**CHAPTER THREE**

**3. Post optimality Analysis**

✪Dear learner, in the previous chapters we discussed about how to solve LP problems using either graphic or simplex approach which is a base for this section. This unit deals with the other version of the simplex model, dual and the sensitivity of the optimal solution to changes in different parameters.

**3.1. DUALITY**

Section objective:

Up on completion of this sub section the learner will be able to:

* Define duality
* Describe the importance of duality
* Explain how to convert dual in to primal and the vice versa
* Formulate dual when the constraints are mixed
* Elucidate how to read the solution for primal from dual’s solution and vice versa.
* Analyze the impact of addition of a new product on the decision

The term **‘dual’** in a general sense **implies two or double**. Every linear programming problem can have **two forms.** The **original formulation of a problem** is referred to as its ***Primal form***. The other form is referred to as its **dual** LP problem or in short ***dual form***.

In the context of LP, duality implies that each LP problem can be analyzed in **two different ways,** **but having equivalent solution.** For example, consider the problem of production planning. By using the primal LP problem, the production manager **attempts to optimize resource** allocation by determining quantities for each product to be produced that will maximize profit. But through **dual LP problem approach**, he attempts to **achieve production plan** that **optimizes resource allocation** so that each product is produced at that quantity such that its marginal opportunity cost equals its marginal return. Thus, the main focus of dual is to find for each resource its best **marginal value or shadow price**. This value reflects the **scarcity** of resources, i.e., the maximum additional prices to be paid to obtain one additional unit of the resources to maximize profit under the resource constraints. If resource is not completely used, i.e., there is slack, then its marginal value is zero.

The shadow price is also defined as the **rate** of change in the optimal objective function value with the respect to the unit change in the availability of a resource. Precisely for any constraint, we have,

 Shadow Price = change in the optimal objective function value

 unit change in the availability of a resource

3.1.1. Formulating the Dual

There are two important forms of primal and dual problems, namely the symmetrical ( or canonical) and the standard form.

**Symmetrical form**

Suppose the primal LP problem is given in the form

 Maximize Zx = c1x1+ c2x2 + ... +cnxn

 Subject to

 a11x1+a12x2+... + a1nxn b1

 a21x1+a22x2+... + a2nxn b2

 am1x1+am2x2+... + amnxn bm

 x1,x2... xn 0

and then the corresponding dual LP problem is defined as:

Minimize Zy = b1y1+ b2y2 + ... +bmym

 Subject to

 a11y1+a21y2+... + am1ym c1

 a12y1+a22y2+... + am2ym c2

 a1ny1+an2y2+... + amnym cn

 y1,y2... ym 0

The **following rules which guide the formulation of the dual problem** will give you the summary of the general relationship between primal and dual LP problems.

1. If the **primal’** objective is to **minimize**, the **dual’s will be to maximize**; and the vice versa
2. The **coefficient’s of the primal’s objective** function become **the RHS values** for the **dual’s constraints**.
3. The **primal’s RHS values** become the **coefficients of the dual’s** objective function.
4. The **coefficients of the first “row**” of the primal’s constraints become the **coefficients of the first “column” of the dual’s** constraint …..
5. The ≤ constraints become ≥ and the vice versa.

Consider this **Primal problem**:

Minimize 40x1 + 44x2 + 48x3

Subject to: 1x1 + 2x2 + 3x3 > 20

 4x1 + 4x2 + 4x3 > 30

 x1, x2, x3 > 0

The **dual** of this problem is:

Maximise Z = 20y1 + 30y2

Subject to:

1y1 + 4y2 < 40

2y1 + 4y2 < 44

3y1 + 4y2 < 48

y1, y2 > 0

We can see from the above table that the original objective was **to minimize**, whereas the **objective of the dual is to maximize**. In addition, the **coefficients** of the **primal’s** **objective** function become the **right-hand side values** for the **dual’s constraints**, whereas the **primal’s right-hand side values** become the **coefficients of the dual’s objective function**.

When the **primal** problem is a **maximization problem** with all < constraints, the **dual** is a **minimization** problem with all > constraints.

**3.1.2. Formulating the Dual when the Primal has Mixed Constraints**

In order to transform a primal problem into its dual, it is easier if all constraints in a maximization problem are of the < variety, and in a minimization problem, every constraint is of the > variety.

To change the direction of a constraint, multiply both sides of the constraints **by -1 based on objective function.** For example, if objective function is **maximization**

 -1(2x1 + 3x2 > 18) is -2x1-3x2 < -18

If a constraint is **an equality,** it must be **replaced with two constraints**, one with a < sign and the other with a > sign. For instance,

 4x1 + 5x2 = 20

 will be replaced by

 4x1 + 5x2 < 20

 4x1 + 5x2 > 20

Then the **second constraint** of these must be multiplied by -1, **depending** on whether the primal is **maximization or a minimization** problem.

**EXAMPLE:**

Formulate the dual of this LP model.

 Maximize Z = 50x+ 80x

 Subject to:

 C 3x+ 5x≤ 45

 C 4x+ 2x ≥16

 C3 6x+6x= 30

 x,x ≥ 0

 **SOLUTION**

Since the problem is a **max problem**, put all the constraints in to **the ≤ form**. Subsequently, C and C3 will be **first adjusted in to ≤ constraints**.

* Cwill be multiplied by -1:

 -1(4x+ 2x ≥16) becomes -4x- 2x≤ -16

- C3 is **equality**, and must be restated as **two separate constraints**. Thus, it becomes:

 6x+6x≤ 30 and 6x+6x ≥30. Then the **second** of these must be multiplied by -1.

 -1(6x+6x ≥30) becomes -6x-6x≤ -30

**Adjusted LP model is**

Maximize z = 50x+ 80x

 Subject to:

 C 3x+ 5x≤ 45

 C -4x- 2x≤ -16

 C3 6x+6x≤ 30

 C4 -6x-6x≤ -30

 x, x ≥ 0

The **dual of the above problem** will be:

 **Minimize** Z= 45y1 - 16y2 + 30y3 – 30y4

 Subject to:

 C 3y1 -4y2 + 6y3 – 6y4 ≥ 50

 C 3y1 - 2y2 + 6y3 – 6y4 ≥ 80

 y1, y2, y3, y4 ≥ 0

3.1.3. Economic Interpretation of the Dual

Suppose the microcomputer firm has been approached by a representative of a department store chain that wants the firm to make computers that will be sold under the store’s brand name. The microcomputer company has only a limited capacity for producing computers, and therefore, must, decide whether to produce its own computers or produce computers for the department store.

For convenience, the original problem is repeated here:

 x1 = number of Model I

 x2 = number of Model II’s

Maximize 60x1 + 50x2

Subject to: Assembly time: 4x1 + 10x2 < 100

 Inspection time: 2x1 + 1x2 < 22

 Storage space: 3x1 + 3x2 < 39

 x1, x2 >= 0

The manager of the firm would reason in the following way. For each unit of Model I that the firm sacrifices to produce computers for the department store, it will gain 4 hours of assembly time, 2 hours of inspection time, and 3 cubic feet of storage space, which can be applied to the store computers. However, it will also give up a unit profit of $60. Therefore, in order for the firm to realistically consider the store’s offer, the amounts of scarce resources that will be given up must produce a return to the firm that is at least equal to the foregone profit. Hence, the value of 4 assembly hours + 2 inspection hours + 3 cubic feet of storage space > $60. By similar reasoning, giving up one unit of Model II will require that the value received by giving up 10 assembly hours + 1 inspection hour + 3 cubic feet of storage must equal or exceed the Model II profit of $50 per unit. These, then, become the constraints of the dual problem. Thus:

 Value Received Resources Minimum Profit

 Per Unit of Freed Up Required

 Model I 4y1 + 2y2 + 3y3 $60

 Model II 10y1 + 1y2 + 3y3 $50

Hence the constraints of the dual refer to the value of capacity (i.e., the scarce resources). The formulation indicates that in order to switch from making units of Models I and II to making computers for the department store, the value received from that switch must be at least equal to the profit foregone on the microcomputer models. The variables y1, y2 and y3 are the marginal values of scarce resources (assembly time, inspection time, and storage space). Solving the dual will tell us the imputed values of the resources given our optimal solution.

Naturally, the department store would want to minimize the use of the scarce resources, because the computer firm almost certainly would base its charges on the amount of resources required. Consequently, the objective function for the dual problem focuses on minimizing the use of the scarce resources. Thus:

 Minimize: 100y1 + 22y2 + 39y3

Looking at the optimal solution to the dual of the microcomputer problem, we can see the marginal values of y2 and y3 in the Quantity column, but not the value of y1. This is because the optimal solution to the primal did not completely use up all of the assembly capacity. Consequently, no amount of either x1 or x2 would need to be given up to obtain one free hour of assembly time. Thus, the marginal value of one hour of assembly time is $0.

Finally, the optimal dual solution always yields the same value of the objective function as the primal optimal. In this case, it is 740. The interpretation is that the imputed value of the resources that are required for the optimal solution equals the amount of profit that the optimal solution would produce.

3.1.4. Comparison of the Primal and Dual Simplex Solutions

Cross -referencing the values in the primal and dual final simplex tableaus is shown as follows.

|  |  |  |
| --- | --- | --- |
|  **PRIMAL** | **HOW LEBELED/ WHERE****FOUND IN THE PRIMAL** | **CORRESPONDENCE IN** **THE DUAL** |
| -Decision variable | x,xx,….. | s1,s2,s3,……. |
| - slack variable | s1,s2,s3,……. | y1,y2,y3,…….. |
| - shadow prices | Z row under slack column | Quantity column in decision variable rows. |
| - solution quantities | Quantity column |  C-Z row under slack and decision variable columns |

Example:

To show that the flip-flopping of values between the primal and the dual carries over to their final simplex tableaus, let us look at the following tables. The first table contains the final tableau for the dual and the second one contains the final tableau for the primal.

Final tableau of Dual solution to the Microcomputer problem

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| CBasis | 100 | 22 | 39 | 0 | 0 | M | M | QuantityPrimal shadow prices |
| y1 | y2 | y3 | s1 | s2 | a1 | a2 |
| y3 | 39 | 16/3 | 0 | 1 | 1/3 | -2/3 | -1/3 | 2/3 | 40/3 |
| y2 | 22 | -6 | 1 | 0 | -1 | 1 | 1 | -1 | 10 |
| ZC-Z | 76 | 22 | 39 | -9 | -4 | 9 | 4 | 740 |
| 24Primal solution quantities | 0 | 0 | 9 | 4 | 9-M | 4-M |

Final tableau of Primal solution to the Microcomputer problem:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| CBasis | 60 | 50 | 0 | 0 | 0 | Quantity |
| x1 | x2 | s1 | s2 | s3 |
| s1 | 0 | 0 | 0 | 1 | 6 | -16/3 | 24 |
| x1 | 60 | 1 | 0 | 0 | 1 | -1/3 | 9 |
| x2 | 50 | 0 | 1 | 0 | -1 | 2/3 | 4 |
| ZC-Z | 60 | 50 | 0 | 10 | 40/3 | 740 |
| 0 | 0 | 0 | -10 | -40/3 |

The primary concern with a simplex solution is often three fold:

* 1. Which variables are in solution?
	2. How much of each variable is in the optimal solution?
	3. What are the shadow prices for the constraints?

Let’s consider how we can obtain the answers to these questions from the dual solution. Notice that the solution quantities of the dual are equal to the shadow prices of the primal (i.e. 40/3 and 10). Next, notice that the values of the solution quantities of the primal (i.e., 24, 9, and 4) can be found in the bottom row of the dual. Now, in the primal solution, s1 equals 24. In the dual, the 24 appears in the y1 column. The implication is that a slack variable in the primal solution becomes a real variable (i.e., a decision variable) in the dual. The reverse is also true: A real variable in the primal solution becomes a slack variable in the dual. Therefore, in the primal solution we have x1 = 9 and x2 = 4; in the dual, 9 appears under s1 in the bottom row, and 4 appears under s2. Thus, we can read the solution to the primal problem from the bottom row of the dual: In the first three columns of the dual, which equate to slack variables of the primal, we can see that the fist slack equals 24 and the other two are zero. Under the dual’s slack columns, we can read the primal values of the decision variables. In essence, then, the variables of the primal problem become the constraints of the dual problem, and vice versa.

3.1.5. Managerial Significance of Duality

The importance of the dual LP problem in terms of the information which it provides about the value of the resources. The economic analysis is concerned with deciding whether or not to secure more resource and how much to pay for these additional resources. The significance of the study dual is as follows;

* The dual variables provide the decision maker a basis for deciding how much to pay for additional unit of resources.
* The maximum amount that should be paid for one additional unit of a resource is called its shadow price (also called simplex multiplier).
* The total marginal value of the resources equals the optimal objective function value. The dual variables equal the marginal value of resources (shadow prices).

***Note:*** The value of ith dual variable is the rate at which the primal objective function value will increase as the ith resource increases, assuming that all other data is unchanged.

3.1.6. Advantages of Duality

* It is advantageous to solve the dual of a primal having less number of constraints, because the number of constraints usually equals the number of iterations required to solve the problem.
* It avoids the necessity of adding surplus or artificial variables and solve the problem quickly.
* The dual variables provide an important economic interpretation of the final solution of an LP problem.
* It is quite useful when investigating change in the parameters of an LP problem ( called Sensitivity analysis)