Chapter Three

Matrices, Determinant and Systems of Linear Equation

Matrices, which are also known as rectangular arrays of numbers or functions, are the main tools of linear algebra. Matrices are very important to express large amounts of data in an organized and concise form. Furthermore, since matrices are single objects, we denote them by single letters and calculate with them directly. All these features have made matrices very popular for expressing scientific and mathematical ideas. Moreover, application of matrices are found in most scientific fields; such as economics, finance, probability theory and statistics, computer science, engineering, physics, geometry, and other areas.

Main Objectives of this Chapter
At the end of this chapter, students will be able to:-

- Understand the notion of matrices and determinants
- Use matrices and determinants to solve system of linear equations
- Apply matrices and determinants to solve real life problems

3.1 Definition of Matrix

Consider an automobile company that manufactures two types of vehicles, Trucks and Passenger cars in two different colors, red and blue. The company's sales for the month of January are 15 Trucks and 20 Passenger cars in red color, and 10 Trucks and 16 Passenger cars in blue color. This data is presented in Table 1.

Table 1

	Trucks	Passenger Cars
Red	15	20
Blue	10	16

The information in the table can be given in the form of rectangular arrays of numbers as

$$\begin{array}{c c} & \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{array}{c} R_1 \\ R_2 \end{array} & \left[\begin{array}{cc} 15 & 20 \\ 10 & 16 \end{array}\right]. \end{array}$$

In this arrangement, the horizontal and vertical lines of numbers are called **rows** (R_1, R_2) and **columns** (C_1, C_2), respectively. The columns C_1 and C_2 represent the Trucks and Passenger cars, respectively, which are sold in January. And the rows R_1 and R_2 represent the red and blue colored vehicles, respectively. An arrangement of this type is called a **matrix**. Note that the above matrix has two rows and two columns. This shows us the usefulness of matrix to organize information.

Definition 3.1 (**Definition of Matrix**). *If m and n are positive integers, then by a* ***matrix*** *of size m by n, or an $m \times n$ matrix, we shall mean a rectangular array consisting of mn numbers, or symbols, or expressions in a boxed display consisting of m rows and n columns. This can be denoted by*

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & & C_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \end{array}$$

where $(R_1, R_2, R_3, ..., R_m)$ and $(C_1, C_2, C_3, ..., C_n)$ represent the m rows and n columns, respectively.

Remark.

1. Note that the first suffix denotes the number of a row (or position) and the second suffix that of a column, so that a_{ij} appears at the intersection of the i-th row and the j-th column.

2. Matrix A of size $m \times n$ may also be expressed by

$$A = [a_{ij}]_{m \times n},$$

where a_{ij} represents the (i, j)-th entry of the matrix $[a_{ij}]$.

Example 3.1. The following are matrices of different size.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix $\qquad B = \begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}$ is 3×3 matrix

$C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$ is 4×2 matrix $\qquad D = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is 4×1 matrix

$E = \begin{bmatrix} a & b & c & d \\ b & c & d & e \end{bmatrix}$ is 2×4 matrix, $\qquad F = \begin{bmatrix} b & c & d & e \end{bmatrix}$ is 1×4 matrix

Definition 3.2. *Matrices which are $n \times 1$ or $1 \times n$ are called* ***vectors****. Thus, the $n \times 1$ matrix*

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{2n} \end{bmatrix}$$

is called a column vector, and the $1 \times n$ matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \end{bmatrix}$$

is called a row vector.

Definition 3.3 (Submatrix). *Let A be an $m \times n$ matrix. A submatrix of matrix A is any matrix of size $r \times s$ with $r \leq m$ and $s \leq n$, which is obtained by deleting any collection of rows and/or columns of matrix A.*

Example 3.2. For the given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$,

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ is a submatrix of A, which is obtained by deleting the third row of A.

(ii) $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \end{bmatrix}$ is a submatrix of A, which is obtained by deleting the second column of A.

(iii) $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ is a submatrix of A, which is obtained by deleting the first column and first row of A.

Definition 3.4 (Equality of Matrices). *Two matrices of the same size, $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, are said to be equal (and write $A = B$) if and only if*

$$a_{ij} = b_{ij}, \quad \text{for all } ij.$$

Example 3.3.

(a) Determine the values of a, b, c and d for which the matrices A and B are equal:

$$A = \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Solution: By Definition 3.4, we have $a_{11} = b_{11}$ implies $a = 5$, $a_{12} = b_{12}$ implies $b = 4$, $a_{21} = b_{21}$ implies $c = 0$ and $a_{22} = b_{22}$ implies $d = 2$.

(b) Find the values of α and β for which the given matrices A and B are equal.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha - \beta & 2 \\ \alpha & -1 \end{bmatrix}$$

Solution: Similarly, we have $a_{11} = b_{11}$ implies $\alpha - \beta = 1$, $a_{21} = b_{21}$ implies $\alpha = 3$, and hence $\beta = 2$.

Definition 3.5 (Zero Matrix). *An $m \times n$ matrix $A = [a_{ij}]$ is said to be the zero matrix if $a_{ij} = 0$ for all ij.*

Example 3.4. The following are zero matrices.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 3.1.

1. Write out the matrix of size 3×3 whose entries are given by $x_{ij} = i + j$.
2. Write out the matrix of size 4×4 whose entries are given by

$$x_{ij} = \begin{cases} 1 & if\ i > j \\ 0 & if\ i = j \\ -1 & if\ i < j. \end{cases}$$

3. For the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, give all the submatrices of size 2×2.

3.2 Matrix Algebra

In this section, we discuss addition of matrices, scalar multiplication, and matrix multiplication.

3.2.1 Addition and Scalar Multiplication

Addition and scalar multiplication are the basic matrix operations. To see the usefulness of these operations, let us observe the following simple application.

Consider again an automobile company that manufactures two types of vehicles, Trucks and Passenger cars in two different colors, red and blue. If the sales for the months of January and February, respectively, are given by

$$J = \begin{bmatrix} 15 & 20 \\ 10 & 16 \end{bmatrix} \text{ and } F = \begin{bmatrix} 12 & 28 \\ 20 & 14 \end{bmatrix},$$

then the total sales for two months can be given as follows. The total number of red Trucks sold in two months is $15 + 12 = 27$. Similarly, the total number of blue Trucks, red Passenger cars and blue Passenger cars sold in the two months are given by $10 + 20 = 30$, $20 + 28 = 48$ and $16 + 14 = 30$, respectively.

The preceding computations are examples of matrix addition. We can write the sum of two 2×2 matrices indicating the sales of January and February as

$$J + F = \begin{bmatrix} 15 & 20 \\ 10 & 16 \end{bmatrix} + \begin{bmatrix} 12 & 28 \\ 20 & 14 \end{bmatrix} = \begin{bmatrix} 15 + 12 & 20 + 28 \\ 10 + 20 & 14 + 16 \end{bmatrix} = \begin{bmatrix} 27 & 48 \\ 30 & 30 \end{bmatrix}.$$

Definition 3.6. *Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of the same size. Then the sum of A and B, denoted by $A + B$, is the $m \times n$ matrix defined by the formula*

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different sizes is undefined.

Example 3.5. For the given matrices A, B, C, D compute $A + B$ and $C + D$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution: Using Definition 3.6, we have

$$A + B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a + w & b + x \\ c + y & d + z \end{bmatrix}$$

and

$$C + D = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ -1 & -1 & 4 \end{bmatrix}.$$

Theorem 3.1 (Laws of Matrix Addition). *Let A, B, C be matrices of the same size $m \times n$, **0** the $m \times n$ zero matrix. Then*

1. ***Closure Law of Addition:*** *$A + B$ is an $m \times n$ matrix.*
2. ***Associative Law:*** *$(A + B) + C = A + (B + C)$.*
3. ***Commutative Law :*** *$A + B = B + A$.*
4. ***Identity Law :*** *$A + \mathbf{0} = A$.*
5. ***Inverse Law :*** *$A + (-A) = \mathbf{0}$.*

Definition 3.7 (Scalar Multiplication). *Let $A = [a_{ij}]$ be an $m \times n$ matrix, and α a scalar. Then the product of the scalar α with matrix A, denoted by αA, is defined by*

$$\alpha A = [\alpha a_{ij}]_{m \times n}.$$

Example 3.6. Consider the automobile manufacturing company once again. Suppose the company's sales for the months of January and March, respectively, are given by

$$J = \begin{bmatrix} 15 & 20 \\ 10 & 16 \end{bmatrix}, \text{ and } M = \begin{bmatrix} 18 & 22 \\ 14 & 20 \end{bmatrix}.$$

(a) If the sales of January is to be doubled in February, then the sales of February should be

$$2J = \begin{bmatrix} 2(15) & 2(20) \\ 2(10) & 2(16) \end{bmatrix} = \begin{bmatrix} 30 & 40 \\ 20 & 32 \end{bmatrix}.$$

(b) If the sales of March is to be declined by 50% in April, then the sales of April should be

$$(\frac{1}{2})J = \begin{bmatrix} \frac{1}{2}(18) & \frac{1}{2}(22) \\ \frac{1}{2}(14) & \frac{1}{2}(20) \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 7 & 10 \end{bmatrix}.$$

Example 3.7. Given the matrices A and B, compute $4A$ and $A + (-1)B$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Solution: Using Definition 3.7, we have

$$4A = 4\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4(1) & 4(2) \\ 4(3) & 4(4) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}.$$

And, from the definitions 3.6 and 3.7, we have

$$A + (-1)B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}.$$

From this example, we observe that the difference of two matrices A and B, which is denoted by $A - B$, can be defined by the formula

$$A - B = A + (-1)B = [a_{ij} - b_{ij}]_{m\times n}.$$

Theorem 3.2 (Laws of Scalar Multiplication). *Let A, B be matrices of the same size $m \times n$, and α and β scalars. Then*

1. ***Closure Law of Scalar Multiplication:*** *αA is an $m \times n$ matrix.*
2. ***Associative Law:*** *$\alpha(\beta A) = (\alpha\beta)A$.*
3. ***Distributive Law:*** *$\alpha(A + B) = \alpha A + \alpha B$.*
4. ***Distributive Law:*** *$(\alpha + \beta)A = \alpha A + \beta A$.*
5. ***Monoidal Law:*** *$1A = A$.*

Example 3.8. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the given matrices. Then,

$$2(A + B) = 2\begin{bmatrix} 1+2 & 2+0 \\ 0+1 & 1+1 \end{bmatrix} = 2\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (2)3 & (2)2 \\ (2)1 & (2)2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix}$$

and

$$2A + 2B = \begin{bmatrix} (2)1 & (2)2 \\ (2)0 & (2)1 \end{bmatrix} + \begin{bmatrix} (2)2 & (2)0 \\ (2)1 & (2)1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix}.$$

Thus, we have $2(A + B) = 2A + 2B$.

Example 3.9. Solve for X in the matrix equation $2X + A = B$, where

$$A = \begin{bmatrix} 4 & 0 \\ -2 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 6 & -4 \\ 8 & 0 \end{bmatrix}.$$

Solution: We begin by solving the equation for X to obtain

$$2X = B - A \text{ implies } X = (\frac{1}{2})(B - A).$$

Thus, we have the solution

$$X = \frac{1}{2}\begin{bmatrix} 6-4 & -4-0 \\ 8-(-2) & 0-2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 10 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix}.$$

3.2.2 Matrix Multiplication

An other important matrix operation is matrix multiplication. To see the usefulness of this operation, consider the application below, in which matrices are helpful for organizing information.

A football stadium has three concession areas, located in South, North and West stands. The top-selling items are, peanuts, hot dogs and soda. Sales for one day are given in the first matrix below, and the prices (in dollar) of the three items are given in the second matrix (note that the price per Peanuts, Hot dogs and Soda are given by \$2.00, \$3.00 and \$2.75, respectively).

$$\begin{array}{c} \\ \text{South Stand} \\ \text{North Stand} \\ \text{West Stand} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Peanuts} & \text{Hot dogs} & \text{Sodas} \end{array} \\ \left[\begin{array}{ccc} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{array}\right] \end{array} \begin{array}{c} \\ \left[\begin{array}{c} 2.00 \\ 3.00 \\ 2.75 \end{array}\right] \end{array}.$$

To calculate the total sales of the three top-selling items at the South stand, multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. Thus, we have

$$120(2.00) + 250(3.00) + 305(2.75) = 1828.75\$ \text{ (South stand sales)}.$$

Similarly, the sales for the other two stands are given below:

$$207(2.00) + 140(3.00) + 419(2.75) = 1986.25\$ \text{ (North stand sales)}.$$

$$29(2.00) + 120(3.00) + 190(2.75) = 940.5\$ \text{ (West stand sales)}.$$

The preceding computations are examples of matrix multiplication. We can write the product of the 3×3 matrix indicating the number of items sold and the 3×1 matrix indicating the selling prices as shown below.

$$\left[\begin{array}{ccc} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{array}\right] \left[\begin{array}{c} 2.00 \\ 3.00 \\ 2.75 \end{array}\right] = \left[\begin{array}{c} 1828.75 \\ 1986.25 \\ 940.5 \end{array}\right]$$

The product of these matrices is the 3×1 matrix giving the total sales for each of the three stands.

Definition 3.8 (Matrix Multiplication). *Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices. Then the product of A and B, denoted by AB, is an $m \times p$ matrix whose (i, j)-th entry is defined by the formula*

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + ... + a_{in}b_{nj.}$$

In the other words, the (i, j)-th entry of the product AB is obtained by summing the products of the elements in the i-th row of A with corresponding elements in the j-th column of B.

The above definition can be understood as follows. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \end{bmatrix}$$

has only one row (R_1), and

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$$

has only one column (C_1), then product AB is given by

$$AB = [R_1C_1] = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + ... + a_{1n}b_{n1}.$$

If A has m rows $R_1, R_2, ..., R_m$, and B has n columns $C_1, C_2, ..., C_p$, then the product AB can be given by the formula

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \dots & R_1C_p \\ R_2C_1 & R_2C_2 & \dots & R_2C_p \\ \vdots & \vdots & \dots & \vdots \\ R_mC_1 & R_mC_2 & \dots & R_mC_p \end{bmatrix}.$$

That is, the (i, j)-th entry of AB is R_iC_j.

Remark. The product AB of two matrices A and B is defined only if the number of columns in A and the number of rows in B are equal.

Example 3.10. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$ be two matrices. Clearly, the product AB is defined in this case, since the number of column of A and the number of rows of B are equal. Thus, we have

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

In this example, the matrices A and B, respectively, are 2×3 and 3×2 matrices, whereas the product AB is a 2×2 matrix.

Example 3.11. Compute the product AB of the given matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Solution: The product AB is defined since the number of columns in matrix A and the number of rows in matrix B are equal. Thus, we have AB is given by

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(1) + (3)(1) & (1)(1) + (2)(-1) + (3)(2) \end{bmatrix} = \begin{bmatrix} 6 & 5 \end{bmatrix}.$$

Note that the product BA is not defined in this case.

Example 3.12. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ be the given matrices. Then, we have

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In this example, we observe that both the products AB and BA are defined. This is true in general i.e., the products AB and BA are defined for any two square matrices A and B of the same size. For the matrices A and B given above, we have $AB \neq BA$. Hence, matrix multiplication is **not commutative**.

Example 3.13. Consider the following diagonal matrices.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}$$

The product AB is given by

$$AB = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{bmatrix}$$

Similarly, we have

$$BA = \begin{bmatrix} b_{11}a_{11} & 0 & 0 \\ 0 & b_{22}a_{22} & 0 \\ 0 & 0 & b_{33}a_{33} \end{bmatrix}.$$

In this case, we have $AB = BA$, and hence the given matrices A and B commute. More generally, if A and B are any two diagonal matrices of the same size, then $AB = BA$.

Theorem 3.3. *Matrix multiplication is associative, i.e., whenever the products are defined, we have $A(BC) = (AB)C$.*

From Theorem 3.3, we shall write ABC for either $A(BC)$ or $(AB)C$. Also, for every positive integer n, we shall write A^n for the product $AAA...A$ (n terms).

Theorem 3.4. *If all multiplications and additions make sense, the following hold for matrices, A, B, C and α, β scalars.*

1. $A(\alpha B + \beta C) = \alpha(AB) + \beta(AC)$.
2. $(\alpha B + \beta C)A = \alpha(BA) + \beta(CA)$.

Exercise 3.2.

1. Find your own examples:
   (i) 2×2 matrices A and B such that $A \neq 0$, $B \neq 0$ with $AB \neq BA$.
   (ii) 2×2 matrices A and B such that $A \neq 0$, $B \neq 0$ but $AB = 0$.
   (iii) 2×2 matrix A such that $A^2 = I_2$ and yet $A \neq I_2$ and $A \neq -I_2$.
2. Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.
3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & c \end{bmatrix}$. Is it possible to choose c so that $AB = BA$? If so, what should be the value of c?
4. Given the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$ and α a scalar
   i. Compute the products $A(BC)$, $(AB)C$, and verify that $A(BC) = (AB)C$.
   ii. Compute the products $\alpha(AB)$, $(\alpha A)B$, $A(\alpha B))$, and verify that
   $$\alpha(AB) = (\alpha A)B = A(\alpha B).$$

5. Consider the automobile producer whose agency's sales for the month of January were given by

$$J = \begin{bmatrix} 15 & 20 \\ 10 & 16 \end{bmatrix}.$$

Suppose that the price of a Truck is \$200 and that of a Passenger car is \$100. Use matrix multiplication to find the total values of the red and blue vehicles for the month of January.

3.3 Types of Matrices

There are certain types of matrices that are so important that they have acquired names of their own. In this section we are going to discuss some of these matrices and their properties.

Definition 3.9 (Square Matrix). *A matrix A is said to be square if it has the same number of rows and columns. If A has n-rows and n-columns, we call it a square matrix of size n.*

Example 3.14. The following are square matrices.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ (Square matrix of size 2)}$$

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 4 & 2 & -2 \end{bmatrix} \text{ (Square matrix of size 3)}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \text{ (Square matrix of size } n\text{)}$$

Definition 3.10 (Identity Matrix). *A square matrix $A = [a_{ij}]_{n\times n}$ is called an identity matrix if*

$$a_{ij} = \begin{cases} 1, & if\ i = j \\ 0, & otherwise \end{cases}$$

and it is denoted by I_n.

Example 3.15. The following are identity matrices.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (Identity matrix of size 2)}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (Identity matrix of size 3)}$$

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \text{ (Identity matrix of size } n\text{)}$$

Definition 3.11 (Diagonal Matrix). *A square matrix $D = [d_{ij}]_{n\times n}$ is said to be diagonal if $d_{ij} = 0$ whenever $i \neq j$. Less formally, D is said to be diagonal when all the entries off the main diagonal are 0.*

Example 3.16. The following are diagonal matrices.

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ (Diagonal matrix of size 2)}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ (Diagonal matrix of size 3)}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ (Diagonal matrix of size 3)}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (Diagonal matrix of size 3)}$$

Note that the identity matrix is the special case of diagonal matrix where all the entries in the main diagonal are 1.

Definition 3.12 (Scalar Matrix). *A diagonal matrix in which all diagonal entries are equal is called a* ***scalar matrix****.*

Example 3.17. The following are scalar matrices.

$$(a) \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad (b) \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 3.13 (Triangular Matrix). *A square matrix $A = [a_{ij}]_{n\times n}$ is said to be* ***lower triangular*** *if and only if $a_{ij} = 0$ whenever $i < j$. A is said to be* ***upper triangular*** *if and only if $a_{ij} = 0$ whenever $i > j$.*

Example 3.18.

(i) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (Upper triangular matrices).

(ii) $\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (Lower triangular matrices).

Remark.

(a) In the lower triangular matrix all the entries above the main diagonal are zero, whereas in the upper triangular matrix all the entries below the main diagonal are zero.

(b) Any diagonal matrix is both upper and lower triangular.

Definition 3.14 (Transpose of Matrix). *Let $A = [a_{ij}]$ be an $m \times n$ matrix . Then by the transpose of A we mean the $n \times m$ matrix, denoted by A^t, whose (i, j)-th entry is the (j, i)-th entry of A. More precisely, if $A = [a_{ij}]_{m\times n}$, then $A^t = [a_{ji}]_{n\times m}$. That is,*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \textit{then} \quad A^t = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nm} \end{bmatrix}.$$

Note that the k-th row of matrix A becomes k-th column of A^t, and the k-th column of A becomes k-th row of A^t.

Example 3.19. Compute the transposes of the following matrices.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Solution: First let us consider matrix A. Now, row 1 of matrix A becomes column 1 of A^t, and row 2 of A becomes column 2 of A^t. Thus, we have

$$A^t = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 3 \end{bmatrix}.$$

Similarly,

$$B^t = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}.$$

Definition 3.15 (Symmetric Matrix). *A square matrix A is said to be* ***Symmetric*** *if $A = A^t$.*

Example 3.20. Distinguish whether the given matrix is symmetric or not.

$$(a)\;\; A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix} \quad (b)\;\; B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

Solution:

(a) For the matrix $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$, $A^t = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$. Thus, we have $A \neq A^t$, and hence A is not symmetric.

(b) For the matrix $B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}$, $B^t = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 3 & 2 & 3 \end{bmatrix}$. Thus, we have $B = B^t$, and hence B is symmetric.

Theorem 3.5 (Properties of Matrix Transpose). *When the relevant sums and products are defined, and α is a scalar. Then*

1. $(A^t)^t = A$.

2. $(A + B)^t = A^t + B^t$.

3. $(\alpha A)^t = \alpha(A^t)$.

3. $(AB)^t = B^t A^t$.

Exercise 3.3.

For the given matrices $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$, and $B = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$:

(a) Show that $(A^t)^t = A$.

(b) Show that $(A + B)^t = A^t + B^t$.

(c) Show that $(4A)^t = 4(A^t)$.

(d) Show that$(AB)^t = B^t A^t$.

3.4 Elementary row operations

Elementary row operations are useful to find the rank of a matrix (see Section 3.6), to compute the determinants of matrices (see Section 3.7), and to find the inverse of a matrix (see Section 3.8). Furthermore, elementary row operations are widely used in solving systems of linear equations (see Section 3.9).

In this section, we introduce the elementary row operations and apply these operations to transform the given matrix into different form.

Definition 3.16 (Elementary Row Operations).
Let A be an $m \times n$ matrix. The following are known as elementary row operations.

1. ***Interchanging two rows:*** $R_i \leftrightarrow R_j$.*(Rule of Interchanging)*
2. ***Multiplying a row by a nonzero scalar:*** $R_i \to \alpha R_i$ *(α is a nonzero scalar). (Rule of Scaling)*
3. ***Adding a multiple of one row to another:*** $R_i \to R_i + \alpha R_j$ *(α is a nonzero scalar). (Rule of Replacement)*

Example 3.21. Use elementary row operations to transform the given matrix A into, (a) an upper triangular matrix, (b) an identity matrix.

$$A = \begin{bmatrix} 3 & 12 & 6 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: Consider the given matrix A:

(a) First let us transform the matrix A into an upper triangular. This can be done as follows:

$$A = \begin{bmatrix} 3 & 12 & 6 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \; R_1 \to (\tfrac{1}{3})R_1 \; \begin{bmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix} \text{(Scaling } R_1\text{)}$$

$$R_2 \to R_2 + (-1)R_1, \; R_3 \to R_3 + (-1)R_1 \; \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix} \text{(Replacing } R_2 \text{ and } R_3\text{)}$$

$$R_2 \to (-\tfrac{1}{3})R_2 \; \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} \text{(Scaling } R_2\text{)}$$

$$R_3 \to R_3 + 2R_2 \; \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \text{(Replacing } R_3\text{)}$$

Hence, the matrix $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular, which is obtained from A by elementary row operations.

(b) To transform the matrix A into a diagonal matrix, we simply change all the entries above the main diagonal into zeros and the entries in the main diagonal into 1. Let us denote the above upper triangular matrix by B. Then we have

$B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $R_3 \to (\frac{1}{3})R_3$ $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (Scaling R_3)

$R_2 \to R_2 + (-1)R_3,\ R_1 \to R_1 + (-2)R_3$ $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Replacing R_1 and R_2)

$R_1 \to R_1 + (-4)R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Replacing R_1). Thus, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity matrix obtained from A.

Definition 3.17. *Two matrices are said to be* ***raw equivalent*** *if one can be obtained from the other by a sequence of elementary row operations.*

Example 3.22. Let A, B, I_3 be the matrices in Example 3.21. Then, A is row equivalent to both B and the identity matrix I_3. Also the matrix B is row equivalent to the identity matrix I_3.

Exercise 3.4.

1. Given the matrix $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, use elementary row operations to find the lower triangular matrix which are row equivalent to A.

2. Given the matrix $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, use elementary row operations to find an identity matrix which is row equivalent to B.

3.5 Row Echelon Form and Reduced Row Echelon Form of a Matrix

In order to find the rank, or to compute the inverse of a matrix, or to solve a linear system, we usually write the matrix either in its row echelon form or reduced row echelon form.

Definition 3.18. *An $m \times n$ matrix is said to be in* ***echelon form (or row echelon form)*** *if the following conditions are satisfied:*

1. *All nonzero rows are above any rows of all zeros.*
2. *Each leading entry of a row is in a column to the right of the leading entry of the row above it. (A* ***leading entry*** *refers to the left most nonzero entry in a nonzero row)*
3. *All entries in a column below a leading entry are zeros.*

If a matrix in row echelon form satisfies the following additional conditions, then it is in ***reduced echelon form (or reduced row echelon form)***

4. *The leading entry in each nonzero row is 1.*
5. *Each leading 1 is the only nonzero entry in its column.*

A matrix in ***row echelon*** *form is said to be in* ***reduced row echelon*** *when every column that has a leading 1 has zeros in every position above and below the leading entry.*

Example 3.23. The given matrices A, B, C, D are in row echelon form

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the following are in reduced row echelon form.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 3.6 (Uniqueness of the Reduced Echelon Form). *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Definition 3.19. *A* ***pivot position*** *in a matrix A is a location in A that corresponds to a leading 1 in the reduced row echelon form of A. A* ***pivot column*** *is a column of A that contains a pivot position. A* ***pivot element*** *is a nonzero number in a pivot position that is used as needed to create zeros via row operations.*

To write a matrix in reduced echelon form:

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot column and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Example 3.24. Find the reduced row echelon form of the matrix A.

$$A = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 5 \end{bmatrix}.$$

Solution:
Step 1: Here, the left most nonzero column is the second column.
Step 2: By row interchanging rule, we can obtain the pivot position as follows;

$$\begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 5 \end{bmatrix} \; R_1 \leftrightarrow R_3 \; \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Step 3:
Now, the leading entry is 1, and to create zeros in all positions below the pivot, we use the replacement rule:

$$R_2 \to R_2 + (-2)R_1 \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -9 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

Step 4:
Now we proceed to the second row. Here, the leading entry is -2. Using a scaling rule we obtain a leading 1:

$$R_2 \to (-\frac{1}{2})R_2 \quad \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{9}{2} \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

And applying row replacement rule:

$$R_3 \to R_3 + (-2)R_2 \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{9}{2} \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

And scaling R_3,

$$R_3 \leftrightarrow (-\frac{1}{6})R_3 \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & \frac{9}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Beginning with the rightmost pivot column, we create zeros above each pivot element. That is, we start from the fourth column:

$$R_1 \to R_1 + (-5)R_3 \, , R_2 \to R_2 + (-\frac{9}{2})R_3 \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And using row replacement (to create zeros above the pivot element in the third column),

$$R_1 \to R_1 + (-1)R_2, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, the required matrix in reduced row echelon form is given by

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exercise 3.5.

1. Determine which matrices are in reduced row echelon form.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

2. Give the row echelon form and also the reduced row echelon form of the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

3.6 Rank of matrix using elementary row operations

The ranks of matrices are useful in determining the number of solutions for linear systems.

Definition 3.20 (Rank of Matrix). *Rank of an $m \times n$ matrix A, denoted by $rank(A)$, is the number of nonzero rows of the reduced row echelon form of A.*

Example 3.25. Determine the ranks of the following matrices which, are in reduced row echelon form.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Clearly, all the matrices are in reduced row echelon form. Hence, by Definition 3.20, we have $rank(A) = 3$ (since the number of nonzero rows in matrix A is 3). Similarly, $rank(B) = 2$ (since the number of nonzero rows in matrix B is 2), $rank(C) = 2$ (since the number of nonzero rows in matrix C is 2), and $rank(D) = 1$ (since the number of nonzero rows in matrix D is 1).

Example 3.26. Find $rank(A)$, where $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 3 & 2 \end{bmatrix}$.

Solution: After a sequence of elementary row operations, we obtain the reduced echelon form of A, which is given by

$$\widetilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $rank(A) = 2$.

Remark. The matrix A and its transpose A^t have the same rank. That is

$$rank(A) = rank(A^t).$$

Example 3.27. Verify that the given matrix A and its transpose A^t have the same rank.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0- \end{bmatrix}, \text{ and } A^t = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution: Observe that the matrix A is in its row echelon form, and hence its rank is 2. Now, we apply elementary row operations to reduce matrix A^t into its row echelon form, and and we get that

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $rank(A^t) = 2 = rank(A)$.

Exercise 3.6. Determine the rank of the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -2 \\ 3 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 2 & 1 & 3 \\ -3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 0 & 3 \end{bmatrix}$$

3.7 Determinant and its properties

The determinant is a function that takes a square matrix as an input and produces a scalar as an output. It has many beneficial properties for studying, matrices and systems of equations.

Definition 3.21 (Determinant of 2×2 matrix). *The determinant of a 2×2 matrix $A = \begin{bmatrix} a & c \\ d & b \end{bmatrix}$, denoted by $det(A)$ (or $|A|$), is defined by the formula*

$$det(A) = \begin{vmatrix} a & c \\ d & b \end{vmatrix} = ab - cd.$$

Example 3.28. Find the determinant of a matrix $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution: Using Definition 3.21, the determinant of matrix A is given by

$$det(A) = \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = (5)(4) - (3)(2) = 14.$$

The determinant of a 3×3 matrix can be defined using the determinants of 2×2 matrices.

Definition 3.22 (Determinant of 3×3 Matrix). *Let*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a 3×3 matrix, and A_{ij} (for $i, j = 1, 2, 3$) be the 2×2 submatrix of A obtained by deleting the i^{th}-raw and the j^{th}-column of A. Then determinant of A is denoted by $det(A)$ (or $|A|$), and is defined as:

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}| \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

Example 3.29. Compute the determinant of a matrix $A = \begin{vmatrix} 2 & 4 & 0 \\ 3 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}$.

Solution: Using Definition 3.22, the determinant is given by

$$det(A) = \begin{vmatrix} 2 & 4 & 0 \\ 3 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= 2(-1-2) - 4(3-4) + 0(3+2) = -6 + 4 + 0 = -2.$$

So far we discussed the determinants of 2×2 and 3×3 matrices. Next we define the determinant of an $n \times n$ matrix for each positive integer n.

Definition 3.23 (Minors and Cofactors).
Let $A = (a_{ij})_{n\times n}$, and A_{ij} be the submatrix of A obtained by deleting the i^{th}-raw and j^{th}-column of A for $i, j = 1, 2, 3, ..., n$. Then

(a) The minor for A at location (i, j), denoted by $M_{ij}(A)$, is the determinant of the submatrix A_{ij}. That is, $M_{ij}(A) = det(A_{ij})$.

(b) The cofactor, denoted by $C_{ij}(A)$, for A at location (i, j) is the sighed determinant of the submatrix A_{ij}. That is, $C_{ij}(A) = (-1)^{i+j} det(A_{ij})$.

Remark. In Definition 3.23, the cofactor $C_{ij}(A)$ at location (i, j) can be computed using the following formula:

$$C_{ij}(A) = \begin{cases} det(A_{ij}), \text{if } \ i+j \ \text{ is even} \\ -det(A_{ij}), \text{if } \ i+j \ \text{ is odd.} \end{cases}$$

Example 3.30. Compute the matrix of cofactors for the given matrix.

$$(a) \ \ A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \ \ (b) \ \ B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution: (a) The minors of A are

$$M_{11}(A) = 2, \ \ M_{21}(A) = 1, \ \ M_{12}(A) = -1, \ \ M_{22}(A) = 1,$$

and the cofactors are

$$C_{11}(A) = (-1)^{1+1}M_{11}(A) = (1)(2) = 2, \ \ C_{21}(A) = (-1)^{2+1}M_{21}(A) = (-1)(1) = -1,$$

$$C_{12}(A) = (-1)^{1+2}M_{12}(A) = (-1)(-1) = 1, \ \ C_{22}(A) = (-1)^{2+2}M_{12}(A) = (1)(1) = 1.$$

Thus, the matrix of cofactors for A is

$$[C_{ij}(A)] = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

(b) The minors of B are
$M_{11}(B) = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1,\ M_{21}(B) = \begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0,\ M_{31}(B) = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -2,$
$M_{12}(B) = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = -5, M_{22}(B) = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,\ M_{32}(B) = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1,$
$M_{13}(B) = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2,\ M_{23}(B) = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} = 0$ and $M_{33}(B) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1,$
and the confactors are

$$\begin{aligned} &C_{11}(B) = (-1)^{1+1}M_{11}(B) = 1,\ C_{21}(B) = (-1)^{2+1}M_{21}(B) = 0, \\ &C_{31}(B) = (-1)^{3+1}M_{31}(B) = -2,\ C_{12}(B) = (-1)^{1+2}M_{12}(B) = 5, \\ &C_{22}(B) = (-1)^{2+2}M_{22}(B) = -3,\ C_{32}(B) = (-1)^{3+2}M_{32}(B) = -1, \\ &C_{13}(B) = (-1)^{1+3}M_{13}(B) = -2,\ C_{23}(B) = (-1)^{2+3}M_{23}(B) = 0, \\ &\text{ and } C_{33}(B) = (-1)^{3+3}M_{33}(B) = 1. \end{aligned}$$

Thus, the matrix of cofactors for B is

$$[C_{ij}(B)] = \begin{bmatrix} 1 & 5 & -2 \\ 0 & -3 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

Definition 3.24 (Determinants of $n \times n$ Matrix). *The determinant of a square matrix $A = [a_{ij}]$ of size $n \times n$, denoted by $det(A)$ (or $|A|$), is defined recursively as follows: if $n = 1$ then $det(A) = a_{11}$; otherwise, we suppose that determinants are defined for all square matrices of size less than n and specify that*

$$det(A) = \sum_{k=1}^{n} a_{k1}C_{k1}(A) = a_{11}C_{11}(A) + a_{21}C_{21}(A) + ... + a_{n1}C_{n1}(A), \quad (3.1)$$

where $C_{ij}(A)$ is the (i, j)-th cofactor of A. The formula (3.1) *is called* ***a cofactor expansion across the 1^{st} column of** A.*

Example 3.31. Consider the matrices given in Example 3.30,

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}.$$

The cofactors of matrices A and B, respectively, are given by

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 5 & -2 \\ 0 & -3 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

Now, using Definition 3.24, we have

$$det(A) = a_{11}C_{11} + a_{21}C_{21} = (1)(2) + (-1)(-1) = 3, \text{ and}$$

$$det(B) = b_{11}C_{11} + b_{21}C_{21} + b_{31}C_{31} = (1)(1) + (1)(0) + (2)(-2) = -3.$$

Example 3.32. Compute the determinant of matrix A:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

(a) by expanding the cofactrs across the 1^{st} row

(b) by expanding the cofactrs across the 1^{st} column

Solution: We have the matrix of cofactors $C_{ij}(A)$, given by

$$[C_{ij}(A)] = \begin{bmatrix} -2 & 1 & -2 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

(a) Now, expanding the cofactors across the 1^{st} row, we have

$$det(A) = a_{11}C_{11}(A) + a_{12}C_{12}(A) + a_{13}C_{13}(A) = (1)(-2) + (1)(1) + (0)(-2) = -1.$$

(b) Similarly, expanding cofactors across the 1^{st} column, we have

$$det(A) = a_{11}C_{11}(A) + a_{21}C_{21}(A) + a_{31}C_{31}(A) = (1)(-2) + (0)(0) + (1)(1) = -1.$$

Observe that the determinant has the same value for expansions of cofactors across the 1^{st} row as well as the 1^{st} column. This is true in general, i.e., the determinant value is the same for the expansions of cofactors across any row or any column. This is briefly stated in the following theorem.

Theorem 3.7. *The determinant of an $n \times n$ matrix A can be computed by cofactor expansion across any row or any column. The expansion across i^{th} row is*

$$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}(A) = a_{i1}C_{i1}(A) + a_{i2}C_{i2}(A) + ... + a_{in}C_{in}(A)$$

$$= (-1)^{i+1} a_{i1}|A_{i1}| + (-1)^{i+2} a_{i2}|A_{i2}| + ... + (-1)^{i+n} a_{in}|A_{in}|$$

and the expansion across j^{th} column is

$$det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}(A) = a_{1j}C_{1j}(A) + a_{2j}C_{2j}(A) + ... + a_{nj}C_{nj}(A)$$

$$= (-1)^{1+j} a_{1j}|A_{1j}| + (-1)^{2+j} a_{2j}|A_{2j}| + ... + (-1)^{n+j} a_{nj}|A_{nj}|$$

Remark. In Theorem 3.7, if the matrix A (for instance) is of size 3×3, then the determinants can be easily computed as follows.

(i) The expansion across 2^{nd} row is

$$|A| = -a_{21}|A_{21}| + a_{22}|A_{22}| + a_{23}|A_{23}|.$$

(ii) The expansion across 3^{rd} column is

$$|A| = a_{13}|A_{13}| - a_{23}|A_{23}| + a_{33}|A_{33}|.$$

(iii) The sign $+$ or $-$ can be determined using the pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

(iv) The computation of determinants becomes easier by expanding the cofactors across a row or column with the most zero entries.

Example 3.33. Compute the determinant of matrix A by expanding the cofactors across an appropriate row or column.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution: Here, we observe that the 3^{rd} column has more number of zero entries than any other columns and rows. Thus, the determinant of A (by expanding the cofactors across the

3^{rd} column) is given by

$$det(A) = a_{13}|A_{13}| - a_{23}|A_{23}| + a_{33}|A_{33}| = 0 - 1 + 0 = -1.$$

Properties of determinats: Let A be the square matrix of size n.

1. If an entire row (or an entire column) consists of zeros, then $det(A) = 0$.
2. If two rows (or columns) are equal, then $det(A) = 0$.
3. If one row (or column) is a scalar multiple of another row (or column), then $det(A) = 0$.
4. If A, B and C, respectively, are the upper triangular, lower triangular, and diagonal matrices, given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \; D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix},$$

then

$$det(A) = a_{11}a_{22}a_{33}, \;\; det(B) = b_{11}b_{22}b_{33}, \;\; \text{and} \;\; det(D) = d_{11}d_{22}d_{33}.$$

That is, the determinants of the triangular and diagonal matrices are simply the products of the entries in the main diagonal.

Example 3.34. Determine the determinants of the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 1 & 3 \\ -1 & -1 & -3 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution: We have, $det(A) = 0$ (since the entire second row of matrix A consists of zeros), $det(B) = 0$ (since the entire third column of matrix A consists of zeros), $det(C) = 0$ (since the first and third rows of C are equal), and $det(D) = 0$ (since the second row of D is a scalar multiple of the first row).

Example 3.35. Compute the determinants of the following matrices.

$$A = \begin{bmatrix} 4 & 3 & -6 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 1 & 5 \end{bmatrix}, \; D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Solution: Using the properties of determinants, we have

$$det(A) = \begin{vmatrix} 4 & 3 & -6 \\ 0 & 2 & 9 \\ 0 & 0 & 3 \end{vmatrix} = (4)(2)(3) = 24, \;\; det(B) = \begin{vmatrix} 3 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 1 & 5 \end{vmatrix} = (3)(4)(5) = 60, \;\; \text{and}$$

$$det(D) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix} = (2)(3)(5) = 30.$$

Theorem 3.8. *For any square matrix A, $det(A) = det(A^t)$ (Transposition doesn't alter determinants).*

Example 3.36. For the given matrix A, verify that $det(A) = det(A^t)$.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution: The transpose of matrix A is given by

$$A^t = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Now, we have the determinants of A and A^t are

$$det(A) = 2, \text{ and } det(A^t) = 2.$$

Thus, $det(A) = det(A^t)$.

Theorem 3.9 (Effects of elementary row operations).

I. *If matrix B is obtained from a square matrix A by interchanging any two rows (i.e., $R_i \leftrightarrow R_j$), then $det(B) = -det(A)$. (**Interchanging**)*

II. *If matrix B is obtained from a square matrix A by multiplying the i^{th} row by a nonzero scalar α (i.e., $R_i \to \alpha R_i$), then $det(B) = \alpha det(A)$. (**Scaling**)*

III *If matrix B is obtained from a square matrix A by adding scalar multiple of one row to the other (i.e., $R_i \to R_i + \alpha R_j$), then $det(B) = det(A)$. (**Replacement**)*

Example 3.37. Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ be the given matrix with $det(A) = -2$.

(a) If a matrix B is obtained from A by interchanging the first and second rows (i.e., $R_1 \leftrightarrow R_2$), then we have

$$det(B) = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 2.$$

Thus, $det(B) = -det(A)$. Here, we observe that if the row interchanging has been made two times, then $det(B) = (-1)^2 det(A) = det(A)$. In general, if the row interchanging has been made n times, then $det(B) = (-1)^n det(A)$. Thus, $det(B) = det(A)$ if n is even, and $det(B) = -det(A)$ if n is odd.

(b) If a matrix B is obtained from A by multiplying the second row by 4 (i.e., $R_2 \to 4R_2$), then we have

$$B = \begin{vmatrix} 3 & 1 & 0 \\ 4 & 0 & 4 \\ 0 & 1 & -1 \end{vmatrix} = -8.$$

Thus, $det(B) = 4det(A)$. If each row of matrix A is multiplied by 4, then we have

$$det(B) = 4^3 det(A).$$

More generally, if A is an $n \times n$ matrix, and B is obtained by multiplying each row of A by a nonzero scalar c, then we have $det(B) = det(cA) = c^n det(A)$.

(c) If a matrix B is obtained by replacing row 2 (i.e., $R_2 \to R_2 + 2R_1$), then $det(B) = \begin{vmatrix} 3 & 1 & 0 \\ 7 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 2$. Thus, $det(B) = det(A)$.

> *Remark.* Property (III) of determinants in Theorem 3.9 is particularly more interesting, since it doesn't change the determinant of the original matrix. This property can be used to transform the given matrix into triangular matrix (upper or lower) for which the computation of determinants is much easier than computing the determinant of the original matrix directly, which is tedious and computationally inefficient.

Example 3.38. Compute the determinants of the matrices A and B using elementary row operations.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & 5 & 6 \\ 1 & 3 & 5 & 3 \\ 1 & 1 & 3 & 6 \end{bmatrix}$$

Solution:

(a) Consider the given matrix A. Applying the row replacement; $R_2 \to R_2 - 2R_1$ and then $R_3 \to R_3 - R_2$, we obtain the following upper triangular matrix.

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \end{bmatrix}$$

Therefore, by Theorem 3.9 we have $det(A) = det(\tilde{A}) = (1)(1)(7) = 7$.

(b) Similarly, by applying the row replacement

$$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - R_1,$$

we obtain the following row equivalent matrix.

$$\tilde{B} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Now, the determinant of the matrix $\tilde{B}$ (by expanding the cofactors across the 1^{st} column and using the determinant of matrix A computed above) is given by

$$\tilde{B} = \begin{vmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 4 \end{vmatrix} = (1)(7) = 7.$$

Therefore, by Theorem 3.9 we have $det(B) = det(\tilde{B}) = 7$.

Theorem 3.10 (Product Rule).
If A and B are two matrices for which the product AB is defined, then

$$det(AB) = det(A)det(B).$$

Example 3.39. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ be the given matrices. Then verify that

$$det(AB) = det(A)det(B).$$

Solution: Here, we have

$$AB = \begin{bmatrix} 4 & 8 \\ 5 & -4 \end{bmatrix},\ det(AB) = -56,\ det(A) = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7, \text{ and } det(B) = \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = 8.$$

Thus,

$$det(A)det(B) = (-7)(8) = -56 = det(AB).$$

Definition 3.25 (Definition of rank using Determinant). *Let A be an $m \times n$ matrix. Then $rank(A) = r$, where r is the largest number such that some $r \times r$ submatrix of A has a nonzero determinant.*

Example 3.40. Compute the the rank of matrix $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -3 & 2 & 0 \end{bmatrix}$ using determinants.

Solution: Observe that, the largest possible size of any square submatrix of A is 2×2. We have (say) a submatrix $\begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$ (which is obtained by deleting the last two columns of A) with $\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3 \neq 0$. Therefore, $rank(A) = 2$.

Exercise 3.7.

1. Compute the determinants of the following matrices using elementary row operations.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 5 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}$$

2. Compute the determinants of the following matrices by expanding cofactors across any appropriate row or column.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 0 \\ 6 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & 1 \\ 5 & 0 & 0 & 0 \\ 4 & 1 & 1 & 2 \end{bmatrix}, C = \begin{bmatrix} 0 & 3 & 1 & 0 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 5 & 1 & -1 & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \end{bmatrix}$$

3. Compute the matrix of cofactors for the given matrices.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 4 \\ 2 & 0 & 3 \end{bmatrix}, \ C = \begin{bmatrix} 3 & 2 & 1 & 1 \\ -1 & 0 & 2 & 0 \\ 4 & 1 & -1 & 0 \\ 3 & 0 & 1 & 0 \end{bmatrix}$$

4. Determine the ranks of the following matrices using determinants.

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 2 & 1 & 3 & 2 & 4 \\ -1 & 2 & 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

3.8 Adjoint and Inverse of a Matrix

The inverses of matrices are useful to solve linear systems. In this section, we define the inverse of a matrix, we discuss different methods to compute an inverse, and the properties of inverses.

Definition 3.26 (Adjoint of a Matrix). *Let A be an $n \times n$ matrix. If $[C_{ij}(A)]$ denotes the matrix of cofactors for A, then the adjugate (or adjoint) matrix of A, denoted by $Adj(A)$, is defined by the formula*

$$Adj(A) = [C_{ij}(A)]^t$$

That is, adjoint of matrix A is the transpose of the matrix of cofactors for A.

Example 3.41. Compute the adjoints of the given matrices.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

Solution: The matrix of cofactors for A is

$$[C_{ij}(A)] = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Thus, the adjoint of matrix A is

$$Adj(A) = [C_{ij}(A)]^t = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

The matrix of cofactors for B is given by

$$[C_{ij}(B)] = \begin{bmatrix} 1 & 5 & -2 \\ 0 & -3 & 0 \\ -2 & -1 & 1 \end{bmatrix}.$$

Thus, the adjoint of matrix B is

$$Adj(B) = [C_{ij}(B)]^t = \begin{bmatrix} 1 & 0 & -2 \\ 5 & -3 & -1 \\ -2 & 0 & 1 \end{bmatrix}.$$

Definition 3.27 (Inverse of a Matrix). *Let A be an $n \times n$ square matrix. The inverse of matrix A is an $n \times n$ matrix B such that*

$$AB = I_n = BA,$$

where I_n is the $n \times n$ identity matrix. If such a Matrix B exists, then the matrix A is said to be ***invertible*** *(or* ***nonsingular****), and its inverse is denoted by A^{-1} (i.e. $B = A^{-1}$). A matrix that does not have an inverse is said to be* ***noninvertible*** *(or* ***singular****).*

Example 3.42. Consider the following matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = BA$$

That is, the products AB and BA give us the identity matrix I_2. Therefore, matrix B is the inverse of A i.e., $A^{-1} = B$.

Similarly, we have

$$CD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = DC.$$

Thus, the matrix D is the inverse of C i.e., $C^{-1} = D$.

Theorem 3.11. *Let A be an $n \times n$ matrix. If A is invertible (non singular) then $det(A) \neq 0$, and the inverse A^{-1} is given by the formula*

$$A^{-1} = \frac{1}{det(A)} Adj(A).$$

Example 3.43. Compute the inverse of the given matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: We have, $det(A) = 6$,

$$[C_{ij}(A)] = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ and } Adj(A) = [C_{ij}(A)]^t = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Therefore, by Theorem 3.11, we have

$$A^{-1} = \frac{1}{det(A)} Adj(A) = \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Theorem 3.12 (**Laws of Inverse**). *Let A, B, C be matrices of appropriate sizes so that the following multiplications make sense, I is a suitably sized identity matrix, and α a nonzero scalar. Then*

i. If the matrix A is invertible, then it has one and only one inverse, A^{-1}.

ii. If A is invertible matrix of size $n \times n$, then so is A^{-1} and hence, $(A^{-1})^{-1} = A$.

iii If any two of the three matrices A, B, AB are invertible, then so is the third, and moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

iv If the matrix A is invertible, then so is αA. Moreover, $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.

v If the matrix A is invertible, then so is A^t. Moreover $(A^t)^{-1} = (A^{-1})^t$.

vi Suppose A is invertible. If $AB = AC$ or $BA = CA$, then $B = C$.

Example 3.44. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ be the given matrix. Then we have

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \text{ and } (A^{-1})^t = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Now,

(a) $2A = \begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}$ and $(2A)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} = \frac{1}{2}A^{-1}$. Thus, we have $(2A)^{-1} = \frac{1}{2}A^{-1}$.

(b) $A^t = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and $(A^t)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Thus, we have $(A^t)^{-1} = (A^{-1})^t$.

Computation of Inverse Using Elementary Row Operations: Gauss-Jordan Elimination

Let A be an $n \times n$ invertible matrix and I_n be the identity matrix of size $n \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & \dots & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Then the inverse A^{-1} can be obtained using elementary row operations as follows.

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

1. Write the $n \times 2n$ matrix that consists of A on the left and the $n \times n$ identity matrix I_n on the right to obtain $[A|I_n]$. This process is called adjoining matrix I_n to matrix A.

2. If possible, row reduce A to I_n using elementary row operations on the entire matrix $[A|I_n]$. The result will be the matrix $[I_n|A^{-1}]$. If this is not possible, then A is noninvertible (or singular).

3. Check your work by multiplying to see that $AA^{-1} = I_n = A^{-1}A$.

Example 3.45. Compute the inverses of the given matrices using Gauss-Jordan Elimination.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$. Then we have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] R_2 \to R_2 + (-3)R_1 \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -3 & 1 \end{array}\right]$$

$$R_2 \to \frac{1}{5}R_2 \quad \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array}\right] R_1 \to R_1 + R_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array}\right].$$

Therefore, the transformed matrix is

$$[I_2|A^{-1}] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{1}{5} \end{array}\right]$$

and hence, the inverse of matrix A is given by $A^{-1} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$.

Similarly, for $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] R_2 \to \frac{1}{2}R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to \frac{1}{3}R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

Therefore, the transformed matrix is

$$[I_3|A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

Exercise 3.8.

1. For the given matrices A and B, compute the adjoint matrices.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 4 \\ 2 & 0 & 3 \end{bmatrix}$$

2. Compute the inverse of the given matrix (if it exists).

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

3. Compute the inverse (if it exists) of the given matrix using elementary row operations.

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 1 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

3.9 System of Linear Equations

Consider an oil refinery that produces gasoline, kerosene and jet fuel form light crude oil and heavy crude oil. The refinery produces 0.3, 0.2 and 0.4 of gasoline, kerosene and jet fuel, respectively, per barrel of light crude oil. And it produces 0.2, 0.4 and 0.3 of gasoline, kerosene and jet fuel, respectively, per barrel of heavy crude oil. This is shown in Table 2. Note that 10% of each of the crude oil is lost during the refining process.

Table 2

	Gasoline	Kerosene	Jet fuel
Light crude oil	0.3	0.2	0.4
Heavy crude oil	0.2	0.4	0.3

Suppose that the refinery has contracted to deliver 550 barrels of gasoline, 500 barrels of kerosene, and 750 barrels of jet fuel. The problem is to find the number of barrels of each crude oil that satisfies the demand.

If l and h represent the number of barrels of light and heavy crude oil, respectively, then the given problem can be expressed as a system of linear equations

$$\begin{aligned} 0.3l + 0.2h &= 550 \\ 0.2l + 0.4h &= 500 \\ 0.4l + 0.3h &= 750 \end{aligned}$$

The given linear system has three equations and two unknowns. The matrix

$$\begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}$$

is known as the **coefficient matrix** of the system, and the right side of the system is a matrix

$$\begin{bmatrix} 550 \\ 500 \\ 750 \end{bmatrix}.$$

With the column vector of unknowns $\begin{bmatrix} l \\ h \end{bmatrix}$, the above information can be organized in matrix form

$$\begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} l \\ h \end{bmatrix} = \begin{bmatrix} 550 \\ 500 \\ 750 \end{bmatrix}.$$

Example 3.46. Consider the following system of two equations and two unknowns x, y

$$\begin{aligned} ax + by &= b_1 \\ cx + dy &= b_2 \end{aligned}.$$

If we interpret (x, y) as coordinates in the xy-plane, then each of the two equations represents a straight line, and (x^*, y^*) is a solution if and only if the point P with coordinates x^*, y^* lies on both lines. In this case, there are three possible cases: there exists only one solution if the lines intersect (see Figure 1 a), there are infinitely many solutions if the lines coincide (see Figure 1 b) and the system has no solution if the lines are parallel (see Figure 1 c).

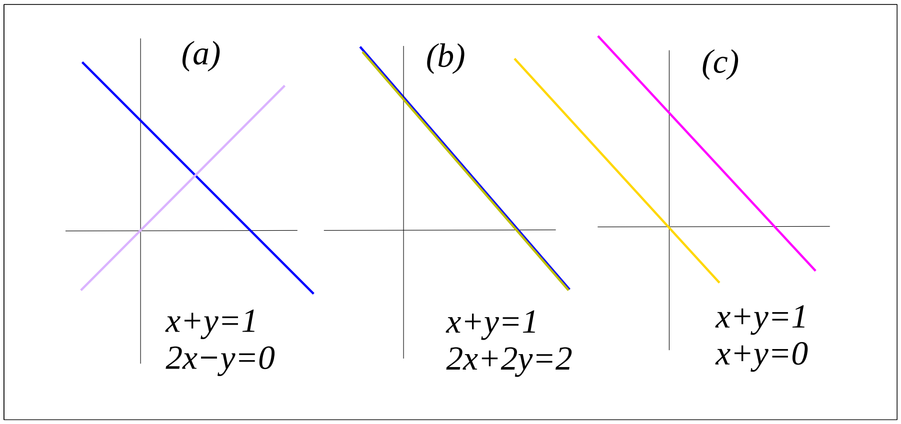

Figure 1: In this figure: (a) represents the case where the lines intersect (b) represents the case where the lines coincide (c) represents the case where the lines are parallel

Let us briefly discuss the three different cases: In part (a) the linear system is given by

$$\begin{aligned} x+y&=1\\ 2x-y&=0. \end{aligned}$$

This system has only solution, namely (x,y)=$(\frac{1}{3},\frac{2}{3})$.

In part (b) the linear system is given by

$$\begin{aligned} x+y&=1\\ 2x+2y&=2. \end{aligned}$$

This system has infinitely many solutions. In fact, the point $(\alpha, 1-\alpha)$ is a solution for each real number α.

And finally, in part (c) the linear system is given by

$$\begin{aligned} x+y&=1\\ x+y&=0, \end{aligned}$$

which has no solutions, since the expressions in the left side of the two equations are the same, but different values in the right side of the two equations.

Definition 3.28. *A linear system (or system of linear equations) of m-**equations** in n-**unknowns** $x_1, x_2, x_3, ..., x_n$ is a set of equations of the form*

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m,
\end{aligned} \tag{3.2}$$

*where a_{ij}'s (for $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n$), are given numbers, called the **coefficients** of the system, and $b_1, b_2, b_3, ..., b_m$ on the right side are also numbers.*

A solution of (3.2) is a set of numbers $x_1, x_2, x_3, ..., x_n$ that satisfies all the m-equations simultaneously.

Matrix Form of a Linear System

From the definition of matrix multiplication, we see that the m-**equations** of (3.2) may be written as a single vector equation

$$Ax = b, \tag{3.3}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

are known as the coefficient matrix, the column vector of unknowns and the column vector of numbers, respectively. We assume that the coefficients a_{ij} are not all zero, so that A is not a zero matrix. Note that x has n components, whereas b has m components.

For the system of linear equations in (3.2), precisely one of the statements below is true:

1. **It admits a unique Solution**: There is one and only one vector $x = (x_1, x_2, x_3, ..., x_n)$ that satisfies all the m-equations simultaneously (the system is **consistent**).

2. **It has infinitely Many Solutions:** There are infinitely many different values of x that satisfy all the m-equations simultaneously (the system is said to be **consistent**).

3. **Has no Solution**: There is no vector x that satisfies all equations simultaneously, or the solution set is empty (the system is said to be **inconsistent**).

3.9.1 Gaussian Elimination

Gaussian elimination, also known as row reduction, is used for solving a system of linear equations. It is usually understood as a sequence of elementary row operations performed on the corresponding matrix of coefficients.

Consider the linear system given in (3.2). The augmented matrix which represents the system is given by

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ . & . & \dots & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Then, the idea here is, we solve the linear system whose augmented matrix is in row echelon form, which is row equivalent to the original system. And, we have the following theorem on the row equivalent linear systems.

Theorem 3.13. *Row-equivalent linear systems have the same set of solutions.*

Thus, if the augmented matrix is initially in row echelon form, then we simply solve it by using back substitution. If it is not, then first rewrite it as a row equivalent system whose augmented matrix is in its row echelon form, and then apply Theorem 3.13.

Example 3.47. Rewrite the following linear system as a row equivalent system, and then solve it.

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= 4. \end{aligned}$$

Solution: Here, the augmented matrix of the given system is

$$[A|b] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 2 & 4 \end{array}\right],$$

which has row echelon form (after a sequence of elementary operations)

$$\widetilde{[A|b]} = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

Thus, the row equivalent system is

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_2 &= 1. \end{aligned}$$

Clearly, solving the above linear system (whose augmented matrix is in row echelon form) is much easier than solving the original system. The only solution of the linear system (represented by an augmented matrix in row echelon form) is $(x_1, x_2) = (2, 1)$. And, hence by Theorem 3.13, a vector $(x_1, x_2) = (2, 1)$ also solves the original linear system.

Gaussian Elimination:

(a) Write the augmented matrix for the linear system.

(b) Use elementary row operations to rewrite the matrix in row echelon form.

(c) Write the system of linear equations corresponding to the matrix in row echelon form, and use back-substitution to find the solution.

Example 3.48. Consider an oil refinery's problem which is given as a system of linear equations

$$\begin{aligned} 0.3l + 0.2h &= 550 \\ 0.2l + 0.4h &= 500 \\ 0.4l + 0.3h &= 750 \end{aligned}$$

where l and h represent the number of barrels of light and heavy crude oil, respectively. The augmented matrix of the given linear system is

$$[A|b] = \left[\begin{array}{cc|c} 0.3 & 0.2 & 550 \\ 0.2 & 0.4 & 500 \\ 0.4 & 0.3 & 750 \end{array}\right],$$

where

$$A = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 550 \\ 500 \\ 750 \end{bmatrix}.$$

And the matrix in row echelon form is given by

$$\widetilde{[A|b]} = \left[\begin{array}{cc|c} 0.1 & 0.2 & 250 \\ 0 & 0.1 & 50 \\ 0 & 0 & 0 \end{array}\right].$$

Now, rewriting the given linear system as row equivalent system we have

$$\begin{aligned} 0.1l + 0.2h &= 250 \\ 0.1h &= 50. \end{aligned}$$

The only solution of the above system (in row echelon form) is $(l, h) = (1500, 500)$, which is also a solution for the original system. Thus, an oil refinery needs 1500 barrels of light crude oil and 500 barrels of heavy crude oil in order to satisfy the demand.

Example 3.49. Solve the given linear system by using the method of Gaussian elimination.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ x_1 - x_2 - 2x_3 &= -1. \end{aligned}$$

Solution: The augmented matrix representing the given system is

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 1 & -1 & -2 & -1 \end{array}\right].$$

Now, by replacing R_2 (i.e., $R_2 \to R_2 - R_1$), we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -3 & -3 \end{array}\right]$$

and by Scaling R_2 ($i.e., R_2 \to (-\frac{1}{3})R_2$), we have

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right].$$

The last matrix is in its row echelon form, and hence the row equivalent system is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ x_2 + x_3 &= 1. \end{aligned}$$

In this case, the system has infinitely many solutions, and the set of solutions is be given by

$$\{(1-\alpha, \alpha, 1-\alpha) : \alpha \in R\}.$$

Example 3.50. Solve the following system of linear equations using the method of Gaussian elimination.

$$\begin{aligned} 4x_2 + 3x_3 &= 8 \\ 2x_1 - x_3 &= 2 \\ 3x_1 + 2x_2 &= 5 \end{aligned}$$

Solution: The augmented matrix of the given system is

$$[A|b] = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Applying the following elementary row operations:
$R_1 \leftrightarrow R_3$ (Interchanging R_1 and R_3)

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \end{array}\right]$$

$R_2 \leftrightarrow R_3$ (Interchanging R_2 and R_3)

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \end{array}\right]$$

$R_3 \to R_3 + (-\frac{2}{3})R_1$ (Replacing R_3)

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 0 & -\frac{4}{3} & -1 & -\frac{4}{3} \end{array}\right]$$

$R_3 \to R_3 + \frac{1}{3}R_2$ (Replacing R_3)

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 5 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & \frac{4}{3} \end{array}\right].$$

The last matrix is in row echelon form, and hence the row equivalent system is given by

$$\begin{aligned} 3x_1 + 2x_2 &= 5 \\ 4x_2 + 3x_3 &= 8 \\ 0 &= \tfrac{4}{3} \end{aligned}$$

We observe that the last equation in the linear system above is a contradiction to the fact that $0 \neq \frac{4}{3}$. Consequently, the given linear system has no solution.

Theorem 3.14. *Consider the system of linear equations in* (3.2)*. If A and b are the matrices of coefficients and the column vector of numbers, respectively. Then the following statements are true.*

(i) If $rank(A) = rank([A|b]) =$ number of unknowns, then the linear system has only one solution.

(ii) If $rank(A) = rank([A|b]) <$ number of unknowns, then the linear system has infinitely many solutions.

(iii) If $rank(A) < rank([A|b])$, then the linear system has no solution.

Remark.

(a) From Theorem 3.14, we observe that the linear system (3.2) has no solution if an echelon form of the augmented matrix has a row of the form $[0, 0, ..., 0 \quad b]$ with b nonzero.

(b) A linear system has unique solution when there are no free variable, and it has infinitely many solutions when there is at least one free variable.

Example 3.51. Use matrix rank to determine the number of solutions for the system.

$$(a) \quad \begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_2 + 4x_3 &= 2 \\ 2x_1 + 7x_3 &= 5 \end{aligned} \quad , \quad (b) \quad \begin{aligned} x_1 + x_2 + 2x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 2 \end{aligned} \qquad (c) \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_2 + 2x_3 &= -2 \\ -2x_2 - 2x_3 &= 3 \end{aligned}$$

Solution:

(a) We have a linear system

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_2 + 4x_3 &= 2 \\ 2x_1 + 7x_3 &= 5 \end{aligned}$$

and the augmented matrix given by

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 7 & 5 \end{array}\right].$$

After a sequence of elementary row operations, we obtain its row echelon form

$$\widetilde{[A|b]} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

From the transformed matrix, we can see that the matrix A in its row echelon form is

$$\widetilde{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, we have $rank(A) = rank([A|b]) =$ *number of unknowns*. Hence, the given linear system has only one solution.

(b) We have a linear system

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 2 \end{aligned}$$

In this case, the augmented matrix and its row echelon form, respectively, are given by

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{array}\right] \text{ and } \widetilde{[A|b]} = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix A in its row echelon form is

$$\widetilde{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, the matrices $\widetilde{A}$, and $\widetilde{[A|b]}$ have only two nonzero rows. Thus, $rank(A) = rank([A|b]) <$ *number of unknowns*. Therefore, by Theorem 3.14, the given system has infinitely many solutions.

(c) Here, we have a linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_2 + 2x_3 &= -2 \\ -2x_2 - 2x_3 &= 3. \end{aligned}$$

The augmented matrix $[A|b]$ and its row echelon form $\widetilde{[A|b]}$, respectively, are given by

$$[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & -2 & -2 & 3 \end{array}\right] \text{ and } \widetilde{[A|b]} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Here, the number of nonzero rows of the row echelon form of A and that of $[A|b]$ are 2 and 3, respectively. Therefore, the given linear system has no solution.

Exercise 3.9. Solve the following linear systems using the method of Gaussian elimination.

$$(a) \quad \begin{aligned} -x_1 + x_2 &= 4 \\ -2x_1 + x_2 &= 0 \end{aligned}$$

$$(b) \quad \begin{aligned} x_1 + x_2 &= -1 \\ x_1 - x_2 &= 0 \\ -2x_1 + x_2 &= 3 \end{aligned}$$

$$(c) \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 4x_1 + 5x_2 + 6x_3 &= 3 \\ 7x_1 + 8x_2 + 9x_3 &= 6. \end{aligned}$$

$$(d) \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_2 + 3x_2 - 2x_3 &= 0 \end{aligned}$$

3.9.2 Cramer's rule

Cramer's Rule is a method for solving linear systems where the number of equations and the number of unknowns are equal. Cramer's rule relies on determinants. Consider the following linear system of n-**equations** in n-**unknowns** $x_1, x_2, x_3, ..., x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n \end{aligned} \tag{3.4}$$

which has a matrix notation

$$Ax = b.$$

Let us define the determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}, \quad D_j = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \dots & a_{1n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & b_n & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix} \tag{3.5}$$

for $j = 1, 2, 3, ..., n$. Here, D is the determinant of the coefficient matrix A, and for each j D_j represents the determinant of a matrix which is obtained from A after replacing the j-th column by the column vector b.

Theorem 3.15 (Cramer's rule)**.**

(a) If a linear system (3.4) *of n-equations in the same number of unknowns $x_1, x_2, x_3, ..., x_n$, has a nonzero coefficient determinant $D = |A|$, then the system has precisely one solution. This solution is given by*

$$x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, ..., x_n = \frac{D_n}{D}$$

where D and D_j for $j = 1, 2, 3, ..., n$ are defined in (3.5).

(b) If the system (3.4) *is homogeneous and $D \neq 0$, then it has only the trivial solution $x_1 = 0, x_2 = 0, x_3 = 0, ..., x_n = 0$. If $D = 0$ the homogeneous system also has nontrivial solutions.*

Example 3.52. Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

Solution: Here, the coefficient matrix A and the column vector b, respectively, are

$$\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}, \text{ and } \begin{bmatrix} 10 \\ 11 \end{bmatrix}.$$

And the determinants D, D_1, D_2 are

$$D = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = (-20) - (-6) = -14, \quad D_1 = \begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix} = (-50) - (-22) = -28,$$

$$D_2 = \begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix} = (44) - (30) = 14.$$

Therefore, by Theorem 3.15, the unique solution of the given linear system is

$$(x_1, x_2) = \left(\frac{D_1}{D}, \frac{D_2}{D} \right) = (2, -1).$$

Example 3.53. Solve the following system of linear equations using Cramer's rule

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + x_3 &= 1 \end{aligned}$$

Solution: With the coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and column vector } b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

the determinants D, D_1, D_2 and D_3 are computed as follows;

$$D = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 1, \quad D_1 = \begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 1, \quad D_2 = \begin{vmatrix} 2 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

and

$$D_3 = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 3.$$

Thus, by Theorem 3.15, the only solution of the given linear system is

$$(x_1, x_2, x_2) = \left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D} \right) = (1, 2, 3).$$

Remark. Cramer's rule doesn't work if the determinant of the coefficient matrix is zero or the coefficient matrix is not square.

Exercise 3.10. Solve the following linear systems using Cramer's rule (if possible).

$$(a) \quad \begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

$$(b) \quad \begin{aligned} -x_1 + 2x_2 - 3x_3 &= 1 \\ 2x_1 + x_3 &= 0 \\ 3x_1 - 4x_2 + 4x_3 &= 2. \end{aligned}$$

$$(c) \quad \begin{aligned} x_1 \qquad &= 7 \\ 2x_2 \quad &= 8 \\ 3x_3 &= 24. \end{aligned}$$

3.9.3 Inverse method

The Inverse method is one of the important methods to solve a linear system with n equations in n unknowns.

Example 3.54. Consider a linear system

$$\begin{aligned} x - y &= 1 \\ x + y &= 3. \end{aligned}$$

Using matrix notation, it can be rewritten as

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \tag{3.6}$$

And if we denote the coefficient matrix by A, then we have

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \text{ and } A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Now, multiplying (from the left) both sides of equation (3.6) by A^{-1}, we have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

And using the fact $A^{-1}A = I_2$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \text{ This implies } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, $(x, y) = (2, 1)$ is the only solution of the given system of linear equations. This shows the usefulness of the matrix inverse to solve linear systems.

Consider the following linear system with n**-equations** in n**-unknowns** $x_1, x_2, x_3, ..., x_n$;

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n. \end{aligned} \tag{3.7}$$

The matrix notation of the linear system (3.7) is

$$Ax = b,$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Theorem 3.16 (Inverse Method). *If A is an invertible matrix, then for each $b \in R^n$, the linear system $Ax = b$ has a unique solution, which is given by*

$$x = A^{-1}b.$$

Example 3.55. Solve the following system of linear equations using matrix inverse method.

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ 3x_1 + 2x_2 &= 12 \end{aligned}$$

Solution: The matrix of coefficients A, the inverse A^{-1}, and the column vector b, respectively, are given by

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 12 \end{bmatrix}.$$

Thus, by Theorem 3.16, the only solution of the given linear system is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}b = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

.

Example 3.56. Use matrix inversion to solve the following linear system.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ x_1 + 2x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

Solution: The coefficient matrix A, the column vector b and the inverse A^{-1}, respectively, are given by

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, by Theorem 3.16, the unique solution of the given linear system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 2 & -3 & -2 \\ -1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

Exercise 3.11. Solve the following linear systems using the method of matrix inversion (if possible).

$$(a) \quad \begin{aligned} 3x_1 + 4x_2 &= -4 \\ 5x_1 + 3x_2 &= 4 \end{aligned}$$

$$(b)\quad \begin{aligned} 4x_1 - x_2 - x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 - 2x_3 &= -1. \end{aligned}$$

$$(c)\quad \begin{aligned} 3x_1 \qquad\qquad &= 12 \\ 4x_2 \qquad &= 16 \\ 5x_3 &= 20. \end{aligned}$$

Review exercises

1. For every square matrix A, show that $A + A^t$ is symmetric.

2. Given matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

   (i) Compute the products $A(BC)$, $(AB)C$, and verify that $A(BC) = (AB)C$.

   (ii) Compute the products $\alpha(AB)$, $(\alpha A)B$, $A(\alpha B)$, and verify that

$$\alpha(AB) = (\alpha A)B = A(\alpha B).$$

3. A fruit grower raises two crops, apples and peaches. The grower ships each of these crops to three different outlets. In the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}$$

   a_{ij} represents the number of units of crop i that the grower ships to outlet j. The matrix $B = \begin{bmatrix} \$3.5 & \$6.00 \end{bmatrix}$ represents the profit per unit. Find the product BA and state what each entry of the matrix represents.

4. A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. In the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}$$

   a_{ij} represents the number of guitars of type i produced at factory j in one day. Find the production levels when production increases by 20%.

5. Find the value of x for which the matrix is equal to its own inverse

$$(a)\ \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix} \quad (b)\ \begin{bmatrix} 2 & x \\ -1 & -2 \end{bmatrix} \quad (c)\ \begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$$

6. If $A = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$, then

    i. show that $A = A^{-1}$

    ii. show that $A^n = \begin{bmatrix} cos(n\theta) & sin(n\theta) \\ -sin(n\theta) & cos(n\theta) \end{bmatrix}$.

7. If $A = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}$, and $B = \begin{bmatrix} cos(\phi) & sin(\phi) \\ -sin(\phi) & cos(\phi) \end{bmatrix}$, then show that

$$AB = \begin{bmatrix} cos(\theta+\phi) & sin(\theta+\phi) \\ -sin(\theta+\phi) & cos(\theta+\phi) \end{bmatrix}.$$

8. Determine the values of α for which the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & \alpha \end{bmatrix}$ is invertible and find A^{-1}.

9. Show that if A is invertible, then so is A^m for every positive integer m; moreover, $(A^m)^{-1} = (A^{-1})^m$.

10. If A and B are $n \times n$ matrices with A is invertible, then show that

$$(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B).$$

11. Solve the following systems of linear equations using Gaussian elimination

$$(a) \quad \begin{array}{l} x_1 - x_2 + 2x_3 = 4 \\ x_1 + x_3 = 6 \\ 2x_1 - 3x_2 + 5x_3 = 4 \\ 3x_1 + 2x_2 - x_3 = 1 \end{array}$$

$$(b) \quad \begin{array}{l} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 = -4 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{array}$$

$$(c) \quad \begin{array}{l} 2x_1 + x_2 - x_3 + 2x_4 = -6 \\ 3x_1 + 4x_2 + x_4 = 1 = 2 \\ x_1 + 5x_2 + 2x_3 + 6x_4 = -3 \\ 5x_1 + 2x_2 - x_3 - x_4 = 1 \end{array}$$

12. Use Cramer's rule (if possible) to solve the following linear systems.

$$(a) \quad \begin{array}{l} x_1 + 2x_2 = 5 \\ -x_1 + x_2 = 1 \end{array}$$

$$(b) \quad \begin{aligned} 4x_1 - x_2 - x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 10 \\ 5x_1 - 2x_2 - 2x_3 &= -1 \end{aligned}$$

$$(c) \quad \begin{aligned} 4x_1 - 2x_2 + 3x_3 &= -2 \\ 2x_1 + 2x_2 + 5x_3 &= 16 \\ 8x_1 - 5x_2 - 2x_3 &= 4 \end{aligned}$$

13. Use matrix inversion method (if possible) to solve the following linear systems.

$$(a) \quad \begin{aligned} 2x_1 + 3x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + x_3 &= -2 \end{aligned}$$

$$(b) \quad \begin{aligned} 2x_1 + 3x_2 + x_3 &= 4 \\ 3x_1 + 3x_2 + x_3 &= 8 \\ 2x_1 + 4x_2 + x_3 &= 5 \end{aligned}$$

$$(c) \quad \begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + 2x_2 + 5x_3 &= 0 \\ 8x_1 - 5x_2 - 2x_3 &= 0 \end{aligned}$$