**Chapter 3**

**Functions**

Our everyday lives are filled with situations in which we encounter relationships between two sets. For example,

* To each automobile, there corresponds a license plate number
* To each circle, there corresponds a circumference
* To each number, there corresponds its square

In order to apply mathematics to a variety of disciplines, we must make the idea of a “relationship” between two sets mathematically precise.

On completion of this chapter students will be able to:

* understand the notion of relation and function
* determine the domain and range of relations and functions
* find the inverse of a relation
* define polynomial and rational functions
* perform the fundamental operations on polynomials
* find the inverse of an invertible function
* apply the theorems on polynomials to find the zeros of polynomial functions
* apply theorems on polynomials to solve related problems
* sketch and analyze the graphs of rational functions
* define exponential, logarithmic, trigonometric and hyperbolic functions
* sketch the graph of exponential, logarithmic, trigonometric and hyperbolic functions
* use basic properties of logarithmic, exponential, hyperbolic and trigonometric functions to solve physical problems

In this chapter, we first look at the definitions of relations and functions, and study real valued functions and their properties, types of functions, polynomial functions, zeros of polynomial functions, rational functions and their graphs, logarithmic, exponential, trigonometric and hyperbolic functions and their graphs. Let’s begin with the review of relations and functions.

3.1. **Review of relations and functions**

After completing this section, the student should be able to:

* define Cartesian product of two sets
* understand the notion of relation and function
* know the difference between relation and function
* determine the domain and range of relations and functions
* find the inverse of a relation

The student is familiar with the phrase ordered pair. In the ordered pair  and ;  and  are the first coordinates while  and  are the second coordinates.

* **Cartesian Product**

Given sets  and . Then, the set  is the Cartesian product of  and , and it is denoted by .

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| **Definition 3.1:** Suppose  and  are sets. The Cartesian product of  and , denoted by , is the set which contains every ordered pair whose first coordinate is an element of  and second coordinate is an element of , i.e.  and . |

**Example 3.1**: For  and , we have

1. , and
2. .

**Example 3.2**: Let  and . Then,

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From example 3.1, we can see that  and  are not equal. Recall that two sets are equal if one is a subset of the other and vice versa. To check equality of Cartesian products we need to define equality of ordered pairs.

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| **Definition 3.2: (Equality of ordered Pairs)**  Two ordered pairs  and  are equal if and only if  and . |

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| **Definition 3.3: (Relation from  into )**  If  and  are sets, any subset of  is called a relation from *A* into *B*. |

Suppose *R* is a relation from a set *A* to a set *B*. Then, *R*⊆ *A*×*B* and hence for each , we have either  or . If , we say “*a* is *R-*related (or simply related) to *b*”, and write . If , we say that “*a* is not related to *b*”.

In particular if *R* is a relation from a set *A* to itself, then we say that *R* is a relation on *A*.

**Example 3.3**:

1. Let  and . Let  be the relation “less than” from  to . Then, .
2. Let  and .
3. The following are relations from  into ;
4. 
5. 
6. 
7. The following are relations from  to ;
8. 
9. 
10. 

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| **Definition 3.4:** Let  be a relation from  into . Then,   1. the domain of , denoted by , is the set of first coordinates of the elements of , i.e      1. the range of , denoted by , is the set of second coordinates of elements of , i.e |

**Remark**: If  is a relation from the set  to the set , then the set  is called the codomain of the relation . The range of relation is always a subset of the codomain.

**Example 3.4**:

1. The set  is a relation from set  to set . The domain of  is , the range of  is  and the codomain of  is .
2. The set of ordered pairs  is a relation between the sets  and , where  is the domain and  is the range.

**Remark**:

1. If  for a relation , we say  is related to (or paired with) . Note that  may also be paired with an element different from . In any case,  is called the image of  while  is called the pre-image of under .
2. If the domain and/or range of a relation is infinite, we cannot list each element assignment, so instead we use set builder notation to describe the relation. The situation we will encounter most frequently is that of a relation defined by an equation or formula. For example,



is a relation for which the range value is 3 less than twice the domain value. Hence, and  are examples of ordered pairs that are of the assignment.

**Example 3.5**:

1. Let  Let  be the relation on  defined by is a factor of . Find the domain and range of .

**Solution**: We have

.

Then,  and .

1. Let  and .

Let is cube root of . Find a)  b)  c) 

**Solution**: We have  and  and 64 are in  whereas 125 is not in . Thus, ,  and .

**Remark**:

1. A relation  on a set  is called
2. a universal relation if 
3. identity relation if 
4. void or empty relation if 
5. If is a relation from  into , then the inverse relation of , denoted by , is a relation from  to  and is given by:

.

Observe that  and . For instance, if  is a relation on a set , then 

**Example 3.6**: Let  be a relation defined on  by .

Find a)  b)  c)  d) 

**Solution**: The smallest natural number is 1.











Therefore, , ,  and .

* **Functions**

Mathematically, it is important for us to distinguish among the relations that assign a unique range element to each domain element and those that do not.

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| **Definition 3.5: (Function)**  A function is a relation in which each element of the domain corresponds to exactly one element of the range. |

**Example 3.7**: Determine whether the following relations are functions.

1.  b) {(2,4),(3,4),(6,-4)}

**Solution**:

1. Since the domain element 3 is assigned to two different values in the range, 5 and 7, it is not a function.
2. Each element in the domain, , is assigned no more than one value in the range, 2 is assigned only 4, 3 is assigned only 4, and 6 is assigned only – 4. Therefore, it is a function.

**Remark**: Map or mapping, transformation and correspondence are synonyms for the word function. If  is a function and , we say *x* is mapped to *y* by *f*.

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| **Definition 3.6**: A relation from *A* into *B* is called a function from *A* into *B*, denoted by  or  if and only if    2. No element of *A* is mapped by  to more than one element in *B*, i.e. if  and , then . |

**Remark**: 1. If to the element *x* of  corresponds  under the function , then we write  and  is called the image of *x* under  and *x* is called a pre-image of  under .

2. The symbol  is read as “ of *x*” but not “ times *x*”.

1. In order to show that a relation  from *A* into *B* is a function, we first show that the domain of  is *A* and next we show that  well defined or single-valued, i.e. if  in *A*, then  in *B* for all .

**Example 3.8**:

1. Let  and . Which of the following are functions from *A* to .
2.  defined by 
3.  defined by 
4.  defined by 
5.  defined by 
6.  defined by 

**Solution**:

a)  is a function because to each element of *A* there corresponds exactly one element of .

b)  is not a function because there is no element of *B* which correspond to *4*(*A*).

c)  is a function because to each element of *A* there corresponds exactly one element

of *B*. In the given function, the images of all element of *A* are the same.

d) is not a function because there are two elements of  which correspond to 2.

In other words, the image of 2 is not unique.

e)  is a function because to each element of  there corresponds exactly one element

of 

As with relations, we can describe a function with an equation. For example, *y=2x+1* is a function, since each *x* will produce only one .

1. Let . Then,  maps:

1 to 1 -1 to 1

2 to 4 -2 to 4

3 to 9 -3 to 9

More generally any real number *x* is mapped to its square. As the square of a number is unique,  maps every real number to a unique number. Thus,  is a function from  into .

We will find it useful to use the following vocabulary: The independent variable refers to the variable representing possible values in the domain, and the dependent variable refers to the variable representing possible values in the range. Thus, in our usual ordered pair notation , *x* is the independent variable and  is the dependent variable.

1. Let *f* be the subset of defined by . Is *f* a function?

**Solution**: First we note that . Then,  satisfies condition (i) in the definition of a function. Now, ,  and  but . Thus  is not well defined. Hence,  is not a function from  to .

1. Let  be the subset of  defined by . Is  a function?

**Solution**: First we show that  satisfies condition (i) in the definition. Let  be any element of . Then, . Hence, . This implies that. Thus, . However,  and so . Now,  and . Thus,  and  are in . Hence we find that  and . This implies that  is not well defined, i.e,  does not satisfy condition (ii). Hence,  is not a function from  to .

* **Domain, codomain and range of a function**

For a function 

1. The set *A* is called the domain of 
2. The set *B* is called the codomain of 
3. The set  of all image of elements of *A* is called the range of 

**Example 3.9**:

1. Let  and . Let  be the correspondence which assigns to each element in , its square. Thus, we have . Therefore,  is a function and ,  and codomain of  is .
2. Let . Let  and  represent the elements in the sets  and , respectively. Let  be a function defined by .

The variable  can take values 2, 4, 6, 7, 9. Thus, we have

.

This implies that  and codomain

of is *IN*.

1. Determine whether the following equations determine  as a function of , if so, find the domain of the function.
2.  b)  c) 

**Solution**:

1. To determine whether  gives  as a function of , we need to know whether each *x-value* uniquely determines a *y-value*. Looking at the equation , we can see that once  is chosen we multiply it by – 3 and then add 5. Thus, for each *x* there is a unique . Therefore,  is a function. It’s domain is the set of all real numbers.
2. Looking at the equation  carefully, we can see that each *x-value* uniquely determines a *y-value* (one *x-value* can not produce two different *y-values*). Therefore,  is a function.

As for its domain, we ask ourselves. Are there any values of  that must be excluded? Since  is a fractional expression, we must exclude any value of  that makes the denominator equal to zero. We must have



Therefore, the domain consists of all real numbers except . Thus, .

1. For the equation , if we choose  we get , which gives . In other words, there are two values associated with . Therefore,  is not a function.
2. Find the domain of the function .

**Solution**: Since  is defined and is real when the expression under the radical is non-negative, we need  to satisfy the inequality

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This is a quadratic inequality, which can be solved by analyzing signs:

Sign of  

Since we want  to be non-negative, the sign analysis shows us that the domain is  or .

**Exercise 3.1**

1. Let *R* be a relation on the set  defined by .
2. List the elements of 
3. Is 
4. Let *R* be a relation on the set  defined by  divides .
5. List the elements of 
6. Find 
7. Find the elements of 
8. Find 
9. Let . Define a relation on  by . Write down the domain, codomain and range of . Find .
10. Find the domain and range of the relation .
11. Let  and . Which of the following are functions from  to ?
12.  c) 
13.  d) 
14. Determine the domain and range of the following relations. Which relation a function?
15.  d) 
16.  e) 
17.  f) {(5,0),(5,1),(5,2),(5,3),(5,4),(5,5)}
18. Find the domain and range of the following functions.
19.  c) 
20.  d) 
21. Given .

Find a)  b) c) 

* 1. **Real Valued functions and their properties**

After completing this section, the student should be able to:

* perform the four fundamental operations on polynomials
* compose functions to get a new function
* determine the domain of the sum, difference, product and quotient of two functions
* define equality of two functions

Let  be a function from set  to set . If  is a subset of the set of real numbers , then  is called a real valued function, and in particular if  is also a subset of , then  is called a real function.

**Example 3.10**: 1. The function  defined by ,  is a real function.

2. The function  defined as  is also a real valued function.

* **Operations on functions**

Functions are not numbers. But just as two numbers  and  can be added to produce a new number , so two functions  and  can be added to produce a new function . This is just one of the several operations on functions that we will describe in this section.

Consider functions  and  defined by  and . We can make a new function  by having it assign to the value , that is,

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| **Definition 3.7: Sum, Difference, Product and Quotient of two functions**  Let  and  be two functions. We define the following four functions:   1. The sum of the two functions 2. The difference of the two functions 3. The product of the two functions 4. The quotient of the two functions (provided   Since an value must be an input into both  and , the domain of  is the set of all  common to the domain of  and . This is usually written as . Similar statements hold for the domains of the difference and product of two functions. In the case of the quotient, we must impose the additional restriction that all elements in the domain of  for which  are excluded. |

**Example 3.11**:

1. Let  and . Find each of the following and its domain
2.  b)  c)  d) 

**Solution**:

1. 
2. 
3. 
4. 

We have





1. Let  and , with respective domains  and . Find formulas for  and  and give their domains.

**Solution**:

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| **Formula Domain** |
|  |

There is yet another way of producing a new function from two given functions.

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| **Definition 3.8: (Composition of functions)**  Given two functions  and , the composition of the two functions is denoted by  and is defined by:  .  is read as  composed with  of .The domain of  consists of those s in the domain of  whose range values are in the domain of , i.e. those s for which  is in the domain of . |

**Example 3.12**:

1. Suppose  and . The function  is found by taking elements in the domain of  and evaluating as follows:



If we attempt to find  we get , but 5 is not in the domain of  and so we cannot find . Hence, . The figure below illustrates this situation.

*g*

*a*

*b*

*c*

2

*Domain*

*of f*

5

3

*z*

*q*

*f*

*Domain of g*

*Range of g*

*Range of f*

1. Given  and , find
2.  b)  c)  d) 

**Solution**:

1. …… First evaluate 





1. …….First evaluate 





1. ……. But 







1. ……. But 







1. Given  and , find
2.  and its domain b)  and its domain

**Solution**: a) . Thus, .

1. . Since  must first be an input into  and so must be in the domain of , we see that .
2. Let  and . Find  and  and its domain.

**Solution**: We have .

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The domain of  is .

We now explore the meaning of equality of two functions. Let  and  be two functions. Then,  and  are subsets of . Suppose . Let  be any element of . Then,  and thus . Since  is a function and , we must have  Conversely, assume that  for all . Let . Then, . Thus, , which implies that . Similarly, we can show that . It now follows that . Thus two functions  and  are equal if and only if  for all . In general we have the following definition.

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| **Definition 3.9: (Equality of functions)**  Two functions are said to be equal if and only if the following two conditions hold:   1. The functions have the same domain; 2. Their functional values are equal at each element of the domain. |

**Example 3.13**:

1. Let  and  be defined by  and . Now, for all , . Thus, .
2. Let , and . The function  and  are not equal because 

**Exercise 3.2**

1. For  and , find each value:
2.  c)  e) 
3.  d)  f) 
4. If  and , find a formula for each of the following and state its domain.
5.  c) 
6.  d) 
7. Let  and .
8. Find  and its domain.
9. Find  and its domain
10. Are  and  the same functions? Explain.
11. Let . Find  so that .
12. Let  Find  so that .
13. If  is a real function defined by . Show that .
14. Find two functions  and  so that the given function , where
15.  c) 
16.  d) 
17. Let  and . Find
18.  c)  e) 
19.  d)  f) 
    1. **Types of functions and inverse of a function**

After completing this section, the student should be able to:

* define one to oneness and ontoness of a function
* check invertibility of a function
* find the inverse of an invertible function

In this section we shall study some important types of functions.

* **One to One functions**

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| **Definition 3.10:** A function  is called **one to one**, often written 1 – 1, if and only if for all ,  implies . In words, no two elements of  are mapped to one element of . |

**Example 3.14**:

1. If we consider the sets  and  and if ,  and , then both  and  are functions from  into . Observe that  is not a 1 – 1 function because  but . However,  is a 1 – 1 function.
2. Let  and . Consider the functions
3.  defined as 
4.  defined as 

Then,  is not 1 – 1, but  is a 1 – 1 function.

* **Onto functions**

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| **Definition 3.11:** Let  be a function from a set  into a set . Then  is called an **onto function(or maps onto**  if every element of  is an image of some element in , i.e, |

**Example 3.15**:

1. Let and . The function  defined by , ,  is not onto because there is no element in , whose image under  is 4. The function  given by  is onto because each element of  is an image of at least one element of  .

Note that if  is a non-empty set, the function  defined by  for all  is a 1 – 1 function from  onto .  is called the **identity map** on .

1. Consider the relation  from  into  defined by  for all . Now, domain of  is . Also, if , then , i.e. . Hence,  is well defined and is a function. However,  and , which implies that  is not 1 – 1. For all ,  is a non-negative integer. This shows that a negative integer has no preimage. Hence,  is not onto. Note that  is onto .
2. Consider the relation  from  into  defined by  for all . As in the previous example, we can show that  is a function. Let  and suppose that . Then  and thus . Hence,  is 1 – 1. Since for all ,  is an even integer; we see that an odd integer has no preimage. Therefore,  is not onto.

* **1 – 1 Correspondence**

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| **Definition 3.12:** A function  is said to be a 1 – 1 correspondence if  is both 1 – 1 and onto. |

**Example 3.16**:

1. Let  and . Suppose  given by  for all . One can easily see that every element of  has a preimage in  and hence  is onto. Moreover, if , then , i.e. . Hence,  is 1 – 1. Therefore,  is a 1 – 1 correspondence between  and .
2. Let  be a finite set. If  is onto, then it is one to one.

**Solution**: Let . Then . Since  is onto we have .Thus, , which implies that , , ,  are all distinct. Hence,  implies  for all . Therefore,  is 1 – 1.

* **Inverse of a function**

Since a function is a relation , the inverse of a function  is denoted by  and is defined by:



For instance, if , then . Note that the inverse of a function is not always a function. To see this consider the function . Then, , which is not a function.

As we have seen above not all functions have an inverse, so it is important to determine whether or not a function has an inverse before we try to find the inverse. If the function does not have an inverse, then we need to realize that it does not have an inverse so that we do not waste our time trying to find something that does not exist.

A one to one function is special because only one to one functions have inverse. If a function is one to one, to find the inverse we will follow the steps below:

1. Interchange  and  in the equation 
2. Solving the resulting equation for , we will obtaining the inverse function.

Note that the domain of the inverse function is the range of the original function and the range of the inverse function is the domain of the original function.

**Example 3.17**:

1. Given . Find  and its domain.

**Solution**: We begin by interchanging  and , and we solve for .

|  |  |
| --- | --- |
|  | Interchange  and |
|  | Take the cube root of both sides |
|  | This is the inverse of the function |

Thus, . The domain of  is the set of all real numbers.

1. Let . Find .

**Solution**: Again we begin by interchanging  and , and then we solve for .

**** Interchange  and 

 Solving for 



Thus, .

**Remark**: Even though, in general, we use an exponent of  to indicate a reciprocal, inverse function notation is an exception to this rule. Please be aware that  is not the reciprocal of . That is,

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If we want to write the reciprocal of the function  by using a negative exponent, we must write

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**Exercise 3.3**

1. Consider the function  from  into . Is  one to one? Is it onto?
2. Let . List all one to one functions from  onto .
3. Let . Let  be the inverse relation, i.e. .
4. Show by an example that  need not be a function.
5. Show that  is a function from  into  if and only if  is 1 – 1.
6. Show that  is a function from  into  if and only if  is 1 – 1 and onto.
7. Show that if  is a function from  into , then .
8. Let  and . Show that  defined by  is a 1 – 1 function from  onto .
9. Which of the following functions are one to one?
10.  defined by 
11.  defined by 
12.  defined by 
13.  defined by 
14.  defined by 
15. Which of the following functions are onto?
16.  defined by 
17.  defined by 
18.  defined by 
19.  defined by 
20. Find  if
21.  d)  g) 
22.  e)  h) 
23.  f) 

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| * 1. **Polynomials, zeros of polynomials, rational functions and their graphs**   After completing this section, the student should be able to:   * define polynomial and rational functions * apply the theorems on polynomials to find the zeros of polynomial functions * use the division algorithm to find quotient and remainder * apply theorems on polynomials to solve related problems * sketch and analyze the graphs of rational functions |

The functions described in this section frequently occur as mathematical models of real-life situations. For instance, in business the demand function gives the price per item, , in terms of the number of items sold, . Suppose a company finds that the price (in Birr) for its model GC-5 calculator is related to the number of calculators sold, (in millions), and is given by the demand function 

The manufacturer’s revenue is determined by multiplying the number of items sold () by the price per item (). Thus, the revenue function is

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These demand and revenue functions are examples of polynomial functions. The major aim of this section is to better understand the significance of applied functions (such as this demand function). In order to do this, we need to analyze the domain, range, and behavior of such functions.

* **Polynomial functions**

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| **Definition 3.13**: A polynomial function is a function of the form    Each  is assumed to be a real number, and  is a non-negative integer,  is called the leading coefficient. Such a polynomial is said to be of degree n. |

**Remark**:

1. The domain of a polynomial function is always the set of real numbers.
2. (Types of polynomials)

* A polynomial of degree 1 is called a linear function.
* A polynomial of degree 2 is called quadratic function.
* A polynomial of degree 3 is called a cubic function.

i.e 

**Example 3.18**: ,  and  are examples of polynomial functions.

* **Properties of polynomial functions**

1. The graph of a polynomial is a smooth unbroken curve. The word smooth means that the graph does not have any sharp corners as turning points.
2. If  is a polynomial of degree , then it has at most  zeros. Thus, a quadratic polynomial has at most 2 zeros.
3. The graph of a polynomial function of degree  can have at most turning points. Thus, the graph of a polynomial of degree 5 can have at most 4 turning points.
4. The graph of a polynomial always exhibits the characteristic that as  gets very large,  gets very large.

* **Zeros of a polynomial**

The zeros of a polynomial function provide valuable information that can be helpful in sketching its graph. One can find the zeros by factorizing the polynomial. However, we have no general method for factorizing polynomials of degree greater than 2. In this subsection, we turn our attention to methods that will allow us to find zeros of higher degree polynomials. To do this, we first need to discuss about the division algorithm. Recall that a number  is a zero of a polynomial function  if .

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| **Division Algorithm** |
| Let  and  be polynomials with , and with the degree of  less than or equal to the degree of . Then there are polynomials  and  such that  , where either  or the degree of  is less than degree of . |

**Example 3.19**: Divide .

**Solution**: Using long division we have



This long division means .

With the aid of the division algorithm, we can derive two important theorems that will allow us to recognize the zeros of polynomials.

If we apply the division algorithm where the divisor, , is linear (that is of the form ), we get

|  |  |
| --- | --- |
|  |  |

Note that since the divisor is of the first degree, the remainder , must be a constant. If we now substitute, into this equation, we get



Therefore, .

The result we just proved is called the remainder theorem.

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| **The Remainder Theorem** |
| When a polynomial  of degree at least 1 is divided by , then the remainder is . |

**Example 3.20**: The remainder when  is divided by  is .

As a consequence of the remainder theorem, if  is a factor of , then the remainder must be 0. Conversely, if the remainder is 0, then , is a factor of . This is known as the Factor Theorem.

|  |
| --- |
| **The Factor Theorem** |
| is a factor of  if and only if . |

The next theorem, called location theorem, allows us to verify that a zero exists somewhere within an interval of numbers, and can also be used to zoom in closer on a value.

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| **Location theorem** |
| Let  be a polynomial function and  and  be real numbers such that . If , then there is at least one zero of  between  and . |

The Factor and Remainder theorems establish the intimate relationship between the factors of a polynomial  and its zeros. Recall that a polynomial of degree *n* can have at most *n* zeros.

Does every polynomial have a zero? Our answer depends on the number system in which we are working. If we restrict ourselves to the set of real number system, then we are already familiar with the fact that the polynomial  has no real zeros. However, this polynomial does have two zeros in the complex number system. (The zeros are  and ). Carl Friedrich Gauss (1777-1855), in his doctoral dissertation, proved that within the complex number system, every polynomial of degree  has at least one zero. This fact is usually referred to as the Fundamental theorem of Algebra.

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| **Fundamental Theorem of Algebra** |
| If  is a polynomial of degree whose coefficients are complex numbers, then  has at least one zero in the complex number system. |

Note that since all real numbers are complex numbers, a polynomial with real coefficients also satisfies the Fundamental theorem of Algebra. As an immediate consequence of the Fundamental theorem of Algebra, we have

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| **The linear Factorization Theorem** |
| If , where  and , then  , where the  are complex numbers (possible real and not necessarily distinct). |

From the linear factorization theorem, it follows that every polynomial of degree  has exactly  zeros in the complex number system, where a root of multiplicity  counted  times.

**Example 3.21**: Express each of the polynomials in the form described by the Linear Factorization Theorem. List each zero and its multiplicity.

1. 
2. 
3. 

**Solution**:

1. We may factorize  as follows:



The zeros of  are 0, 8, and – 2 each of multiplicity one.

1. We may factorize  as follows:



Thus, the zeros of  are  and 2, each of multiplicity one.

1. We may factorize  as follows:



Thus, the zeros of f(x) are 0 with multiplicity two and  and  each with multiplicity one.

**Example 3.22**:

1. Find a polynomial  with exactly the following zeros and multiplicity.

|  |  |
| --- | --- |
| zeros | multiplicity |
|  | 3 |
| 2 | 4 |
| 5 | 2 |

Are there any other polynomials that give the same roots and multiplicity?

1. Find a polynomial *f* (*x*) having the zeros described in part (a) such that *f*(1) = 32.

**Solution**:

1. Based on the Factor Theorem, we may write the polynomial as:



which gives the required roots and multiplicities.

Any polynomial of the form , where  is a non-zero constant will give the same roots and multiplicities.

1. Based on part (1), we know that . Since we want , we have



Thus, .

Our experience in using the quadratic formula on quadratic equations with real coefficients has shown us that complex roots always appear in conjugate pairs. For example, the roots of  are  and . In fact, this property extends to all polynomial equations with real coefficients.

|  |
| --- |
| **Conjugate Roots Theorem** |
| Let  be a polynomial with real coefficients. If complex number  (where  and  are real numbers) is a zero of , then so is its conjugate . |

**Example 3.23**: Let  Given that  is a zero, find the other zero of .

**Solution**: According to the Conjugate Roots Theorem, if  is a zero, then its conjugate,  must also be a zero. Therefore, and  are both factors of , and so their product must be a factor of . That is, is a factor of . Dividing  by , we obtain



Thus, the zeros of  are , ,  and 1.

The theorems we have discussed so far are called existence theorems because they ensure the existence of zeros and linear factors of polynomials. These theorems do not tell us how to find the zeros or the linear factors. The Linear Factorization Theorem guarantees that we can factor a polynomial of degree at least one into linear factors, but it does not tell us how.

We know from experience that if  happens to be a quadratic function, then we may find the zeros of  by using the quadratic formula to obtain the zeros



The rest of this subsection is devoted to developing some special methods for finding the zeros of a polynomial function.

As we have seen, even though we have no general techniques for factorizing polynomials of degree greater than 2, if we happen to know a root, say , we can use long division to divide  by  and obtain a quotient polynomial of lower degree. If we can get the quotient polynomial down to a quadratic, then we are able to determine all the roots. But how do we find a root to start the process? The following theorem can be most helpful.

|  |
| --- |
| **The Rational Root Theorem** |
| Suppose that , where  is an degree polynomial with integer coefficients. If  is a rational root of , where  and  have no common factor other than , then  is a factor of  and  is a factor of . |

To get a feeling as to why this theorem is true, suppose  is a root of .

Then,  which implies that

 multiplying both sides by 8



If we look at equation (1), the left hand side is divisible by 3, and therefore the right hand side must also be divisible by 3. Since 8 is not divisible by 3,  must be divisible by 3. From equation (2),  must be divisible by 2.

**Example 3.24**: Find all the zeros of the function 

**Solution**: According to the Rational Root Theorem, if  is a rational root of the given equation, then  must be a factor of  and  must be a factor of 2. Thus, we have

possible values of : 

possible values of : 

possible rational roots : 

We may check these possible roots by substituting the value in . Now  and . Since  is negative and  is positive, by location theorem,  has a zero between  and 1. Since , then  is a factor of . Using long division, we obtain



Therefore, the zeros of *p*(*x*) are ,  and 3.

* **Rational Functions and their Graphs**

A rational function is a function of the form  where both *n*(*x*) and *d*(*x*) are polynomials and .

**Example 3.25**: The functions ,  and  are examples of rational function.

Note that the domain of the rational function  is 

**Example 3.26**: Find the domain and zeros of the function .

**Solution**: The values of  for which  are excluded from the domain of Since , we have . To find the zeros of , we solve the equation



Therefore, to find the zeros of , we solve , giving . Since  does not make the denominator zero, it is the only zero of .

The following termsand notations are useful in our next discussion.

Given a number *a*,

* **** approaches *a* from the right means *x* takes any value near and near to *a* but *x* > *a*. This is denoted by: *x*→*a*+ (read: ‘*x* approaches *a* from the right’ ).

For instance, *x*→ 1+  means *x* can be 1.001, 1.0001, 1.00001, 1.000001, etc.

*  approaches *a* from the left means *x* takes any value near and near to *a* but *x* < *a*.

This is denoted by: *x*→*a*–  (read: ‘*x* approaches *a* from the left’ ).

For instance, *x*→1– means *x* can be 0.99, 0.999, 0.9999, 0.9999, etc.

* *x*→∞ (read: ‘*x* approaches or tends to *infinity*’) means the value of *x* gets indefinitely larger and larger in magnitude (keep increasing without bound). For instance, *x* can be 106, 1010, 1012, etc.
* *x*→ –∞ (read: ‘*x* approaches or tends to negative *infinity*’) means the value of *x* is negative and gets indefinitely larger and larger negative in magnitude (keep decreasing without bound). For instance, *x* can be –106, –1010, –1012, etc.

The same meanings apply also for the values of a function *f* if we wrote  *f*(*x*)→∞ or *f*(*x*)→−∞. The following figure illustrates these notion and notations.

|  |
| --- |
| *y*→∞  x  *y*→−∞  *x*→∞  *x*→*a***–** *x*→*a***+**  *x*→ –∞  y  *f*(*x*)→∞,  as*x*→*a*−  *a*  *y* =*f*(*x*)  *f*(*x*)→ –∞, as*x*→*a*+  *f*(*x*)→ –∞,  as*x*→–∞  *f*(*x*)→∞,  as*x*→∞  *a*    Fig. 2.1. Graphical illustration of the idea of *x*→*a*+, *f*(*x*)→∞, etc. |

We may also write *f*(*x*)→*b* (read: ‘*f*(*x*) approaches *b*’) to mean the function values, *f*(*x*), becomes arbitrarily closer and closer to *b* (i.e., approximately *b*) but not exactly equal to *b*. For instance, if , then *f*(*x*)→0 as *x*→∞; i.e.,  is approximately 0 when *x* is arbitrarily large.

The following steps are usually used to sketch (or draw) the graph of a rational function *f*(*x*).

1. Identify the domain and simplify it.
2. Find the intercepts of the graph whenever possible. Recall the following:

* y–intercept is the point on y-axis where the graph of *y* = *f*(*x*) intersects with the *y*-axis. At this point *x*=0. Thus, *y* = *f*(0), or (0, *f*(0) ) is the y-intercept if 0∈Dom(*f*).
* x–intercept is the point on x-axis where the graph of *y* = *f*(*x*) intersects with the *x*-axis. At this point *y*=0. Thus, *x*=*a* or (*a*, 0) is x-intercept if *f*(*a*)=0.

1. Determine the asymptotes of the graph. Here, remember the following.

* Vertical Asymptote: The vertical line *x*=*a* is called a vertical asymptote(VA) of *f*(*x)* if

1. *a*∉dom(*f*), i.e., *f*  is not defined at *x*=*a*; and
2. *f*(*x*)→∞ or *f*(*x*)→ –∞ when *x*→*a*+ or *x*→*a*– . In this case, the graph of *f* is almost vertically rising upward (if *f*(*x*)→∞) or sinking downward (if *f*(*x*)→−∞) along with the vertical line *x*=*a* when *x* approaches *a* either from the right or from the left.

**Example 3.27**: Consider  where *a* ≥ 0 and *n* is a positive integer.

Obviously *a*∉Dom(*f*). Next, we investigate the trend of the values of *f*(*x*) near *a*. To do this, we consider two cases, when *n* is even or odd:

Suppose *n* is even: In this case (*x* – *a*)*n* > 0 for all *x*∈ℜ\{*a*}; and since (*x* – *a*)*n* →0 as *x*→*a*+  or *x*→*a*– . Hence,  as *x*→*a*+ or *x*→*a*– . Therefore, *x=a* is a VA of *f*(*x*). Moreover, *y*= 1/*an* or (0, 1/*an* ) is its y-intercept since *f*(0)=1/*an*. However, it has no x-intercept since *f*(*x*) >0 for all *x* in its domain (See, Fig. 2.2 (A)).

Suppose *n* is odd: In this case (*x* – *a*)*n*> 0 for all *x*>*a* and 1/ (*x* – *a*)*n* →∞ when *x*→*a*+  as in the above case. Thus, *x=a* is its VA. However, 1/(*x*–*a*)*n*→ –∞ when *x*→*a*– since (*x* – *a*)*n*< 0 for *x*<*a*. Moreover, *y*= –1/*an* or (0, –1/*an* ) is its y-intercept since *f*(0) = –1/*an*. However, it has no x-intercept also in this case. (See, Fig. 2.2 (B)).

Note that in both cases, as *x*→∞or *x*→ –∞.

*n*-**even**

Fig. 2.2 (A)

*a*

1*/an*

−1*/an*

*x=a*

VA

*n*-**odd**

*x=a*

VA

*a*

*x*

*y*

*x*

*y*

Fig. 2.2 (B)

|  |
| --- |
| **Remark**: Let  be a rational function. Then,  1. if and , then *x=a* is a VA of *f* .  2. if , then *x=a* may or may **not** be a VA of *f* . In this case, simplify *f*(*x*) and look for VA of the simplest form of *f*. |

* Horizontal Asymptote: A horizontal line *y*=*b* is called horizontal asymptote (HA) of *f*(*x*) if the value of the function becomes closer and closer to *b* (i.e., *f*(*x*)→*b*)as *x*→∞ or as *x*→ –∞.

In this case, the graph of *f* becomes almost a horizontal line along with (or near) the line *y=b* as *x*→∞ and as *x*→–∞. For instance, from the above example, the HA of is *y*=0 (the x-axis) , for any positive integer *n* (See, Fig. 2.2).

**Remark**: A rational function  has a HA only when *degree*(*n*(*x*)) ≤*degree*(*d*(*x*)).

In this case, (i) If *degree*(*n*(*x*)) <*degree*(*d*(*x*)), then *y* = 0 (the x-axis) is the HA of *f*.

(ii) If *degree*(*n*(*x*)) =*degree*(*d*(*x*))*=n*, i.e., ,

then is the HA of *f*.

* Oblique Asymptote: The oblique line *y*=*ax*+*b*, *a*≠0, is called an oblique asymptote (OA) of *f* if the value of the function, *f*(*x),* becomes closer and closer to *ax*+*b*(i.e., *f*(*x*) becomes approximately *ax*+*b*) as either *x*→∞ or *x*→ –∞. In this case, the graph of *f* becomes almost a straight line along with (or near) the oblique line *y*=*ax*+*b* as *x*→∞ and as *x*→ –∞.

**Note**: A rational function  has an OA only when *degree*(*n*(*x*)) = *degree*(*d*(*x*)) + 1. In this case, using long division, if the quotient of *n*(*x*) ÷*d*(*x*) is *ax +b*, then *y*=*ax*+*b* is the OA of *f*.

**Example 3.28:** Sketch the graphs of 

**Solution**: (a) Since *x*−1=0 at *x*=1, dom(*f*) = ℜ\{1}.

* Intercepts: y-intercept: *x*=0 ⇒*y=f* (0) = –2. Hence, (0, – 2) is y-intercept.

x-intercept: *y*=0 ⇒*x*+2=0 ⇒*x*= –2. Hence, (–2, 0) is x-intercept.

* Asymptotes:
* VA:Since *x*−1=0 at*x*=1 and *x*+2≠0 at *x*=1, *x*=1 is VA of *f*. In fact, if *x*→1+ , then *x*+2 ≈3 but the denominator *x*–1 is almost 0 (but positive).

Consequently, *f*(*x*)→∞ as  *x*→1+.

Moreover, *f*(*x*)→ –∞ as  *x*→1– (since , if *x*→1– then *x*–1 is almost 0 but negative ) .

(So, the graph of *f* rises up to +∞ at the right side of *x=*1, and sink down to −∞ at the left side of *x=*1)

* HA: Note that if you divide *x*+2 by *x*–1, the quotient is 1 and remainder is 3. Thus,

. Thus, if *x*→∞ (or *x*→ –∞), then →0 so that *f*(*x*)→1. Hence, *y*=1 is the HA of *f*.

Using these information, you can sketch the graph of *f* as displayed below in Fig. 2.3 (A).

(b) Both the denominator and numerator are 0 at *x*=1. So, first factorize and simplify them:

*x*2+3*x*+2=(*x*+2)(*x*+1) and *x*2–1 = (*x –*1)( *x+*1) . Therefore,

, *x*≠ –1

. (So, dom(*g*) = ℜ\{1, –1} )

This implies that only *x*=1 is VA.

Hence, the graph of  is exactly the same as that of  except that *g*(*x*) is not defined at *x*= –1. Therefore, the graph of *g* and its VA are the same as that of *f* except that there should be a ‘hole’ at the point corresponding to *x*= –1 on the graph of *g* as shown on Fig. 2.3(B) below.

*y*=1 (HA)





*x=*1

VA

*x=*1

(B) 

Fig 2.3 (A) 

‘hole’

at*x=−*1

*y*=1

*−*2

*−*2

*−*2

*−*2

−1

**Exercise 3.4**

1. Perform the requested divisions. Find the quotient and remainder and verify the Remainder Theorem by computing .
2. Divide 
3. Divide 
4. Divide 
5. Divide 
6. Given that , factor as completely as possible.
7. Given that and , find the remaining zeros of .
8. Given that 3 is a double zero of , find all the zeros of .
9. a) Write the general polynomial  whose only zeros are 1, 2 and 3, with multiplicity 3, 2 and 1 respectively. What is its degree?

b) Find  described in part (a) if .

1. If is a root of find the remaining zeros of p(*x*).
2. Determine the rational zeros of the polynomials
3. 
4. 
5. 
6. Find the domain and the real zeros of the given function.
7.  b)  c)  d) 
8. Sketch the graph of

a) b)  c) d) 

1. Determine the behavior of  when  is near 3.
2. The graph of any rational function in which the degree of the numerator is exactly one more than the degree of the denominator will have an oblique (or slant) asymptote.
3. Use long division to show that



1. Show that this means that the line  is a slant asymptote for the graph and sketch the graph of .
   1. **Definition and basic properties of logarithmic, exponential, trigonometric and hyperbolic functions and their graphs**

After completing this section, the student should be able to:

* define exponential, logarithmic, trigonometric and hyperbolic functions
* understand the relationship of the exponential and logarithmic functions
* define the hyperbolic functions and be familiar with their properties
* sketch the graph of exponential, logarithmic, trigonometric and hyperbolic functions
* use basic properties of logarithmic, exponential, hyperbolic and trigonometric functions to solve problems
* **Exponents and radicals**

|  |
| --- |
| **Definition 3.14:** For a natural number  and a real number , the power , read “ the  power of ” or “ raised to ”, is defined as follows:    In the symbol,  is called the base and  is called the exponent. |

For example, .

Based of the definition of ,  must be a natural number. It does not make sense for  to be negative or zero. However, we can extend the definition of exponents to include 0 and negative exponents.

|  |
| --- |
| **Definition 3.15: (Zero and Negative Exponents)**  *Definition of zero Exponent Definition of Negative Exponent*    Note:  is undefined. |

As a result of the above definition, we have . We have the following rules of exponents for integer exponents:

|  |
| --- |
| **Rules for Integer Exponents** |
| 1. 4. 2. 5. |

Next we extend the definition of exponents even further to include rational number exponents. To do this, we assume that we want the rules for integer exponents also to apply to rational exponents and then use the rules to show us to define a rational exponent. For example, how do we define ? Consider .

If we apply rule 2 and square , we get . Thus,  is a number that, when squared, yields 9. There are two possible answers: 3 and – 3, since squaring either number will yield 9. To avoid ambiguity, we define (called the principal square root of ) as the non-negative quantity that, when squared, yield . Thus, .

We will arrive at the definition of  in the same way as we did for . For example, if we cube , we get . Thus,  is the number that, when cubed, yields 8. Since  we have . Similarly, . Thus, we define (called the cube root of ) as the quantity that, when cubed yields .

|  |
| --- |
| **Definition 3.16: (Rational Exponent )**  If  is an odd positive integer, then  if and only if  If  is an even positive integer and , then  if and only if |

We call  the principal  root of . Hence,  is the real number (nonnegative when  is even) that, when raised to the  power, yields . Therefore,

 since 

 since 

 since 

 since 

 is not a real number

Thus far, we have defined , where  is a natural number. With the help of the second rule for exponent, we can define the expression , where  and  are natural numbers and  is reduced to lowest terms.

|  |
| --- |
| **Definition 3.17: (Rational Exponent )**  If  is a real number, then (i.e. the  root of a raised to the  power) |

We can also define negative rational exponents:

|  |
| --- |
|  |

**Example 3.29**: Evaluate the following

1.  b)  c) 

**Solution**: We have

1. 
2. 
3. 

Radical notation is an alternative way of writing an expression with rational exponents. We define for real number , the  root of  as follows:

|  |
| --- |
| **Definition 3.18 ( root of ):** = , where  is a positive integer. |

The number  is also called the principal  root of . If the  root of  exists, we have:

|  |
| --- |
| For  a real number and  a positive integer, |

For example,  and .

* **Exponential Functions**

In the previous sections we examined functions of the form , where  is a constant. How is this function different from .

|  |
| --- |
| **Definition 3.19:** A function of the form , where  and , is called an exponential function. |

**Example 3.30**: The functions ,  and  are examples of exponential functions.

As usual the first question raised when we encounter a new function is its domain. Since rational exponents are well defined, we know that any rational number will be in the domain of an exponential function. For example, let . Then as  takes on the rational values  – 2 ,  and , we have

Note that even though we do not know the exact values of  and , we do know exactly what they mean. However, what about  for irrational values of ? For instance, 

We have not defined the meaning of irrational exponents. In fact, a precise formal definition of  where  is irrational requires the ideas of calculus. However, we can get an idea of what  should be by using successive rational approximations to . For example, we have



Thus, it would seem reasonable to expect that . Since 1.414 and 1.415 are rational numbers,  and  are well defined, even though we cannot compute their values by hand. Using a calculator, we get . If we use better approximations to , we get . Using a calculator again, we get . Computing directly on a calculator gives . This numerical evidence suggests that as  approaches , the values of  approach a unique real number that we designate by , and so we will accept without proof, the fact that the domain of the exponential function is the set of real numbers.

|  |
| --- |
| The exponential function , where  and , is defined for all real values of . In addition all the rules for rational exponents hold for real number exponents as well. |

Before we state some general facts about exponential functions , let’s see if we can determine what the graph of an exponential function will look like.

**Example 3.31**:

1. Sketch the graph of the function  and identify its domain and range.

**Solution**: To aid in our analysis, we set up a short table of values to give us a frame of

reference.

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |

O

(1,2)

1

1

2

***x***

***y***

*y =* 2*x*

With these points in hand, we draw a smooth curve through the points obtaining the graph appearing above. Observe that the domain of  is , the graph has no intercepts, as

, the  values are increasing very rapidly, whereas as , the values are getting closer and closer to 0. Thus,  is a horizontal asymptote, the intercept is 1 and the range of  is the set of positive real numbers.

1. Sketch the graph of .

**Solution**: It would be instructive to compute a table of values as we did in example 1 above (you are urged to do so). However, we will take a different approach. We note that . If , then . Thus by the graphing principle for , we can obtain the graph of  by reflecting the graph of  about the .

O

(−1,2)

1

1

2

*x*

*y*

**

−1

Here again the  is a horizontal asymptote, there is no intercept, 1 is intercept and the range is the set of positive real numbers. However, the graph is now decreasing rather than increasing.

The following box summarizes the important facts about exponential functions and their graphs.

|  |
| --- |
| **The Exponential function** |
| 1. The domain of the exponential function is the set of real numbers 2. The range of the exponential function is the set of positive real numbers 3. The graph of  exhibits exponential growth if  or exponential decay if . 4. The intercept is 1. 5. The intercept is a horizontal asymptote 6. The exponential function is 1 – 1. Algebraically if , then |

**Example 3.32**: Sketch the graph of each of the following. Find the domain, range, intercepts, and asymptotes.

1.  b)  c) 

**Solution**:

1. To get the graph of . We start with the graph of , which is the basic exponential growth graph, and shift it up 1 unit.

|  |  |
| --- | --- |
| 10  2  1  1  2  *y*=3*x*+1  *y* = 1 | From the graph we see that     * The intercept is 2 * The line  is a horizontal asymptote |

1. To get the graph of , we start with the graph of , and shift 1 unit to the left.

|  |  |
| --- | --- |
| 1  *y*=3*x+*1  9 | From the graph we see that     * The intercept is 3 * The line  is a horizontal asymptote |

1. To get the graph of , we start with the basic exponential decay . We then reflect it with respect to the , which gives the graph of . Finally, we shift this graph up 3 units to get the required graph of .

x

(−1,−9)

1

1

y

*y* = *−9− x*

−1

−9

−1

x

O

(−1,9)

1

1

9

y

*y* = *9− x*

−1

x

1

3

y

*y* = *−9− x*+3

−1

2

y = 3

From the graph of , we can see that , , the line  is a horizontal asymptote, 2 is the intercept and  is the intercept.

**Remark**: When the base  of the exponential function  equals to the number , where , we call the exponential function the natural exponential function.

* **Logarithmic Functions**

In the previous subsection we noted that the exponential function  (where  and ) is one to one. Thus, the exponential function has an inverse function. What is the inverse of ?

To find the inverse of , let’s review the process for finding an inverse function by comparing the process for the polynomial function  and the exponential function . Keep in mind that  is our independent variable and  is the dependent variable and so whenever possible we want a function solved explicitly for .

|  |  |
| --- | --- |
| **To find the inverse of** | **To find the inverse of** |
| Interchange  and  solve for | Interchange  and  solve for |

There is no algebraic procedure we can use to solve  for . By introducing radical notations we could express the inverse of  explicitly in the form . In words,  and  both mean exactly the same thing:  is the number whose cube is . Similarly, if we want to express  explicitly as a function of , we need to invent a special notation for this. The key idea is to take the equation  and express it verbally.

|  |
| --- |
| means is the exponent to which 3 must be raised to yield |

We introduce the following notation, which expresses this idea in a much more compact form.

|  |
| --- |
| **Definition 3.20:** For and , we write to mean is the exponent to which must be raised to yield . In other words, |

We read as “ equals the logarithm of to the base ”.



|  |
| --- |
| REMEMBER: is an alternative way of writing |

When an expression is written in the form , it is said to be in exponential form. When an expression is written in the form , it is said to be in logarithmic form. The table below illustrates the equivalence of the exponential and logarithmic forms.



|  |  |
| --- | --- |
| Exponential form | Logarithmic form |
|  |  |

**Example 3.33**:

1. Write each of the following in exponential form.
2. b)



**Solution**: We have a) means and b) means



1. Write each of the following in logarithmic form.
2. b)



**Solution**: We have a) means



b) means



1. Evaluate each of the following.
2. b)



**Solution**:

1. To evaluate , we let , and then rewrite the equation in exponential form, . Now, if we can express both sides in terms of the same base, we can solve the resulting exponential equation, as follows:



Let Rewrite in exponential form



Express both sides in terms of the same base



Since the exponential function is 1 – 1



Therefore, .



1. We apply the same procedure as in part (a).

Let Rewrite in exponential form



Express both sides in terms of the same base



Since the exponential function is 1 – 1



Therefore, .



As was pointed out at the beginning of this subsection, logarithm notation was invented to express the inverse of the exponential function. Thus, is a function of . We usually write rather than writing and use parenthesis only when needed to clarify the input to the log function. For example,



If , then , whereas if , then , which is undefined.



**Example 3.34**: Given , find



1. b) c) d)



**Solution**:

1. (since )



1. (since )



1. is not defined (what power of 5 will yield 0?). We say that 0 is not in the domain of .



1. is not defined (what power of 5 will yield -125?). We say that -125 is not in the domain of .



Acknowledging that the logarithmic and exponential functions are inverses, we can derive a great deal of information about the logarithmic function and its graph from the exponential function and its graph.

**Example 3.35**: Sketch the graph of the following functions. Find the domain and range of each.

1. b)



**Solution**: a) Since is the inverse of , we can obtain the graph of by reflecting the graph of about the line , as shown below.



1

1

x

y

*y =* 3*x*

*y =* x

*y =* log3x

1. To get the graph of , we reflect the graph of about the line as shown below.

1

1

x

y

**

**

y=x



Taking note of the features of the two graphs we have the following important informations about the graph of the log function:

|  |
| --- |
| **The Logarithmic Function** |
| 1. Its domain is the set of positive real numbers 2. Its range is the set of real numbers. 3. Its graph exhibits logarithmic growth if and logarithmic decay if .  1. The intercept is 1. There is no intercept.  1. The is a vertical asymptote. |

**Example 3.36**:

1. Sketch the graph of . Find the domain, range, asymptote and intercepts.



**Solution**: We can obtain the graph of by applying the graphing principle to shift the basic logarithmic growth graph 2 units to the right and 1 unit up.



1

1

x

y

*x=* 2

*y =* 1+ log3(x−2)

2

3

We have , and the graph has the line as a vertical asymptote. To find the intercept, we set and solve for . Setting and solving for , we will obtain . Thus, the intercept is .



1. Find the inverse function for
2. b)



**Solution**: Following the procedure for finding an inverse function, we have

|  |  |
| --- | --- |
| (a) Interchange and  solve explicitly for  Write in logarithmic form    Thus, | (b) Interchange and  Write in logarithmic form  solve explicitly for    Thus, |

The following table contains the basic properties of logarithm:

|  |
| --- |
| **Properties of logarithm** |
| Assume that and are positive and . Then  In words, logarithm of a product is equal to the sum of the logs of the factors.  In words, the log of a quotient is the log of the numerator minus the log of the denominator.  In words, the log of a power is the exponent times the log.   1. if *a* is positive and . |

**Example 3.37**:

1. Express in terms of simpler logarithms.
2. b) c)



**Solution**:



1. Examining the properties of logarithms, we can see that they deal with log of a product, quotient and power. Thus, which is the log of a sum cannot be simplified using log properties.



1. We have

=.



1. Show that .



**Solution**: We have .



The logarithmic function was introduced without stressing the particular base chosen. However, there are two bases of special importance in science and mathematics, namely, and .



|  |
| --- |
| **Definition 3.21: (Common Logarithm)**  is called the common logarithm function. We write . |

The inverse of the natural exponential function is called the natural logarithmic function and has its own special notation.

|  |
| --- |
| **Definition 3.22: (Natural Logarithm)**  is called the natural logarithmic function. We write . |

**Example 3.38**:

1. Evaluate



**Solution**: Let . Then, .



1. Find the inverse function of .



**Solution**: Let Interchange and



Solve for



Rewrite in logarithmic form



Thus, .



* **Trigonometric functions and their graphs**

For the functions we have encountered so far, namely polynomial, rational and exponential functions, as the independent variable goes to infinity the graph of each of these three functions either goes to infinity(very quickly) for exponential functions or approaches a finite horizontal asymptote. None of these functions can model the regular periodic patterns that play an important role in the social, biological, and physical sciences: business cycles, agricultural seasons, heart rhythms, and hormone level fluctuations, and tides and planetary motions. The basic functions for studying regular periodic behaviour are the trigonometric functions. The domain of the trigonometric functions is more naturally the set of all geometric angles.

**Angle Measurement**

An angle is the figure formed by two half-lines or rays with a common end point. The common end point is called the vertex of the angle.

**A**

In forming the angle, one side remains fixed and the other side rotates. The fixed side is called the initial side and the side that rotates is called the terminal side. If the terminal side rotates in a counter clockwise direction, we call the angle positive angle, and if the terminal side rotates in a clockwise direction, we call the angle negative angle.



**B**

**B**



What attribute of an angle are we trying to measure when we measure the size of an angle? A moment of thought will lead us to the conclusion that when we measure an angle we are trying to answer the question: Through what part of a complete rotation has the terminal side rotated?

We will use degree (°) as the unit of measurement for angles. Recall that the measure of a full round angle (full circle) is 360°, straight angle is 180°, and right angle is 90°.

An alternative unit of measure for angles which will indicate their size is the radian measure. To see the connection between the degree measure and radian measure of an angle, let us consider an angle and draw a circle of radius with the vertex of at its center . Let represent the length of the arc of the circle intercepted by (as shown below).



**O**

**r**



**s**

Basic geometry tells us that the central angle will be the same fractional part of one complete rotation as will be of the circumference of the circle. For example, if is of a complete rotation, then will be of the circumference of the circle. In other words, we can set up the following proportion:



Thus, we have the following conversion formula:

|  |
| --- |
|  |

**Example 3.39**:

1. Convert each of the following radian measures to degrees.
2. b)



**Solution**: a) By the conversion formula, we have , which implies that .



1. Again using the conversion formula, we get , which implies that .



1. Convert to radian measures

a) b)



**Solution**: a) Let represent the radian measure of . Using the conversion formula, we obtain: , which implies that .



b) Rather than using the conversion formula, we notice that . In part (a) we found that , and so we have .



To define the trigonometric functions, we will view all angles in the context of a Cartesian coordinate system: that is, given an angle , we begin by putting in standard position, meaning that the vertex of is placed at the origin and initial side of is placed along the positive . Thus the location of the terminal side of will, of course, depend on the size of .

**r**

P(x,y)

X

Y



**ϴ**

X

Y

We then locate a point( other than the origin) on the terminal side of and identify its coordinates and its distance to the origin, dented by . Then, is positive.



With in standard position, we define the six trigonometric functions of as follows:



|  |
| --- |
| **Definition 3.23**  **Name of function** **Abbreviation**   **Definition**  Sine  Cosine  Tangent  Cosecant  Secant  Cotangent |

Recall that the radian measure of an angle is defined as , where is angle in radians



is the length of the arc intercepted by and is the length of the radius. Since and are both lengths, the quotient is a pure number without any units attached. Thus, any angle can be interpreted as a real number. Conversely, any real number can be interpreted as an angle. Thus, we can describe the domains of the trigonometric functions in the frame work of the real number systems. If we let , the domain consists of all real numbers for which is defined. Since and is never equal to zero, the domain for is the set of all real numbers. Similarly, the domain of is also the set of all real numbers.



* **The graph of**



To analyze , we keep in mind that once we choose a real number , we draw the angle, in standard position, that corresponds to . To simplify our analysis, we choose the point on the terminal side so that . That is, is a point on the unit circle . Note that .

(0,1)

(x,y)

(1,0)

(0,-1)

(-1,0)

θ

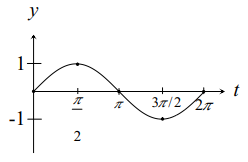


As the terminal side of moves through the first quadrant, increases from 0 (when ) to 1(when ). Thus, as increases from 0 to , steadily increases from 0 to 1.



As increases from to , decreases form 1 to 0. A similar analysis reveals that as increases from to , decreases from 0 to – 1; and as increases from to , increases from – 1 to 0.



Based on this analysis, we have the graph of in the interval as show below.

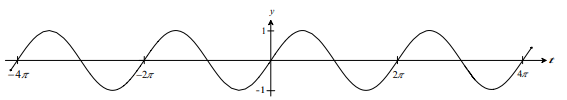


y = sin



Since the values of depend only on the position of the terminal side, adding or subtracting multiples of to will leave the value of unchanged. Thus, the values of will repeat every units. The complete graph of appears below.





The graph of , which is called the basic sine curve.



* **The graph of**



Applying the same type of analysis to , we will able to get a good idea of what its graph looks like. The figure below shows the angle corresponding to as it increases through quadrant I, II, III and IV.



Keeping in mind that , we have the following:



1. As increases from 0 to , decreases from 1 to 0.



1. As increases from to , decreases from 0 to – 1.



1. As increases from to , increases from – 1 to 0.

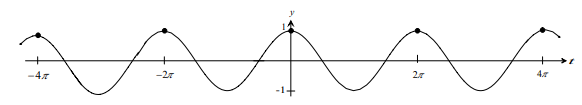


1. As increases from to , increases from 0 to 1.



Based on this analysis, we have the graph of as shown below:





* **The graph of**



Since is undefined whenever , is undefined whenever the terminal side of the angle corresponding to falls on the . This happens for , to which we can add or subtract any multiple of that will again bring the terminal side back to the . Thus, domain of is , where is an integer.



1. As increases from 0 to , decreases from 1 to 0 and increases from 0 to 1; therefore, increases from 0 to .



1. As increases from to , decreases from 0 to – 1 and decreases from 1 to 0; therefore, increases from to 0.



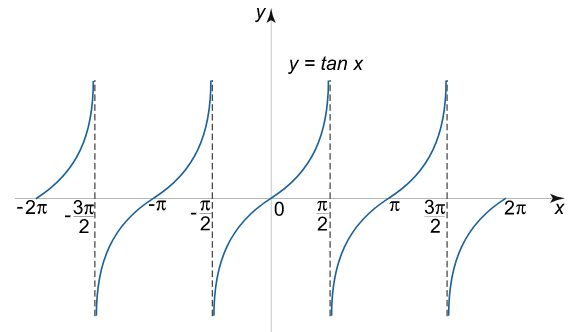
1. As increases from to , increases from – 1 to 0 and decreases from 0 to – 1; therefore, increases from 0 to .



1. As increases from to , increases from to 1 and increases from – 1 to 0; therefore, increases from to 0.



You may want to add some more specific values to this analysis. In any case, we get the following as the graph of the tangent function.



|  |
| --- |
| **Definition 3.24: (Periodic Function)**  A function  is called periodic if there exists a number  such that for all  in the domain of . The smallest such number  iscalled the period of the function |

A periodic function keeps repeating the same set of over and over again. The graph of a periodic function shows the same basic segment of its graph being repeated. In the case of sine and cosine functions, the period is . The period of the tangent function is .



|  |
| --- |
| **Definition 3.25: (Amplitude of a periodic function)**  The amplitude of a periodic function is  maximum value of minimum value of |

Thus, the amplitude of the basic sine and cosine function is 1.

The portion of the graph of a sine or cosine function over one period is called a complete cycle of the graph. In other words, the minimal portion of a sine or cosine graph that keeps repeating itself is called a complete cycle of the graph.

|  |
| --- |
| **Definition 3.26: (Frequency of a periodic function)**  The number of complete cycles a sine or cosine graph makes on an interval of length equal to is called its frequency. |

The frequency of the basic sine curve and the basic cosine curve is 1, because each graph makes 1 complete cycle in the interval .



If a sine function has period of (see the figure below), then the number of complete cycles its graph will make in an interval of length is .

Y

X



A sine graph of period  and frequency 4

Thus if a sine function has a period of , its frequency is 4 and its graph will make 4 complete cycles in an interval of length .



**Example 3.40**: Sketch the graph of and find its amplitude, period and frequency.



**Solution**: We can obtain this graph by applying our knowledge of the basic sine graph. For the basic curve, we have



These quadrantal values serve as guide points, which help us draw the graph. To obtain similar guide points for , we ask for what values of is



and we get



Thus, will have the values 0, 1, 0, , 0 at and , respectively. The graph of will thus complete one cycle in the interval , and will repeat the same values in the interval .



Y

X

From this graph we see that has an amplitude of 1, a period , and a frequency of 2.



For convenience we summarize our discussion on the domains of the trigonometric functions in the table.

|  |  |
| --- | --- |
|  | Domain = All real numbers  Domain = All real numbers  Domain =  Domain = {}  Domain =  Domain =  where is an integer |

In the course of our discussion of the trigonometric functions, we have discussed two types of trigonometric relationships: the reciprocal and quotient relationships. These relationships are examples of trigonometric identities. In the table below we list identities that are satisfied by the trigonometric functions.

|  |
| --- |
| **The reciprocal Identities** 1.  2.  3.  **The quotient Identities** 4.  5. |
| **The Pythagorean Identities** 6.  7.  8.  **The addition formula** 9. (a)  (b)  10. (a)  (b)  11. (a)  (b)  **The double angle formula** 12.  13.  14.  **The half-angle formula** 15.  16.  17. |

* **Hyperbolic functions and their graphs**

The hyperbolic functions are certain combinations of exponential functions, that occur in various applications, with properties similar to those of the trigonometric functions. Among many other applications they are used to describe the formation of satellite rings around the planets, to describe the shape of a rope hanging from two points, and have application in relativity theory. The two basic hyperbolic functions are the hyperbolic sine and hyperbolic cosine functions. They are defined as follows:

|  |  |
| --- | --- |
| **Definition 3.27**: | |
| 1. The **hyperbolic sine** function is defined by:     The domain of is . | 1. The **hyperbolic cosine** function is defined by:     The domain of is also |

**Remark**:

1. is pronounced and is pronounced as.



2. Since for all , we see that for every .



3. If , then . Thus, is an even



function.

4. is an odd function.



3. In contrast to sine and cosine, the hyperbolic functions are not periodic.

**Example 3.28**: Using the above definitions, show that



1. 



**Solution**:

1. We have



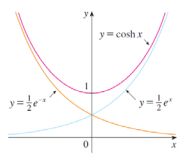
1. Left as an exercise.

* **The graph of**



Since is an even function, its graph is symmetric about the . Its intercept is , because . As tends to infinity, tends to infinity because goes to infinity and approaches to 0. When is a large negative number acts like , because gets close to 0. Thus the graph of looks like:





This graph can also be obtained by geometrically adding the two curves and , and taking half of each resulting . Observe that range of is .

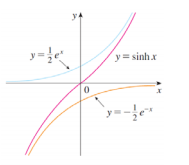


* **The graph of**



Since is an odd function, its graph is symmetric about the origin. The graph passes through the origin because . As gets large acts like and when is a large negative number, acts like . Thus, the graph of looks like:



****

The remaining four hyperbolic functions are defined in terms of and by



analogy with trigonometry.

|  |
| --- |
| (The domain of is .  (The domain of is )  (The domain of is )  (The domain of is ) |

You may sketch the graphs of these four hyperbolic functions (see exercise 19).

The trigonometric terminology and notation for the hyperbolic functions stem from the fact that they satisfy a list of identities that much resemble the familiar trigonometric identities, apart from an occasional difference of sign.

|  |
| --- |
| (1)  (2)  (3)  (4)  (5) |

The trigonometric functions are sometimes called circular functions because the point lies on the circle for all . Similarly, identity (1) tells us that the point lies on the hyperbola , and this is the reason for the name hyperbolic functions.



**Exercise 3.5**

1. Find the domain of the given function.
2. b) c) d)



1. Sketch the graph of the given function. Identify the domain, range, intercepts, and asymptotes.
2. b) c) d)



1. Solve the given exponential equation.
2. b) c) d)



1. Let . Show that .



1. Let . Show that .



1. Let . Show that .



1. Evaluate the given logarithmic expression (where it is defined).
2. c) e)



1. d) f)



1. If , find and the domain of .



1. If , find and the domain of .



1. Show that



1. Sketch the graph of the given function and identify the domain, range, intercepts and asymptotes.
2. b) c) d)



1. Find the inverse of .



1. Let . Find a function so that .



1. Convert the given angle from radians to degrees
2. b) c)



1. Convert the given angle from degrees to radians
2. b) c)



1. Sketch the graph of
2. c) e)



1. d) f)



1. Verify the following identities:



1. Given and , find .



1. Sketch the graphs of
2.  c) 
3.  d) 
4. Prove the identities (2) and (3).
5. Find the exact numerical value of
6. b) c)



1. Prove the following identities:

