **eigenvalue** of A iff there is a non-zero vector X such that AX = λX ……………………………….…. (*)

If λ is an eigenvalue of A then any vector satisfying (*) is called an **eigenvector** of A corresponding to λ.

How to determine the Eigenvalues and corresponding Eigenvectors of a Matrix?

X is an eigenvector with eigenvalue $\lambda \Leftrightarrow AX = \lambda X \Leftrightarrow (A - \lambda I_n) X = 0$ …(**)

$$or \quad \begin{bmatrix} a_{11}-\lambda & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22}-\lambda & ... & . & a_{2n} \\ . & . & & & . \\ . & . & & & . \\ . & . & & & . \\ a_{n1} & a_{n2} & & & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

X = 0 is the trivial solution of (**). Further solutions will exist iff $|A - \lambda I_n| = 0$.

Hence, solving the equation $|A - \lambda I_n| = 0$ gives the eigenvalue(s) of A.

For each eigenvalue λ, the corresponding eigenvector is found by substituting λ back into the equation $|A - \lambda I_n| X = 0$.

Note: i) The polynomial $|A - \lambda I_n|$ is called the **characteristic polynomial of A.**

ii) The equation $|A - \lambda I_n| = 0$ is called the **characteristic equation**.

Example: Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the eigenvalues and the corresponding eignvectors of A.

Solution: $|A - \lambda I_2| = 0 \Leftrightarrow \left| \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 3\lambda - 28 = 0$$

$$\Leftrightarrow \lambda = 7 \text{ or } \lambda = -4$$

The corresponding eigenvectors can now be found as follow:

For λ = 7: $(A - 7I_2)X = 0 \Leftrightarrow \left[\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow y = x$$

Hence, any vector of the type $\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where β is any real number, is an eigenvector corresponding to the eigenvalue 7.

For λ = -4: $(A + 4I_2)X = 0 \Leftrightarrow \left[\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow y = \frac{-5}{6}x$$

Hence, any vector of the type $\beta\begin{bmatrix} 1 \\ \frac{-5}{6} \end{bmatrix}$, where β is any real number, is an eigenvector corresponding to the eigenvalue -4.

Note: If X is an eigenvector with eigenvalue λ, αX is also an eigenvector with the same eigenvalue, where α is a non-zero scalar.

The following theorem summarizes our results so far.

Theorem: If A is an n x n matrix, then the following statements are equivalent:

i) λ is an eigenvalue of A.

ii) There is a non-zero vector $X \in K^n$ such that AX = λX.

iii) The system of equations (A - λI)X = 0 has non-trivial solutions.

iv) λ is a solution of the characteristic equation det (A - λI) in K.

Example: Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Find the eigenvalues and the corresponding eignvectors of A.

Solution: Characteristic equation of A: $\begin{vmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$

$$\Leftrightarrow \quad (3 - \lambda)\ (3 - \lambda)\ (5 - \lambda) - 4(5 - \lambda) = 0$$

$$\Leftrightarrow \quad [(3 - \lambda)^2 - 4]\ (5 - \lambda) = 0$$

$$\Leftrightarrow \quad (\lambda^2 - 6\lambda + 5)\ (\lambda - 5) = 0$$

$$\Leftrightarrow \quad (\lambda - 1)\ (\lambda - 5)^2 = 0$$

So, eigenvalues of A are: λ = 1 and λ = 5.

To find the corresponding eigenvectors, we substitute the values of λ in the equation

(A - λI) X = 0. That is, $\begin{bmatrix} 3-\lambda & -2 & 0 \\ -2 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ …. (*)

For λ = 1, (*) becomes: $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$ x = y, z = 0

Thus, the eigenvectors corresponding to eigenvalue 1 are vectors of the form:

$$X = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ for any x in the set of real numbers.

For λ = 5, (*) becomes: $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow$ x = -y

Thus, eigenvectors of A corresponding to eigenvalues 5 are vectors of the form.

$$X = \begin{bmatrix} x \\ -x \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ for any x and z in the set of real numbers.