**CHAPTER FOUR**

**4*.* Transportation and Assignment Models**

***4.1. Transportation Models***

**Sub section objectives**

Up on completion of this section, the learner will be able to:

* Explain transportation models
* Identify application areas of transportation models
* Formulate LP model for transportation problems
* Describe methods of finding initial feasible solution
* Check for optimality of transportation model using stepping stone and MODI techniques.

***Introduction***

 In this chapter we will try to see special types of LP models which are **Transportation and Assignment Models.**

The purpose of using a LP model for transportation problems is to **minimize transportation costs,** taking into account the **origin supplies, the destination demands,** and the **transportation costs.** The transportation method is similar in certain respects to the simplex technique because both involve an initial feasible solution that is evaluated to determine if it can be improved. However, the transportation method requires considerably less computational effort. In addition, it is not unusual to discover that the initial feasible solution in a transportation problem is the optimum.

4.1.1. FORMULATING THE MODEL

 In order to develop a model of a transportation problem, it is necessary to have the following information:

1. Supply quantity (capacity) of each origin.
2. Demand quantity of each destination.
3. Unit transportation cost for each origin-destination route.

 **Assumptions**

The transportation algorithm requires the assumption that:

* **All goods be homogeneous**, so that any origin is capable of supplying any destination,
* **Transportation costs are a direct linear** function of the quantity shipped
* we shall add one additional requirement that the **total quantity available is equal to the total demand.**

**Example**

 Harley’s Sand and Gravel Pit has contracted to provide topsoil for **three residential housing** developments. Topsoil can be **supplied from three different “farms**” as follows:

 **Farm** Weekly **capacity/supply** (cubic yards)

 A 100

 B 200

 C 200

**Demand** for the topsoil generated by the construction projects is:

 Project Weekly **demand**(cubic yards)

1. 50
2. 150
3. 300

The manager of the sand and gravel pit has estimated **the cost per cubic yard to ship** over each of the possible routes:

 Cost per cubic yard to

From Project #1 Project #2 Project #3

Farm A Birr4 Birr 2 Birr 8

Farm B 5 1 9

Farm C 7 6 3

This constitutes the information needed to solve the problem.

The **next step is to arrange the information into a transportation table**. This is shown in the following table.

Transportation table for Harley’s sand and gravel

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To:From:From: | Project #1 | Project #2 | Project #3 | Supply |
| Farm A |  |  | 4 |  |  | 2 |  |  | 8 | 100 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Farm B |  |  | 5 |  |  | 1 |  |  | 9 | 200 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300 |  |

☞ **Note**: It is possible to write this transportation problem in the form of LPM.

Nine ( 3x3) decision variables are there and the objective function is minimization.

LPM for this problem is:

Zmin: 4x11 + 2x12 +8x13 + 5x21 + x22 + 9x23 + 7x31 + 6x32 + 3x33

Subject to:

 x11 + x12 +x13  100

 x21 + x22 + x23200 Capacity/ Source constraint

 x31 + x32 + x33200

 x11 + x21 + x3 1 50

 x12 + x22 +x32150 Demand/Destination constraint

 x13 + x23 + x33300

This can be rewritten as:

Zmin: 4x11 + 2x12 +8x13 + 5x21 + x22 + 9x23 + 7x31 + 6x32 + 3x33

Subject to:

 x11 + x12 + x13  100

 x21 + x22 + x23 200

 x31 + x32 + x33 200

 x11 + x21 + x31  50

 x12 + x22 + x32 150

 x13 + x23 + x33 300

x11 , x12, x13 ,x21, x22 , x23 , x31, x32 , x33  0

4.1.2. FINDING AND INITIAL FEASIBLE SOLUTION

The starting point of the transportation method is a feasible solution. For an assignment to be feasible, two conditions must be fulfilled:

* A feasible solution is one in which assignments are made in such a way that all supply and demand requirements are satisfied.
* The **number of nonzero (occupied) cells** should equal **one less than the sum of the number of rows and the number of columns in a transportation table.** In the case of a table with 3 rows and 3 columns, the number of occupied cells should be 3 + 3 -1 = 5 in order to be able to use the transportation algorithm.

A number of different approaches can be used to find an initial feasible solution. Three of these are described here:

* The northwest-corner method.
* An intuitive approach/Least cost method
* Vogel’s / Penalty Method

4.1.2.1. The Northwest-Corner Method

The northwest corner method is a systematic approach for developing an initial feasible solution.

The northwest corner method gets its name because the **starting point for the allocation process is the upper left-hand (Northwest) corner of the transportation table.** For the Harley problem, this would be the cell that represents the route from Farm A to Project #1. The following set of principles guides the allocation:

* 1. Begin with the **upper left-hand cell**, and **allocate as many units** as possible to that cell. This will be the smaller of the row supply and the column demand. Adjust the row and column quantities to reflect the allocation.
	2. **Remain** in a row or column until its **supply or demand is completely exhausted** or **satisfied,** allocating the maximum number of units to each cell in turn, until all supply has been allocated (and all demand has been satisfied because we assume total supply and demand are equal).

Initial Feasible Solution for Harley using Northwest-corner method

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To:From: | Project #1 | Project #2 | Project #3 | Supply |
| Farm A |  |  | 4 |  |  | 2 |  |  | 8 | 100 |
| 50(first) | 50(second) |  |
| Farm B |  |  | 5 |  |  | 1 |  |  | 9 | 200 |
|  | 100(third) | 100(fourth) |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200 |
|  |  | 200(last) |
| Demand | 50 | 150 | 300 | 500 |

The **total cost is found by multiplying the quantities in “completed” (i.e. nonempty) cells** by the **cell’s unit cost** and, then, **summing those amounts.** Thus:

Total cost = 50(4) + 50(2) + 100(1) + 100(9) + 200(3) = $1900

As noted earlier, the **main drawback** of the northwest-corner method is that it **does not consider cell (route) costs** in making the allocation.

4.1.2.2. The Intuitive Approach/least cost cell method

This approach, also known as the **minimum-cost method**, uses **lowest cell cost** as the basis for selecting routes. The procedure is as follows:

1. Identify the **cell that has the lowest unit cost.** If there is a tie, select one arbitrarily. Allocate a quantity to this cell that is equal to the lower of the available supply for the row and the demand for the column.
2. Cross out the cells in the **row or column that has been exhausted** (or both, if both have been exhausted), and adjust the remaining row or column total accordingly.
3. Identify the cell with the **lowest cost from the remaining cells**. Allocate a quantity to this cell that is equal to the lower of the available supply of the row and the demand for the column.
4. Repeat steps (ii) and (iii) until all supply and demand have been exhausted.

The initial feasible solution for the Harley problem completed using the above steps is shown below.

Initial Feasible Solution for the Harley problem using the Intuitive approach

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To:From: | Project #1 | Project #2 | Project #3 | Supply |
| Farm A |  |  | 4 |  |  | 2 |  |  | 8 | 100 |
| 50 |  | 50 |
| Farm B |  |  | 5 |  |  | 1 |  |  | 9 | 200 |
|  | 150 | 50 |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200 |
|  |  | 200 |
| Demand | 50 | 150 | 300 | 500 |

We can easily verify that this is a **feasible solution by checking to see that the row and column totals of the assigned cell quantities equal the supply and demand totals for the rows and columns.** Now let us compute the total cost of this solution and compare it to that of the northwest corner solution.

 Total cost = 50(4) + 50(8) + 150(1) + 50(9) + 200(3) = $1800

Compared to the plan generated using the Northwest-corner method, this one has a total cost that is $100 less. This is due to the fact that the previous one did not involve the use of cost information in allocating units.

**4.1.2.3. Vogel’s Approximation Method (VAM)**

 Vogel’s Approximation Method (also called VAM), is based on the concept of **penalty cost orregret.**Ifa decision maker incorrectly chooses from several alternative courses of action, a penalty may be suffered (and the decision maker may regret the decision that was made). In transportation problem, the courses of action are the alternative routes and a wrong decision is allocating to a cell that does not contain the lowest cost.

This method is preferred over the other two methods because the initial feasible solution obtained with VAM is either **optimal or very close to the optimal.** With VAM the basis of **allocation is unit cost penalty** i.e. that column or row which **has the highest unit cost penalty** (**difference between the** **lowest and the next lowest cost**) is selected **first for allocation** and the subsequent allocations in cells are also done keeping in view the **highest unit cost penalty**.

**Steps in VAM**

1. Construct the cost, requirement, and availability matrix i.e. **cost matrix with column and row information.**
2. Compute a penalty for each row and column in the transportation table. The penalty is merely the **difference between the lowest cost and the next lowest** **cost element** in that particular row or column.
3. **Identify the row and column** with the **largest penalty**. In this identified row (column), **choose the cell which has the smallest cost and allocate the maximum possible quantity** **to this cell.** **Delete the row (column**) in which capacity (demand) is exhausted. When there is a tie for penalty, select one arbitrarily. After allocation, cross that row or column and disregard it from further consideration.
4. Repeat steps 1 to 3 for the reduced table until the entire capabilities are used to fill the requirement at different warehouses.
5. From step 4 we will get initial feasible solution. Now for initial feasible solution find the total cost.

The solution for our problem is as follows:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To: From: | Project #1 | Project #2 | Project #3 | Supply |
|  Farm A |  |  | 4 |  |  | 2 |  |  | 8 | 100 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  Farm B |  |  | 5 |  |  | 1 |  |  | 9 | 200 |
|  |  |  |  |  | 1501 |  |  |  |
|  |  |  |  |  |   |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200 |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300 |  |

Difference of highest & next highest

|  |  |  |  |
| --- | --- | --- | --- |
| R-1 | 2 | 4 | 4 |
| R-2 | 4 | 4 | 4 |
| R-3 | 3 | 4 | - |

penalty

|  |  |  |
| --- | --- | --- |
| 1 | 1 | 5 |
| 1 | - | 5 |
| 1 | - | 1 |

Needs revision

 Table2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To: From: | Project #1 | Project #2 | Project #3 | Supply  |
|  Farm A |  |  | 4 |  |  | 2 |  |  | 8 |  100  |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  Farm B |  |  | 5 |  |  | 1 |  |  | 9 |  200 50  |
|  |  |  |  |  | 150 |  |  |  |
|  |  |  |  |  |  |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200  |
|  |  |  |  |  |  | 200 2 |  |
|  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300 100 |  |

 Hc NHC Hc- NHC

8 4 4

 9 5 4

 7 3 4 selected

 Penalty 1 1 1 Second allocation

Table 3

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To: From:3 | Project #1 | Project #2 | Project #3 | Supply   |
|  Farm A |  | 4 |  |  | 2 |  |  | 8 |  100 50  |
|  | 50 |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
|  Farm B  |  |  | 5 |  |  | 1 |  |  | 9 |  200 50  |
|  |  |  |  |  | 150 |  |  |  |
|  |  |  |  |  |   |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200  |
|  |  |  |  |  |  | 200 |  |
|  |  |  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300 100 |  |

Third allocation

 Penalty

 4 selected

 4

 -

 Penalty 1 - 1

*Table 4*

Since there is no penalty for the remaining cells, we allocate for these cells according to their cost.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To: From: | Project #1 | Project #2 | Project #3 | Supply  |
|  Farm A |  |  | 4 |  |  | 2 | 4 | 8 |  100  50  |
|  | 50 |  |  |  | 50 |  |
|  |  |  |  |  |  |  |  |  |
|   Farm B  |  |  | 5 |  |  | 1 | 505 |  | 9 |  200 50  |
|  |  |  |  |  | 150 |  |  |
|  |  |  |  |  |   |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200  |
|  |  |  |  |  |  | 200 |  |
|  |  |  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300 100 50 |  |

 Fourth allocation

 Fifth allocation

Therefore the final allocation will be:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To: From: | Project #1 | Project #2 | Project #3 | Supply  |
|  Farm A |  |  | 4 |  |  | 2 | 4 | 8 |  100   |
|  | 50 |  |  |  | 50 |  |
|  |  |  |  |  |  |  |  |  |
|   Farm B  |  |  | 5 |  |  | 1 | 505 |  | 9 |  200  |
|  |  |  |  |  | 150 |  |  |
|  |  |  |  |  |   |  |  |  |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200  |
|  |  |  |  |  |  | 200 |  |
|  |  |  |  |  |  |  |  |  |
| Demand | 50 | 150 | 300  |  |

Total cost:

 1x150 + 3x200+4x50 + 8x50 + 9x50

 = Birr 1800

**4.1.3. Evaluating a Solution for Optimality**

The test for optimality for a feasible solution involves a cost evaluation of *empty cells* (i.e., routes to which no units have been allocated) to see if an improved solution is possible. We shall consider two methods for cell evaluation:

* The Stepping-stone method

The stepping-stone method involves a good deal of more effort than the MODI method. However, it provides an intuitive understanding of the evaluation process. Moreover, when a solution is not optimal, the distribution plan must be revised by reallocating units into and out of various cells, and only the stepping-stone method can be used for the reallocation.

4.1.3.1. The Stepping-stone method

The Stepping-stone method involves tracing a series of closed paths in the transportation table, using one such path for each empty cell. The path **represents a shift of one unit into an empty cell,** and it enables the manager or analyst to answer a “what-if” question: What impact on total cost would there be if one unit were shifted into an unused route? The result is a cost change per unit shifted into a cell. If the shift would result in a cost savings, the stepping-stone path also can be used to determine the maximum number of units that can be shifted into the empty cell, as well as modifications to other completed cells needed to compensate for the shift into the previously unused cell.

Reconsider the initial feasible solution we found using the northwest-corner method. Only the unoccupied cells need to be evaluated because the question at this point is not how many units to allocate to a particular route but only if converting a cell from zero units to nonzero (a positive integer) would decrease or increase total costs. The unoccupied cells are A-3, B-1, C-1, and C-2. They must be evaluated one at a time, but in no particular order.

Rules for tracing Stepping-stone paths:

1. **All unoccupied cells** must be evaluated. Evaluate **cells one at a time**.
2. Except for the cell being evaluated, only add or subtract in occupied cells. (It is permissible to skip over unoccupied cells to find an occupied cell from which the path can continue.)
3. A path will consist of only **horizontal and vertical moves**, starting and ending with the empty cell that is being evaluated.
4. Alter + and – signs, **beginning with a + sign in the cell being evaluated.**

☞ **Note** that it is not necessary to actually alter the quantities in the various cells to reflect the one-unit change, the + and – signs suffice.

The general implication of the plus **and minus signs is that cells with + signs mean one unit would be added, cells with a – sign indicate one unit would be subtracted.** The net impact of such a one-unit shift can be determined by **adding the cell costs** **with signs attached** and noting the resulting value. Thus, for cell B-1, we have a net change of +2 (i.e., 5+2-4-1) which means that for each unit shifted into cell B-1, the total cost would increase by $2.

Computed in similar way, the evaluations of cells C-1, A-3, and C-2 result in +10, -2, and +11 respectively. The negative value for cell A-3 indicates an improved solution is possible: For each unit we can shift into that cell, the total cost will decrease by $2. The following table shows how empty cell C-1 can be evaluated using the Stepping stone method.

Evaluation path for cell C-1

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  To:From: | Project #1 | Project #2 | Project #3 | Supply |
| Farm A |  |  | 4 |  |  | 2 |  |  | 8 | 100 |
| 50–- | 50+ |  |
| Farm B |  |  | 5 |  |  | 1 |  |  | 9 | 200 |
|  | 100– | 100+ |
| Farm C |  |  | 7 |  |  | 6 |  |  | 3 | 200 |
| + |  | – 200 |
| Demand | 50 | 150 | 300 | 500 |

# 4.2. ASSIGNMENT PROBLEMS

✪In the previous chapter, we discussed about the transportation problem. Now we consider another type of special linear programming problem, called the assignment problem.

Section objectives:

Up on completion of this section, the learner will be able to:

* Define assignment model
* Formulate LP model for assignment problems
* Identify areas of application
* Find optimal allocation or solution to the model

 There are many situations where the a**ssignment of people or machines** and so on, may be called for. **The assignment is a problem because people posses varying abilities for performing different** **jobs** and, therefore, the **costs of performing** the jobs by different people are different. Obviously, if all persons could do **a job in the same time or at the same cost** then it would not matter who of them is assigned the job. Thus, in assignment problem, the question is how should the assignment be made in order that the **total cost involved is minimized (or the total value is maximized when pay-offs are given in terms of, say, profits).**

A typical assignment problem follows:

**Example**

A production supervisor is considering how he should assign the **four jobs that are performed, to four of the workers** working under him. He want to assign the jobs to the workers such that the aggregate time to perform the job in the least.

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 45 | 40 | 51 | 67 |
| 2 | 57 | 42 | 63 | 55 |
| 3 | 49 | 52 | 48 | 64 |
| 4 | 41 | 45 | 60 | 55 |

4.2.1. THE ASSIGNMENT PROBLEM (A.P) definition:

1. The Assignment Problem (A.P) is a **special case of Transportation Problem** (T.P) under the condition that the **number of origins is equal to the number of destinations,**

 i.e. **m=n**

 Hence assignment is made on the basis of 1:1

* Number of jobs equal to number of machines or persons.
* Each man or machine is loaded with one and only one job.
* Each man or machine is independently capable of handling any of the jobs being presented.
* Loading criteria must be clearly specified such as “ minimizing operating time or “ maximizing profit”, or “ minimizing production cost” or minimizing production cycle time e.t.c
1. The Assignment Problem (A.P.) is a special case of Transportation Problem (T.P.) in which the number of sources and destinations are the same, and the objective is to assign the given job (task) to most appropriate machine (person) so as to optimize the objective like minimizing cost.
* **Cost Vector (Cij)**

Cost vector (Cij) is the cost of assigning ith job (Task) to jth machine (person),

* **Assignment Vector (Xij) is defined as follows**

Xij = 0; If ith job (Task) is not assigned by jth machine (person) or

Xij = 1; If ith job (Task) is assigned to jth machine (person).

* **Cost Matrix (Cij)**

Assignment problem is stated in the form of (n\*n) matrix. This is called cost matrix. This is illustrated as given below.

 Cij = cost of assigning ith job to jth machine

 (Symbols j = Job (Task) M = Machine (person)

General Assignment Problem Table

.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  **M** | **1** | **2** | **3** | **…** | **K** | **…** | **N** |
| **J** |
| 1 | C11 | C12 | C13 | … | C1k | … | C1n |
| 2 | C21 | C22 | C23 | … | C2k | … | C2n |
| 3 | C31 | C32 | C33 | … | C3k | … | C3k |
| … | … | … | … | … | … | … | … |
| k | Ck1 | Ck2 | Ck3 | … | Ckk | … | Ckn |
| … | … | … | … | … | … | … | … |
| n | Cn1 | Cn2 | Cn3 | … | Cnk | … | Cnn |

* **Effective Matrix**

A cost Matrix in A.P. is called an “Effectiveness Matrix” **when there is at least one zero in each row and column.** Following is an example of Effectiveness Matrix.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **1** | **2** | **3** | **4** |
| **1** | 0 | 1 | 3 | 4 |
| **2** | 5 | 0 | 6 | 0 |
| **3** | 7 | 8 | 0 | 9 |
| **4** | 0 | 4 | 3 | 8 |

1. **Mathematical Modeling of an A.P.**

Let there be ‘n’ jobs in a manufacturing plant. Let there be ‘n’ machines to process the product. A job i (i = 1, 2,…, n) when processed in a machine j (j = 1,2,…, n), it is assumed to incur a cost of Cij.

The assignment is made in such a way that one job is associated with one machine (see assumption). Hence we have the following:

  

 =

This special structure of A.P. makes solution much easy compared to the conventional T.P.

***Remark:***

1. It may be noted that assignment problem is a variation of transportation problem with two characteristics

(i) the cost matrix is **square matrix**, and

(ii) the optimum solution for the problem would always be such that there would be only one assignment in a given row or column of the cost matrix.

2. In assignment problem if a constant is added or subtracted from every element of any row or column in the given cost matrix then an assignment that minimizes the total cost in one matrix also minimizes the total cost in the other matrix.

4.2.3. HUNGARIAN ASSIGNEMNT METHOD (HAM)

A method, designed specially to handle the assignment problems in an efficient way, called the **Hungarian Assignment Method**, is available, which is based on the **concept of opportunity cost**. For a **typical balanced assignment problem** involving a certain number of persons and an equal number of jobs, and with an **objective function of the minimization type,** the method is applied as listed in the **following steps:**

Step 1. Locate the **smallest cost element in each row** of the cost table. Now **subtract this smallest from each element in that row.** As a result, there shall be at least one zero in each row of this new table, called the **Reduced Cost Table** (**Row Reduction**).

Step 2.In the **reduced cost table obtained, consider each column** and locate the smallest element in it. **Subtract the smallest value from every other entry in the column**. As a consequence of this action, there would be at least one zero in each of the rows and columns of the second **reduced cost table (Column Reduction).**

Step 3. **Draw the minimum number of horizontal and vertical lines** (not diagonal ones) that are required to cover the entire ‘zero’ elements. If the **number of lines drawn is equal to n (the number of rows/columns) the solution is optimal,** and proceeds to step 6. If the number of lines drawn is smallest than n, go to step 4.

Step 4. **Select the smallest uncovered** (by the lines) cost element. **Subtract this element** from **all uncovered elements** including itself and **add this element** to each value located **at the intersection of any lines**. The **cost elements through which only one line passes remain unaltered.**

Step 5. Repeat steps 3 and 4 until an optimal solution is obtained.

Step 6.Given the optimal solution, make the job assignments as indicated by the ‘zero’ elements. This done as follows:

1. **Locate a row which only ‘zero’ elemen**t. Assign the job corresponding to this element to its corresponding person. **Cross out the zero’s**, if any, in the column corresponding to the element, whish is indicative of the fact that the particular job and person are no more available.
2. Repeat (a) for each of such rows which contain only one zero. Similarly, perform the same operation in respect of each **column containing only one ‘zero’ element**, **crossing out the zero(s), if any, in the row in which the element lies.**
3. If there **is no row or column with only a single ’zero’** element left, then select a **row/column arbitrarily and choose one of the jobs (or persons**) and make the assignment. Now cross the remaining zeros in the column and row in respect of which the assignment is made.
4. Repeat steps (a) through (c) until all assignments are made.
5. Determine the total cost with **reference to the original cost table**.

**Example**

Solve the assignment problem given in Illustrative Example 1 for optimal solution using HAM. The information is reproduced in the following table

Time Taken (in minutes) by 4 workers

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 45 | 40 | 51 | 67 |
| 2 | 57 | 42 | 63 | 55 |
| 3 | 49 | 52 | 48 | 64 |
| 4 | 41 | 45 | 60 | 55 |

The solution to this problem is given here in a step-wise manner.

Step 1: The **minimum value of each row is subtracted from all elements** in the row. It is shown in the reduced cost table, also called opportunity cost table, given as follows.

 Table 1**-row reduction**

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 5 | 0 | 11 | 27 |
| 2 | 15 | 0 | 21 | 13 |
| 3 | 1 | 4 | 0 | 16 |
| 4 | 0 | 4 | 19 | 14 |

**Step 2:** For **each column of this table, the minimum value is subtracted from all the other values**. Obviously, the columns that contain a zero would remain unaffected by this operation. Here only the fourth column values would change. The table below shows this.

 **Table 2 column reduction**

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 5 | 0 | 11 | 14 |
| 2 | 15 | 0 | 21 | 0 |
| 3 | 1 | 4 | 0 | 3 |
| 4 | 0 | 4 | 19 | 1 |

Step 3: Draw the **minimum number of lines covering all zeros**. As a general rule, we should first **cover those rows/columns which contain larger number of zeros**. The above table is reproduced in the next table and the lines are drawn.

 Table3 draw line & cover zeros

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 5 | 0 | 11 | 14 |
| 2 | 15 | 0 | 21 | 0 |
| 3 | 1 | 4 | 0 | 3 |
| 4 | 0 | 4 | 19 | 1 |

Step 4: Since the **number of lines drawn is equal to 4(=n), the optimal solution is obtained**. The assignments are made after scanning the rows and columns **for unit zeros**. Assignments made are shown with squares, as shown in the following table.

**Assignment of Jobs**

|  |  |
| --- | --- |
| **Worker** | **Job** |
| A | B | C | D |
| 1 | 5 | 0 | 11 | 14 |
| 2 | 15 |  0 **X** | 21 | 0 |
| 3 | 1 | 4 | 0 | 3 |
| 4 | 0 | 4 | 19 | 1 |

Assignments are made in the following order. Rows 1, 3, and 4 contain only one zero each. So assign 1-B, 3-C and 4-A. Since worker 1 has been assigned job B, only worker 2 and job D are left for assignment. The final pattern of assignments is **1-B, 2-D, 3-C, and 4-A, involving a total** time of **40+55+48+41=184 minutes.** This is the **optimal solution** to the problem-the same as obtained by enumeration and transportation methods.

**Example**

Using the following cost matrix, determine (a) optimal assignment, and (b) the cost of assignments.

 Reduced Cost Table 1

|  |  |
| --- | --- |
| **Machinist** | **Job** |
| 1 | 2 | 3 | 4 | 5 |
| A | 10 | 3 | 3 | 2 | 8 |
| B | 9 | 7 | 8 | 2 | 7 |
| C | 7 | 5 | 6 | 2 | 4 |
| D | 3 | 5 | 8 | 2 | 4 |
| E | 9 | 10 | 9 | 6 | 10 |

**Iteration 1:** Obtain **row reductions**.

Reduced Cost Table 1

|  |  |
| --- | --- |
| **Machinist** | **Job** |
| 1 | 2 | 3 | 4 | 5 |
| A | 8 | 1 | 1 | 0 | 6 |
| B | 7 | 5 | 6 | 0 | 5 |
| C | 5 | 3 | 4 | 0 | 2 |
| D | 1 | 3 | 6 | 0 | 2 |
| E | 3 | 4 | 3 | 0 | 4 |

**Iteration 2:**  Obtain **column reductions and draw the minimum number of lines** to cover all zeros.

Reduced Cost Table2

|  |  |
| --- | --- |
| **Machinist** | **Job** |
| 1 | 2 | 3 | 4 | 5 |
| A | 7 | 0 | 0 | 0 | 4 |
| B | 6 | 4 | 5 | 0 | 3 |
| C | 4 | 2 | 3 | 0 | 0 |
| D | 0 | 2 | 5 | 0 | 0 |
| E | 2 | 3 | 2 | 0 | 2 |

Since the **number of lines covering** **all zeros is less than the number of columns/rows**, we modify this table. The **least of the uncovered cell values is 2**. This value would be **subtracted** from each of **the uncovered** **values** and **added to each value lying at the intersection** of lines (**corresponding to cells A-4, D-4, A-5 and D-5).** Accordingly, the new table would appear as shown as follows.

**Iteration 3**

|  |  |
| --- | --- |
| **Machinist** | **Job** |
| 1 | 2 | 3 | 4 | 5 |
| A | 7 | 0 |  0 **X** | 2 | 6 |
| B | 4 | 2 | 3 | 0 | 3 |
| C | 2------------ | 0 **X--------** | 1----------- | 0  **X-------** | 0------- |
| D | 0 | 2 | 5 | 2 | 2 |
| E |  0 **X** | 1 | 0 | 0 **X** | 2 |

The optimal assignments can be made as the **least number of lines covering all zeros** in Table 6.14 **equals 5.** Considering rows and columns, the assignments can be made in the following order:

1. Select the second row. Assign machinist B to job 4. Cross out zeros at cells C-4 and E-4.
2. Consider row 4, Assign machinist D to job 1. Cancel the zero at cell E-1.
3. Since there is a single zero in the row, put machinist E to job 3 and cross out the zero at A-3.
4. There being only a single left in each of the first and third rows, we assign job 2 to machinist A and job 5 to C.

The total cost associated with the optimal machinist-job assignment pattern A-2, B-4, C-5, D-1 and E-3 is 3+2+4+3+9 = 21

4.2.5. Special Issues

When we solve assignment problems, there are cases which are treated differently from the usual way.

**Unbalanced Assignment Problems**

The Hungarian Method of solving an assignment problem requires that the number of columns should be equal to the number of rows. They are **equal, the problem is balanced** problem, and **when not,** it is called an **unbalanced problem**. Thus, where there are 5 workers and 4 machines, or when there are 4 workers and 6 machines, for instance, we have unbalanced situations in which one-to-one match is not possible. In case the machines are in excess, the excess machine(s) would remain idle.

In **such situations, dummy column(s)/row(s),** whichever is smaller in number, are **inserted** with **zeros as the cost elements**.

**Example:**

A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given below. Determine the job assignments which will minimize the total cost.

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 18 | 24 | 28 | 32 |
| B | 8 | 13 | 17 | 18 |
| C | 10 | 15 | 19 | 22 |

**Introduction of dummy**

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 18 | 24 | 28 | 32 |
| B | 8 | 13 | 17 | 18 |
| C | 10 | 15 | 19 | 22 |
|  | D | 0 | 0 | 0 | 0 |

**Raw reduction**

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 6 | 10 | 14 |
| B | 0 | 5 | 9 | 10 |
| C | 0 | 5 | 9 | 12 |
|  | D | 0 | 0 | 0 | 0 |

Column reduction

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 6 | 10 | 14 |
| B | 0 | 5 | 9 | 10 |
| C | 0 | 5 | 9 | 12 |
|  | D | 0 | 0 | 0 | 0 |

Drawing the lines

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 6 | 10 | 14 |
| B | 0 | 5 | 9 | 10 |
| C | 0 | 5 | 9 | 12 |
|  | D | 0 | 0 | 0 | 0 |

In this case the covering lines are less than no. of rows & columns then **select the least cost(5) from uncovered cost** and subtract it from uncovered and add at the intersection points.

Modified table with **least uncovered cost**

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 1 | 5 | 9 |
| B | 0 | 0 | 4 | 5 |
| C | 0 | 0 | 4 | 7 |
|  | D | 5 | 0 | 0 | 0 |

In this table the smallest uncovered cost is -4

Still there are less numbers of lines continue the same method

Second modified table

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 1 | 1 | 5 |
| B | 0 | 0 | 0 | 1 |
| C | 0 | 0 | 0 | 3 |
|  | D | 5 | 0 | 0 | 0 |

Now the numbers of lines are equal to R = C=4=4

Assign machines to jobs

|  |  |
| --- | --- |
| **Job** | **Machine** |
|  | W | X | Y | Z |
| A | 0 | 1 | 1 | 5 |
| B | 0x | 0 1st | 0x1st | 1 |
| C | 0x | 0x2nd | 0 2nd | 3 |
|  | D | 5 | 0x | 0x | 0 |

The assignment is done as follows

W-A, X-B, Y-C &Z-D=18+13+19+0=50 birr

W-A, X-C, Y-B & Z-D=18+15+17+0=50 birr

**Unique Vs Multiple Optimal Solutions**

 In any of the problems discussed so far, we have not experienced such a situation. Hence, each one of them has had a unique optimal solution. When a problem has a unique optimal solution, it means that no other solution to the problem exists which yields the same objective function value (cost, time, profit e. t. c) as the one obtained from the optimal solution derived. On the other hand in a problem with multiple optimal solutions, there exists more than one solution which all is optimal and equally attractive. Consider the following example.

**Example**:

Solve the following assignment problem and obtain the minimum cost at which all the jobs can be performed.

|  |  |
| --- | --- |
| **Worker** | **Job (cost in ’00 Br.)** |
| **1** | **2** | **3** | **4** | **5** |
| **A** | 25 | 18 | 32 | 20 | 21 |
| **B** | 34 | 25 | 21 | 12 | 17 |
| **C** | 20 | 17 | 20 | 32 | 16 |
| **D** | 20 | 28 | 20 | 16 | 27 |

**Solution:** This problem is **unbalanced since number of jobs is 5** while the **number of workers is 4.** We first balance it by introducing a **dummy worker E**, as shown in table below.

Balancing the Assignment Problem

|  |  |
| --- | --- |
| **Worker** | **Job (cost in ’00 Br.)** |
| **1** | **2** | **3** | **4** | **5** |
| **A** | 25 | 18 | 32 | 20 | 21 |
| **B** | 34 | 25 | 21 | 12 | 17 |
| **C** | 20 | 17 | 20 | 32 | 16 |
| **D** | 20 | 28 | 20 | 16 | 27 |
| **E** | 0 | 0 | 0 | 0 | 0 |

**Step 1:** Obtain reduced cost values by subtracting the minimum value in each row from every cell in the row. This is given in Table below.

**Cost reduction**

|  |  |
| --- | --- |
| **Worker** | **Job (cost in ’00 Br.)** |
| **1** | **2** | **3** | **4** | **5** |
| **A** | 7 | 0 | 14 | 2 | 3 |
| **B** | 22 | 13 | 9 | 0 | 5 |
| **C** | 4 | 1 | 4 | 16 | 0 |
| **D** | 4 | 12 | 4 | 0 | 11 |
| **E** | 0 | 0 | 0 | 0 | 0 |

Since there is at least one zero in each row and column, we test it for optimality. Accordingly, lines are drawn. All zeros are **covered by 4 lines, which is less than 5 (the order of the given matrix).** Hence, we proceed to improve the solution. The **least uncovered value is 4.** **Subtracting from every uncovered value and adding it to every value lying at the intersection** of lines, we get the revised values as shown below.

**Reduce cost 2 and Assignment**

|  |  |
| --- | --- |
| **Worker** | **Job (cost in ’00 Br.)** |
| **1** | **2** | **3** | **4** | **5** |
| **A** | 7 | 0 | 14 | 6 | 3 |
| **B** | 18 | 9 | 5 | 0 | 1 |
| **C** | 4 | 1 | 4 | 20 | 0 |
| **D** | 0 | 8 | 0X |  0 **X** | 7 |
| **E** |  0 **X** | 0 **X** | 0 | 4 | 0 **X** |

The solution given in the reduced cist 2 table is optimal since the number of lines covering zeros matches with the order of the matrix. We can, therefore, proceed to make assignments. To begin with, since each of the columns has multiple zeros, we cannot start making assignments considering columns and have, therefore, to look through rows. The first row has a single zero. Thus, we make assignment A-2 and cross out zero at E-2.

Further, the second and the third rows have one zero each. We make assignments B-4 and C-5, and cross out zeros at D-4 and E-5. Now, **both the rows left two zeros** each and so have both the columns. This **indicates existence of multiple optimal solutions**. To obtain the solutions, we **select zeros arbitrarily** and proceed as discussed below.

1. Select the zero at D-1, make assignment and cross out zeros at D-3 and E-1 (as both, worker D and job 1, are not available any more). Next, assign worker E to job 3, corresponding to the only zero left. Evidently, selecting the zero at E-3 initially would have the effect of making same assignments.
2. Select the zero at D-3, make assignment and cross at D-1 and E-3. Next, make assignment at the only zero left at E-1. Obviously, selecting the zero at E-1 making assignment in the first place would lead to the same assignments.

To conclude, the problem **has two optimal solutions** as given below.

|  |  |  |  |
| --- | --- | --- | --- |
| **Solution 1** | **(’00 Br.)** | **Solution 2** | **(’00 Br.) cost** |
| Worker | Job | Cost | Worker | Job | Cost |
|  A  | 2 | 18 |  A  | 2 | 18 |
| B | 4 | 12 | B | 4 | 12 |
| C | 5 | 16 | C | 5 | 16 |
| D | 1 | 20 | D | 3 | 20 |
| Job left | 3 |  | Job left | 1 |  |
|  | Total | 66 |  | Total | 66 |

**Maximization Case in Assignment Problem**

 In **some situations**, the assignment problem may call for **maximization of profit**, revenue, e.t.c. as the objective. For example, we may be faced with the problem of assignment of salesmen in different regions in which they can display different qualities in making sales

(reflected in amounts of sales executed by them). Obviously, assignment would be made in such a way that the total expected revenue is maximized.

For dealing with a maximization problem, **we first change it into an equivalent minimization problem.** This is achieved by **subtraction** **each of the elements** of the given pay-off matrix from a constant (value) k. Usually, **the largest value of all values** in the given matrix is located and then **each one of the values is subtract from it** (the largest value is taken so as **to avoid the appearance of negative signs**). The rest operations are run as **usua**l

**Example:**

A company plans to assign 5 salesmen to 5 districts in which it operates. Estimates of sales revenue in thousands of birr for each salesman in different districts are given in the following table. In your opinion, what should be the placement of the salesmen if the objective is to maximize the expected sales revenue?

**First select largest REVENUE among the whole entries**

|  |  |
| --- | --- |
| **Salesman** | **District** |
| D1 | D2 | D3 | D4 | D5 |
| S1 | 40 | 46 | 48 | 36 | 48 |
| S2 | 48 | 32 | 36 | 29 | 44 |
| S3 | 49 | 35 | 41 | 38 | 45 |
| S4 | 30 | 46 | 49 | 44 | 44 |
| S5 | 37 | 41 | 48 | 43 | 47 |

**Solution:** Since it is a maximization problem, we would first **subtract each of the entries** in the table **from the largest one, which equals 49** here. The resultant data are given in Table below.

Opportunity Loss Matrix-**reduction of all entries from the largest**

|  |  |
| --- | --- |
| **Salesman** | **District** |
| D1 | D2 | D3 | D4 | D5 |
| S1 | 9 | 3 | 1 | 13 | 1 |
| S2 | 1 | 17 | 13 | 20 | 5 |
| S3 | 0 | 14 | 8 | 11 | 4 |
| S4 | 19 | 3 | 0 | 5 | 5 |
| S5 | 12 | 8 | 1 | 6 | 2 |

 Now, we will proceed as usual.

**Step 1:** **Subtract small value** in each row from every value in the row.

 Table 1-**raw reduction**

|  |  |
| --- | --- |
| **Salesman** | **District** |
| D1 | D2 | D3 | D4 | D5 |
| S1 | 8 | 2 | 0 | 12 | 0 |
| S2 | 0 | 16 | 12 | 19 | 4 |
| S3 | 0 | 14 | 8 | 11 | 4 |
| S4 | 19 | 3 | 0 | 5 | 5 |
| S5 | 11 | 7 | 0 | 5 | 1 |

**Step 2:** Subtract minimum value in each column in reduced cost table 1 from each value in the column. Test for optimality by drawing lines to cover zeros. These are shown in table below (in the Reduced cost Table 2)

 Table 2-**column reduction**

|  |  |
| --- | --- |
| **Salesman** | **District** |
| D1 | D2 | D3 | D4 | D5 |
| S1 | 8 | 0 | 0 X | 7 | 0X |
| S2 | 0 | 14 | 12 | 14 | 4 |
| S3 | 0x | 12 | 8 | 6 | 4 |
| S4 | 19 | 1 | 0 | 0X | 5 |
| S5 | 11 | 5 | 0X | 0 | 1 |

Since the **number of lines covering all zeros is fewer than n=4<5,** then we select **least value from uncovered cell value,** which **equals 4**. With this, we can modify the table as given in the Reduced Cost Table 3.

**Steps 4, 5, 6:** Find improved solution. Test for optimality and make assignments.

Reduced cost Table 3

|  |  |
| --- | --- |
| **Salesman** | **District** |
| D1 | D2 | D3 | D4 | D5 |
| S1 | 12 | 0 |  0 **X** | 7 | 0 **X** |
| S2 | 0 | 10 | 8 | 10 | 0 **X** |
| S3 |  0  **X** | 8 | 4 | 2 | 0 |
| S4 | 23 | 1 | 0 | 0 **X** | 5 |
| S5 | 15 | 5 | 0 **X** | 0 | 1 |

More than one optimal assignment is possible in this case because of the existence of multiple zeros in different rows and columns. The possible assignments are:

 S1-D2, S2-D5, S3-D1, S4-D3, S5-D4; or

 S1-D2, S2-D1, S3-D5, S4-D3, S5-D5; or

 S1-D2, S2-D1, S3-D1, S4-D4, S5-D3; or

 S1-D2, S2-D1, S3-D5, S4-D4, S5-D

Each of these assignment patterns would lead to expected aggregated sales equal to 231 thousand birr.