**CHAPTER TWO**

# 2. LINEAR PROGRAMMING: Application and Model

# Formulation

# INTRODUCTION

The term **linear implies** that all the mathematical relations used in the problem are **linear or straight-line relations,** while the term **programming refers** to the method of determining a particular program or plan of action, i.e., **the use of algorithms** that is a well defined sequence of steps that will lead to an optimal solution. Taken as a whole, the term linear programming refers to a family of mathematical techniques for determining the **optimum allocation of resources and obtaining a particular objective** when there are alternative uses of the limited or constrained resources.

The technique of linear programming is applicable to problems in which the total effectiveness can be expressed as linear function of individual allocations and the limitations on resources give rise to linear equation or inequalities of the individual allocations.

# 2.1. LINEAR PROGRAMMING MODELS

Linear programming models are **mathematical representations** of LP problems. Knowledge of these characteristics enables us to **recognize problems** that are amenable to **a solution using LP models,** and to be able to correctly formulate an LP model. These characteristics can be grouped as **components and assumptions**. The components relate to the **structure** of a model, where as the **assumptions reveal the conditions** under which the model is valid.

# 2.1.1. COMPONENTS OF LP MODELS

There are **four major components** of LP models including: **Objective function, decision variables, constraints and parameters.**

## Objective and Objective Function

The objective in problem solving is the **criterion by which all decisions** **are evaluated**. It provides the focus for problem solving. In linear programming models, a single, quantifiable objective must be specified by the decision maker. Because we are dealing with **optimization,** the objective will be either **maximization or minimization**.

A LP model consists of a mathematical statement of the objective called the objective function.

## Decision variables

They represent **unknown quantities** **to be solved** for. The decision maker can **control** the **value of the objective**, which is achieved through choices in the levels of decision variables. For example, how much of each product should be produced in order to obtain the greatest profit?

## Constraints

However, the ability of a decision maker to select values of the decision variables in a LP problem is subject to **certain restrictions or limits** **coming from a variety of sources.** The **restrictions may reflect availabilities of resources** (e.g., raw materials, labor time, etc.), legal or contractual requirements (e.g., product standards, work standards, etc.), technological requirements (e.g., necessary compressive strength or tensile strength) or they may reflect other limits based on forecasts, customer orders, company policies, and so on. In LP model, the **restrictions are referred to as constraints**. **Only solutions** **that satisfy all constraints** in a model are acceptable and are referred to as **feasible solutions**. The **optimal solution** will be the one that provides the **best value for the objective function**.

Generally speaking, a constraint has **four elements**:

* A **right hand side** (RHS) quantity that specifies **the limit for that constraint**. It must be a constant, not a variable.
* An **algebraic** sign that indicates whether the limit is an **upper bound** that cannot be exceeded, a **lower bound** that is the lowest acceptable amount, or an **equality** that must be met exactly.
* The **decision variables** to which the constraint applies.
* The **impact that one unit of each decision variable** will have on the right-hand side quantity of the constraint.

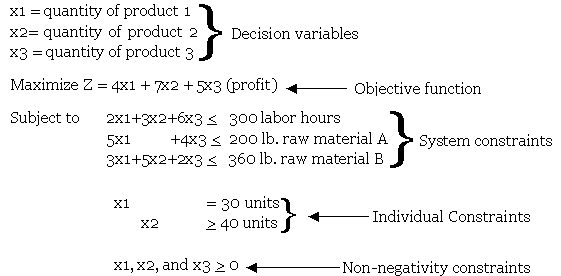
**Constraints** can be arranged into **three groups**:

* *System constraints* – involve **more than one decision** variable,
* *Individual constraints* – involve **only one** variable, and
* *Non-negativity constraints* – specify that **no variable** will be allowed to take on a **negative value.**

## Parameters

The objective function and the constraints consist of symbols that represent the decision variables (e.g., X1, X2, etc.) and **numerical values** called parameters. The parameters are **fixed values** that specify the **impact that one unit of each decision variable** will have on the **objective and on any constraint** it pertains to as well as the numerical value of each constraint.

The following simple example illustrates the components of LP models:



# 2.1.2. ASSUMPTIONS OF LP MODELS

## Linearity (proportionality)

The linearity requirement is that each **decision variable** has a linear impact on the **objective function** and in **each constraint** in which it appears. In terms of a mathematical model, a function or equation is linear when the variables included are **all to the power 1** (not squared, cubed, square root, etc.) and no products (e.g., x1x2) appear. On the other hand, the **amount of** **each resource used (supplied) and its contribution** to the profit (or cost) in the objective function **must be proportional to the value of each decision variable.**

**Divisibility (Continuity)**

The divisibility requirement pertains to potential **values of decision variables**. It is assumed that **non-integer values are acceptable**. However, if the problem concerns, for example, the optimal number of houses to construct, 3.5 do not appear to be acceptable. Instead, that type of problem would seem to require strictly integer solutions. In such cases, integer-programming methods should be used. It should be noted, however, that some obvious integer type situations could be handled under the assumption of divisibility. For instance, suppose 3.5 to be the optimal number of television sets to produce per hour, which is unacceptable, but it would result in 7 sets per two hours, which would then be acceptable.

## Certainty

This requirement involves **two aspects** of LP models. It is assumed that these values are **known and constant.** In practice, production times and other parameters may not be truly constant. Therefore, the model builder must make an assessment as to the degree to which the certainty requirement is met.

**Additivity**

**The value** of the **objective** function and the total amount of each **resource** used (or supplied), must be **equal to the sum of the respective individual contributions** (profit or cost) by decision variables. For example, the total profit earned from the sale of two products A and B must be equal to the sum of the profits earned separately from A and B.

## Non-negativity

It assumes that **negative values of variables are unrealistic** and, therefore, **will not be** considered in any **potential solutions**. Only **positive values and zero** will be allowed and the non-negativity assumption is inherent in LP models.

# 2.1.3. Advantages of Linear Programming

Following are certain advantages of linear programming.

* Linear programming helps in attaining the **optimum use of scarce productive resources.** It also indicates how a decision maker can **employ** productive resources effectively by **selecting and allocating** these resources.
* LP techniques improve **quality of decisions**. The decision making approach of the user of this technique becomes **more objective and less subjective**.
* **Highlighting of bottlenecks in the production process** is one of the most advantages of this technique.
* LP also helps in **re-evaluation of a basic plan** for **changing conditions.** If conditions change when the plan is carried out, they can be determined so as to adjust the remainder of the plan for best results.

# 2.1.4. Limitations of Linear Programming

# In spite of having many advantages and wide area of applications, there are some limitations associated with this technique. These are:

* LP **treats all relation ships** among decision variables **as linear**. However, generally **neither the objective functions nor the constraints in real-life situations** concerning business and industrial problems are linearly related to variables.
* While solving a LP model, there is **no guarantee that we will get integer valued solutions.** For example, in finding out how many men and machines would be required to perform a particular job, a non integer valued solution will be meaningless.
* A LP model **does not take in to consideration the effect of time** and **uncertainty.** Thus, the LP model should be defined in such a way that any change due to internal as well as external factors can be incorporated.
* It deals with **only with a single objective,** whereas in real-life situations we may come across conflicting multi-objective problems.
* Parameters appearing in the model are assumed to **be constant** but, in real life situation they are frequently **neither known nor constant**.

# 2.1.5. FORMULATING LP MODELS

Just as it is to define a problem, careful formulation of the model that will be used to solve the problem is important. Linear programming algorithms (solution techniques) are widely used and understood and computer packages are readily available for solving

Steps in formulating LP models:

* Identify the decision variables.
* Determine the objective function.
* Identify the constraints.
* Determine appropriate values for parameters and determine whether an upper limit, lower limit or equality is called for.
* Use this information to build a model.
* Validate the model.

# 2.2. LINEAR PROGRAMMING APPLICATIONS

There is a wide range of problems that lend themselves to solution by linear programming techniques. This discussion is only meant to give an indication of the LP techniques for managerial decision making and the apparent diversity of situations to which linear programming can be applied. Some of these include **production management** (product mix, blending problems, production planning, Assembly line balancing…), **Marketing management** (media selection, traveling sales man problem, physical distribution), **Financial management** (portfolio selection, profit planning), **agricultural application**, **military applications, personnel management** (staffing problem, Determination of equitable salary and etc.

## Product Mix/production management

Organizations often produce similar services that use the **same resources**. Linear programming answers **what mix of output** (or service) will **maximize profit** given the availability of **scarce resources**

## Diet problem

It usually involves the **mixing of raw materials or other ingredients** to obtain an end product that has certain characteristics. Other applications that fall into this category include mixing feed for livestock, mixing pet foods, mixing building materials (concrete, mortar, paint), and so on.

## Blending problems

They are very similar to diet problems. Strictly speaking, however, blending problems have additional requirement, i.e. to achieve a mix that have specific consistency. For example, how many quarts of the different juices each with different sugar content proportion must be mixed together to achieve one gallon that has a sugar content of 17 percent?

## Portfolio selection/investment

These problems generally involve allocating a **fixed dollar amount among a variety of investments,** such as bonds, real states, etc. The goal usually is to maximize income or total return.

**Examples**

**1 Product Mix**

ABC private limited company is engaged in the production of **power and traction transformers**. Both of these categories of transformers pass through **three basic processes:** **core preparation, core to coil assembly, and vapor phase drying**. A power transformer yields a contribution of Birr 50,000 and traction transformer contributes Birr 10,000. The **time required** in the **production of these two products** in terms of hours for each of the processes is as follows.

**Power transformer Traction Transformer**

Core preparation 75 15

Core to Coil Assembly 160 30

Vapor Phase Drying 45 10

If the **capacities available are 1000, 1500, and 750 machine hours** in each processes respectively, **formulate the problem as LP.**

**Solution**

*Step1***.** **Identify decision variables**

Since the products to be produced are **power and traction transformers** using the available resource to attain the objective set, we consider them as **decision variables**. This is because the organization’s **problem** here is **how many of each product** to produce in order to attain the **objective,** which requires the management decision.

LetX1 = the no of power transformers to be produced.

X2= the no of traction transformer to be produced

*Step2*. **Determine Objective Function**

From the problem above, we understand that the problem is **maximization problem.**

Hence,

Zmax = 50,000X1+ 10,000X2

This is because, each unit of X1 contributes Birr 50,000 and X2 contributes Birr

10,000 to objective function.

***Step 3*. Identify constraints**

Here we have **two constraints**: structural and **non-negativity** constraints. The

Structural constraint is the **amount of** **machine hours** available in each process.

*Step 4.* **Determining Parameters**

Parameters are already identified in the table of the problem.

☞**Note:** Step 3 and 4 can be performed simultaneously as:

75X1 + 15X2 < 1000 hrs- Core preparation process

160X1 + 30X2 < 1500 hrs- Core to Coil Assembly Machine hour constraint

45X1 + 10X2 < 750 hrs- Vapor Phase Drying

X1, X2 > 0

*Step 5***. Building and validating the model**

Zmax = 50,000X1+ 10,000X2

Subject to:

75X1 + 15X2 < 1000 hrs

160X1 + 30X2 < 1500 hrs

45X1 + 10X2 < 750 hrs

X1, X2 > 0

**Eg. 2 Investment Application**

An individual investor has **Birr 70,000 to divide** among several investments. The alternative investments are municipal **bonds** with an 8.5% return, **certificates of deposits** with a 10% return, **Treasury bill** with a 6.5% return, and **income bonds** with a 13% return. The amount of time until maturity is the **same** for each alternative. However, each investment alternative has a **different perceived risk** to the investor; thus it is advisable to diversify. The investor wants to know **how much to invest in each alternative** in order to **maximize the return**. The following **guidelines** have been established for diversifying the investment and lessening the risk perceived by the investor.

1. No more than 20% of the total investment should be in an income bonds.
2. The amount invested in certificates of deposit should not exceed the amount invested in other three alternatives.
3. At least 30% of the investment should be in treasury bills and certificates of deposits.
4. The ratio of the amount invested in municipal bonds to the amount invested in treasury bills should not exceed one to three.

The investor wants to invest the entire Birr 70,000

**Required**: Formulate a LP model for the problem.

Solution

**Decision variables**

Four decision variables represent the monetary amount invested in each investment alternative.

Let X1, X2, X3 and X4 represent investment on municipal bonds, certificates of deposits, Treasury bill, and income bonds respectively.

**Objective function**

The problem is **maximization** because from the word problem we know that the objective of the investor is maximization of return from the investment in the four alternatives.

Therefore, the objective function is expressed as:

Zmax = 0.085 X1+ 0.1X2+ 0.065X3+0.13X4

Where,

Z = total return from all investment

0.085 X1=return from the investment in municipal bonds.

0. 100 X2=return from the investment in certificates in deposit

0.065 X3=return from the investment in treasury bills

0.130 X4=return from the investment in income bonds

**Model Constraints**

In this problem the **constraints** are the **guide lines established** **for diversifying** the total investment. Each guideline is transformed in to mathematical constraint separately.

Guideline 1 above is transformed as:

X4 <20 %(70,000) ⇒ X4 <14,000

Guideline 2

X2< X1 +X3+ X4 ⇒ X2 - X1 -X3- X4< 0

Guideline 3

X2+ X3> 30 %( 70,000) ⇒ X2 + X3 > 21,000

Guideline 4

X1/ X3< 1/3⇒ 3X1<X3⇒3X1-X3< 0

Finally,

X1+ X2+ X3+X4 = 70,000, because the investor’s money equals to the sum of money invested in all alternatives.

This last constraint differs from  and  inequalities previously developed in that there is a specific requirement to invest an exact amount. Therefore, the possibility to invest more or less than Birr 70,000 is not considered.

This problem contains **all three types of constraints** possible in LP problems:  and =. As this problem demonstrates there is no restriction on mixing these types of constraints.

The complete LPM for this problem can be summarized as:

Zmax = 0.085 X1+ 0.1X2+ 0.065X3+0.13X4

Subjected to:

X4 <14,000

X2 - X1 -X3 - X4 < 0

X2 + X3 > 21,000

3X1-X3 < 0

X1 + X2 + X3 + X4 = 70,000,

X1+ X2+ X3+X40

**Eg. Marketing Application**

Supermarket store chain has hired an **advertising firm** to determine the **types and amount** of advertising it should have for its stores. The **three types** of advertising available are **radio television commercials, and news papers advertisements.** The retail chain desires to know the number of each type of advertisement it should purchase in order to **maximize exposure.** It is estimated that each ad or commercial will reach the following potential audience and cost the following amount.

**Exposure**

**Type of Advertisement** (people /ad or commercial) Cost/constraint

Television commercial 20,000 Birr 15,000

Radio commercial 12,000 6,000

News paper advertisement 9,000 4,000

The company must consider the following **resource constraints**.

1. The budget limit for advertising is Birr 100,000
2. The television station has time available for 4 commercials.
3. The radio station has time available for 10 commercials.
4. The news paper has space available for 7 ads.
5. The advertising agency has **time and staff** available for producing **no more than a total of 15 commercials and/or ads.**

*Decision variables*

Let X1 = number of television commercials

X2 = number of radio commercials

X3 = number of news paper ads

**Objective function**

It is not only profit which is to be maximized. In this problem the objective to be maximized is **audience exposure.**

Zmax = 20,000x1 + 12,000x2 + 9,000x3, where

Z = total level of audience exposure

20,000x1 = estimated number of people reached by television commercials

12,000x2 = estimated number of people reached by radio commercials

9,000x3 = estimated number of people reached by news paper ads

**Model constraints**

**Budget constraint**

15,000x1 + 6,000x2 + 4,000x3 Birr 100,000

**Capacity constraint**

Television commercials and radio commercials are limited to 4 and 10 respectively and news paper ads are limited to 7.

X1 4 television commercials

X2  10 radio commercials

X3 7 news paper ads

**Policy constraint**

The total number of commercials and ads can not exceed 15

X1 + X2 + X315

The complete linear programming model for this problem is summarized as:

Zmax = 20,000X1 + 12,000X2 + 9,000X3

Sub. to:

15,000X1 + 6,000X2 + 4,000X3 Birr 100,000

X1  4

X2  10

X3  7

X1 + X2 + X3 15

X1, X2 , X3  0

**Chemical mixture**

A chemical corporation produces a chemical mixture for the customer in 1000- pound batches. The mixture contains **three ingredients- Zinc, mercury and potassium**. The mixture must conform to formula specifications (i.e., a recipe) supplied by a customer. The company wants to know the amount of each ingredient to put in the mixture that will meet all the requirements of **the mix** and **minimize total cost.**

The customer has supplied the following formula specifications for each batch of mixture.

* + - 1. The mixture must contain at least 200 lb of mercury
      2. The mixture must contain at least 300 lb of zinc

3. The mixture must contain at least 100 lb of potassium

The **cost per pound of mixture** is Birr4 of mercury, Birr 8 of zinc, and Birr 9 of potassium.

**Required**: Formulate LPM for the problem

**Solution**

**Decision variables**

The model of this problem consists of three decision variables representing the amount of each ingredient in the mixture.

X1 = number of lb of mercury in a batch

X2 = number of lb of zinc in a batch

X3 = number of lb of potassium in a batch

**Objective function**

Zmin = 4x1 + 8x2 + 9x3

*Constraints*

X1200 lb… specification 1

X2300 lb… specification 2

X3100 lb… specification 3

Finally, the sum of all ingredients must equal 1000 pounds.

x1 + x2 + x3 = 1000 lb

The complete LPM of the problem is:

Zmin = 4x1 + 8x2 + 9x3

Subject to:

X1200 lb

X2300 lb

X3100 lb

x1 + x2 + x3 = 1000 lb

x1, x2 , x30

# 2.3. Solving LP Model

# Following the formulation of a mathematical model, the next stage in the application of LP to decision making problem is to find the solution of the model. An optimal, as well as feasible solution to a LP problem is obtained by choosing from several values of decision variables x1,x2…xn , the one set of values that satisfy the given set of constraints simultaneously and also provide the optimal ( maximum or minimum) values of the given objective function. The most common solution approaches are to solve graphically and algebraically the set of mathematical relationships that form the model, thus determining the values of decision variables.

# 2.3.1. GRAPHICAL LINEAR PROGRAMMING METHODS

Graphical linear programming is a relatively straightforward for determining the optimal solution to certain linear programming problems involving **only two decision variables**. Although graphic method is limited as a solution approach, it is very useful in the presentation of LP, in that it gives a “picture” of how a solution is derived thus a better understanding of the solution. More over, graphical methods provide a visual portrayal of many important concepts.

In this method, the two decision variables are considered as ordered pairs (X1, X2), which represent a point in a plane, i.e, X1 is represented on X-axis and X2 on Y-axis.

Graphical method has the following advantages:

* It is simple
* It is easy to understand, and
* It saves time.

**Important Definitions**

***Solution:*** Theset of valuesof decision variablesxj (j = 1,2,…, n) which **satisfy the constraints** of a LP problem is said to constitute solution to that LP problem.

**Feasible solution** The set of values of decision variables xj (j = 1,2,…, n) which **satisfy all the constraints and non- negativity conditions** of a LP problems simultaneously.

**Infeasible solution** The set of values of decision variables xj (j = 1,2,…, n) which **do not satisfy** **all the constraints and non- negativity** conditions of an LP problems simultaneously is said to constitute the infeasible solution to that LP problem.

**Basic solution** For a set of m simultaneous equations in n variables ( n>m), a solution obtained **by setting ( n-m) variables equal to zero** and solving for remaining m equations in m variables is called a basic solution.

The (n-m) variables **whose value did not appear in this solution** are called non-basic variables and the remaining m variables are called **basic variables.**

**Basic feasible solution** A feasible solution to LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values. Basic feasible solutions are of two types:

* *Degenerate* A basic feasible solution is called degenerate **if value of at least one** **basic variable is zero.**
* *Non-degenerate* A basic feasible solution is called non-degenerate **if value of all m basic variables are non- zero and positive.**

**Optimal Basic feasible solution:** A basic feasible solution which **optimizes** the objective function value of the given LP problem is called an **optimal basic feasible solution**.

**Unbounded solution:** A solution which **can increase or decrease** the value of the LP problem **indefinitely** is called an unbounded solution.

**Example**

In order to demonstrate the method, let us take a microcomputer problem in which a firm is about to start **production of two new microcomputers**, X1 and X2. Each requires limited resources of **assembly time, inspection time, and storage space.** The manager wants to determine **how much of each computer** to produce in order to **maximize the profit** generated by selling them. Other relevant information is given below:

Type 1 Type 2

Profit per unit $60 $50

Assembly time per unit 4 hrs 10 hrs

Inspection time per unit 2 hrs 1 hrs

Storage space per unit 3 cubic feet 3 cubic feet

Availability of company resources:

Resources Amount available

Assembly time 100 hrs

Inspection time 22 hrs

Storage space 39 cubic feet

The model is then:

Maximize 60X1 + 50X2

Subject to Assembly 4 X 1 +10 X 2 < 100 hrs

Inspection 2 X 1 + 1 X 2 < 22 hrs

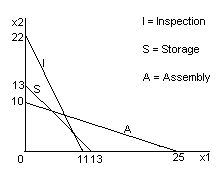
Storage 3 X 1 + 3 X 2 < 39 cubic feet

X1, X2 > 0

**Graphical solution method** involves the following steps.

**Step 1. Plot each of the constraints**

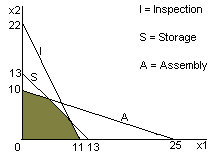
The step begins by plotting the non-negativity constraint, which restricts our graph only to the first quadrant. We then can deal with the task of graphing the rest of the constraints in two parts. For example, let us take the first constraint 4X1 + 10X2 < 100 hrs. First we treat the constraint as equality: 4X1 + 10X2 = 100. Then identify the easiest two points where the line intersects each axis by alternatively equating each decision variable to zero and solving for the other: when X1=0, X2 becomes 10 and when X2=0, X1 will be 25. We can now plot the straight line that is boundary of the feasible region as we have got two points (0, 10) and (25,0).





## Step 2. Identify the feasible region

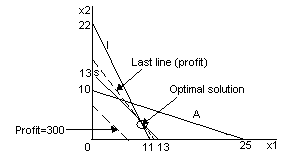
The feasible region is **the solution space that satisfies all the constraints simultaneously**. It is the **intersection of the entire region** represented by all constraints of the problem. We shade in the **feasible region depending on the inequality sign**. In our example above, for all the constraints except the non-negativity constraint, the inequality sign is **‘less than or equal to’** and it represents region of the plane below the plotted lines.



**2.3.1.1. The Objective Function Approach**

This approach involves plotting an objective function line on the graph and then using that line to determine where in the feasible solution space the optimal point is. We use the same logic as plotting a constraint line except that it is not an equation until we equate it to some right hand side quantity. Any quantity will do till we find a line that would last touch the **feasible solution space.**

The **optimal solution to a LP problem will always occur at a corner point** because as the objective function line is moved in the direction that will improve its value (e.g., away from the **origin in our profit maximization** problem), it will last touch one of **these intersections of constraints.** Then we determine which **two constraints intersect there** (in our **case inspection and storage constraints)** and solve the **equations simultaneously to obtain the mix of the two decision variables** that gives the value of the objective function **at the optimum**. Simultaneously solving inspection and storage equations, we find the quantity of type 1 microcomputer to be produced (X1) = 9 and that of type 2 (X2) = 4 giving the maximum profit of 60(9) + 50(4) = Birr740

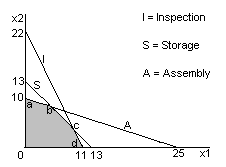


**2.3.1.2. The Extreme Point Approach**

Corner or extreme point graphic method states that for problems that have optimal solutions, a solution will occur at the corner point in the case of unique solution, while in the case of multiple solutions, at least one will occur at a corner point as these multiple solutions will be combinations of those points between two corner points. The necessary steps for this approach is after graphing the problem, we determine the values of the decision variables at each corner point of the feasible region either by inspection or using simultaneous equations. We then substitute the values at each corner point into the objective function to obtain its value at each corner point and select the one with the highest value of the objective function (for a maximum problem) or lowest value (for a minimum problem) as the optimal solution.

i) **Extreme Method of Solving Maximization Problems with < constraint**

Maximization of objective function involves finding the point where the combination of products results in maximum value of objective function. The constraints are connected with < sign. The solution space lies below the slant line and is bounded by the line segments. The origin and other points below the slant lines are in the solution space (i.e., feasible region).



Using this method for our example, simultaneously solving for corner points a, b, c, and d, we find corresponding profit values of 500, 700, 740, and 660, respectively giving us the same solution as the above one at C. Therefore, the optimal solution is x1= 9 units and x2 = 4 units while the optimal value of objective function is 740.

**Interpretation**:

For a firm to maximize its profit (740), it should produce 9 units of the Model I microcomputer and 4 units of model II.

**Minimization case**

**Solving of Minimization problems**

Solving minimization problems involve where the **objective functions (like cost function) will be minimum.** The constraints are connected to **RHS values with > sign**. But, in real world problems, mixes of constraint is also possible. The value of RHS involves the **lowest value** of the constraint. The solution space **lies above the slant lines and it is not enclosed**. It extends indefinitely **above the lines in the first quadrant**. This means simply that cost increases without limit as **more and more units are produced.** The minimum cost will occur at a **point along the inner boundary of the solution space.** The origin and other points below the lines **are not** in the solution space.

**Example**

Zmin = 0.1x+0.07y

Subject to:

6x+2y > 18

8x+10y > 40

y > 1

x,y > 0

Find the values of x and y which makes the objective function minimum?

# Solution

Coordinates

Corner point x y Zmin=0.1x+0.07y

A 0 9 0.63

**B 25/11 24/11 0.38**

C 15/4 1 0.445

# 2.3.1.3. Graphical & objective Solutions for the Special Cases of LP

## i) Unboundedness

Unboundedness occurs when the **decision variable increased indefinitely** without violating any of the constraints. The reason for it may be concluded to be **wrong formulation** of the problem such as incorrectly **maximizing instead of minimizing** and/or errors in the given problem. Checking equalities or rethinking the problem statement will resolve the problem.

Example:

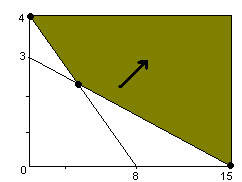
Max Z = 10X1 + 20X2

Subject to 2X1 + 4X2 > 16

X1 + 5X2 > 15

X1, X2 > 0

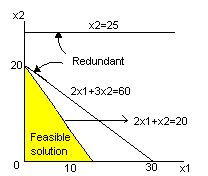
Following the above listed steps of graphical solution method, we find the following graph for the above model:



The shaded area represents the set of **all feasible solutions** and as can be seen from the graph, the **solution is unbounded**.

## ii) Redundant Constraints

In some cases, a constraint **does not form a unique boundary** of the feasible solution space. Such a constraint is called a **redundant constraint**. A constraint is redundant if its removal **would not alter the feasible solution space**. Redundancy of any constraint **does not cause any difficulty** in solving LP problems graphically. Constraints appear redundant when it may be more binding (restrictive) than others.



## iii) Infeasibility

In some cases after plotting all the constraints on the graph, feasible area (common region) that represents all the constraint of the problem **cannot be obtained**. In other words, infeasibility is a condition that arises when **no value of the variables satisfy all** the **constraints simultaneously**. Such a problem arises due to **wrong model formulation** with conflicting constraints.

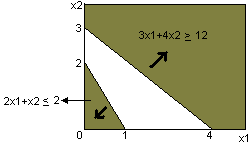
For example,

Max Z = 3X1+2X2

Subject to: 2X1 + X2 < 2

3X1 + 4X2 > 12

X1, X2 > 0



## iv) Multiple optimal solutions

Recall the optimum solution is that extreme point for which the objective function has the largest value. It is of course possible that in a given problem there may be **more than one optimal solution.**

There are two conditions that should be satisfied for an alternative optimal solution to exist:

* The given objective function is **parallel to a constraint** that forms the boundary of the feasible region. In other words, the **slope of an objective function** is the same as that of the **constraint forming** the boundary of the feasible region; and
* The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint should be an active constraint.

☞**Note**: The constraint is said to be an **active or binding or tight**, if at optimality the **left hand side equals the right hand side.** In other words, an **equality constraint** is always active. An inequality sign may or may not be active.

For example

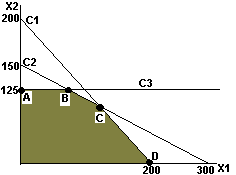
Max Z = 8X1+16X2

Subject to: X1 + X2 < 200 ……. C1

3X1 + 6X2 < 900 ……. C2

X2 < 125 ……. C3

X1, X2 > 0



In the problem above, using extreme point method and solving for values of corner points simultaneously, the objective function assumes its maximum value of **2,400 at two corner points** B (50,125) and C (100,100). Therefore, the optimal solution is *found on the line segment connecting the two corner points*. One benefit of having multiple optimal solutions is that for other (perhaps qualitative) reasons, a manager may prefer one of them to the others, even though each would achieve the same value of the objective function. In practical terms, one of the two corner points is usually chosen because of ease in identifying its values.

**Summary**

* The term linear programming refers to a family of mathematical techniques for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the limited or constrained resources.
* The linear programming models exhibit certain common characteristics: An objective function to be maximized or minimized, a set of constraints, decision variables for measuring the level of activity, and linearity among all constraint relationships and the objective function.
* The graphic approach to the solution LP problems is not efficient means of solving problems. For one thing, drawing accurate graphs is tedious. More over the graphic approach is limited to models with only two decision variables.
* Special cases that one face solving a problem graphically include: Mix of constraints, unbounded solution, infeasibility, redundancy, multiple solution

# 2.3.2. THE SIMPLEX METHOD

Dear learner, in our previous section we tried to thoroughly discuss about graphic approach of LP problems, where the LP consists of **only two** decision variables. But, the **real world problems may consist of even very large number of decision variables** which **can not be solved using graphic approach.** Therefore, in this section we will try to see about simplex method, which is more comprehensive as compared to graphic method.

The graphical method is restricted to problems with **two decision variables**. When the number of **variables and the number of constraints increase**, it becomes **difficult to visualize** the solution space. In order **to avoid this limitation, the simplex method**, or **iterative or step by step method** is efficient method for solving linear programming problems, which was developed by George B. Datzing in 1947.

The simplex method is an **algebraic procedure that starts with a feasible solution that is not optimal and systematically moves from one feasible solution to another until an optimal solution is found.**

Constraints are generally expressed using inequalities either in ‘less than’ or ‘greater than’ or in mixed form. Thus, constraints are **not in standard form**, meaning they should be converted into **equalities**.

To convert the **inequality** constraint into **equality**, **we introduce slack or surplus variables**.

In economic terminology, **slack variables** represent **unused capacity and surplus variables represent excess amount**. The **contribution (cost or profit)** associated with the slack and surplus variables **is zero**. An inequality of the ‘less than or equal to’ type is transformed into **equality by introducing a non-negative slack variable**, as follows:

Example

Non Standard form Standard form

X1+2X2 <= 6 X1+2X2+S1= 6

, where X1 and X2 are decision variables and S1 is a slack variable, **added** to the

smaller side of the inequality.

On the other hand, an inequality with **‘greater than or equal to’** type is changed into equality by **subtracting surplus** variable as follows:

As stated above, since both the **slack and surplus variables** are insignificant with no contribution in the objective function, they are **represented with coefficient of zero in the objective** function.

To find a **unique solution**, the **number of variables should not exceed the number of equations**. When the number of variables is more than the number of equations, the number of solutions is **unlimited.** So as to get a unique solution, we have to set at least (n-m) variables to zero, where n is the number of variables and m is the number of equalities.

To demonstrate the simplex method, we will use the microcomputer problem with the following objective function and constraints.

**Non standard form**

Max. max Z = 60X1+50X2

Subject to 4X1+10X2 ≤ 100

2X1+X2 ≤22

3X1+3X2 ≤ 39

The Microcomputer Problem, which was discussed in graphic approach, can be **standardized** as:

Max. Z = 60X1+50X2+0S1+0S2+0S3

Subject to 4X1+10X2+S1 = 100

2X1+X2+S2 = 22

3X1+3X2+S3 = 39

, where X1 & X2 are decision variables and S1, S2 & S3 are slack variables.

Here, the number of variables (5) is greater than the number of equations (3). Therefore, the decision variables are set to zero.

This solution will serve as an initial feasible solution. An **initial feasible solution** is a first solution used to generate **other basic feasible solutions**. The initial basic feasible solution is obtained by setting all the decision variables to zero. As a result, the initial basic feasible solution is entirely composed of the slack variables. X1 & X2 are **non basic variables** since they **are not in solution**. S1, S2 & S3 are **basic** **variables since they are in solution**.

A *tableau* is a system of displaying the basic feasible solution, the constraints of the standard linear programming problem as well as the objective function in a tabular form. A tableau is useful in summarizing the result of each **iteration,** i.e. the process of moving from one solution/corner to another solution/corner in order to select the optimal solution.

# THE SIMPLEX ALGORITHM

# 2.3.2.1. Maximization case

The solution steps of the simplex method can be outlined as follows:

Step1. **Formulate** the linear programming model of the real world problem, i.e., obtain a mathematical representation of the problem's objective function and constraints.

Step2. Express the mathematical model of L.P. problem in the **standard form** by adding slack variables in the left-hand side of the constraints and assign a zero coefficient to these in the objective function.

Thus we can restate the problem in terms of equations:

Maximize Z = C1X1+ C2X2 + ... +CnXn + OS1 + OS2 +... +0Sm

Subject to a11X1+a12X2+... + a1nxn+s1i =b1

a2lX1+al22X2+... + a2nXn+S2 = b2

amlXl + am2 X2 +... + amNxn + Sm = bm

, where X1, X2... Xn and S1, S2 ... Sm are non-negative.

**Note that the slack variables** have been assigned **zero coefficients** in the **objective function**. The reason is that these variables typically **contribute nothing to the value of the objective** function.

Step 3. Design the **initial feasible solution.** An initial basic feasible solution is obtained by setting the decision variables to zero.

X1= X2 = ... = Xn = 0. Thus, we get S1 = b1, S2 = b2 ... Sm = bm.

Step 4. **Set up the initial simplex tableau**. For computational efficiency and simplicity, the initial basic feasible solution, the constraints of the standard LPP as well as the objective function can be displayed in a tabular form, called the simplex tableau as shown below:

Initial Simplex Tableau

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Cj  (Contribution Per Unit) | | C1 C2 ... Cn,, 0 0 ... 0 | Quantity Column/solution | Ratio  XB/amn |
| Basic variables | CB | X1, X2, ... Xn S1 S2 ... Sm | b(=Xj) |  |
| S1 | CB1 | a11 a12 ... a1n 1 0 ... 0 | b1=Xb1 |  |
| S2 | CB2 | a21 a22 ... a2n 0 1 ... 0 | b2=Xb2 |  |
| Sm | CBm | am1 am2…amn 0 0 ... 1 | bm=Xbm |  |
| Zj=∑CBiXj | | 0 0 ... 0 0 0 ... 0 | ∑CBiXj |  |
| Cj-Zj  (Net contribution per unit) | | C1-Z1 C2-Z2 ... Cn-Zn |  |  |

The **interpretation of the data** in the above tableau is given as under. Other simplex tableau will have similar interpretations.

* In the first row labeled "Cj", we write the **coefficients of the variables in the objective** function. These values will remain the same in subsequent tableaus.
* The second row shows the major column headings.
* In the first column of the second row, under the label *"Basic variables"* (also called *Product mix* column), the basic variables are listed.
* In the second column of the second row, under the label "CB", the coefficients of the current basic variables in the objective function are listed. Thus the **coefficients** of S1, S2... Sm, which are included in the initial feasible solution, are written in the **CB** column.
* The values listed under the non-basic variables (X1, X2… Xn) in the initial simplex tableau consists of the coefficients of the decision variables in the constraint set. They can be interpreted as physical rates of substitution.
* The values listed under the basic variables (S1, S2... Sm) in the initial simplex tableau represents the coefficients of the slack variables in the constraints set.
* In the next column (also called *Quantity column*), we write the **solution values** of the basic variables.
* **To find an entry in the Zj**row under a column, we multiply the **entries** of that column by the **corresponding entries** of ‘CB’ column and **add the results,** i.e**., Zj= ∑CBiX**j. The Zj row entries will all be **equal to zero** in the initial simplex tableau. The other Zj entries represent the decrease in the value of objective function that would result if one of the variables not included in the solution were brought into the solution. The Zj entry under the “*Quantity Column*" gives the current value of the objective function.
* The **last row labeled "Cj-Zj", called the index row or net evaluation row,** is used to determine **whether or not the current solution is optimal or not.** The calculation of Cj-Zj row simply involves subtracting each Zj value from the corresponding Cj value for that column, which is written at the top of that column.

☞***Note****:*  The entries in the Cj - Zj, row represent **the net contribution to the objective function** that results by introducing one unit of each of the respective column variables. A **plus value** indicates that a greater contribution can be made by bringing the variable for that column into the solution. A negative value indicates the amount by which contribution would decrease if one unit of the variable for that column were brought into the solution. Index row elements are also known as the *shadow prices (or accounting prices)*

Step 5. **We test if the current solution is optimum or not**. If all the elements or entries in the Cj- Zj row (i.e., index row) are **negative or zero**, then the current solution is **optimum.** If there exists some **positive number**, the current solution can be further improved by **removing one basic variable** from the basis and replacing it by some **non-basic one.** So start trying to improve the current solution in line with the following steps.

Step 6: Further, **iterate towards an optimum solution**. To improve the current feasible solution, we replace one current basic variable (called the *departing variable*) by a new non-basic variable (called the *entering variable*).

* We now determine the **variable to enter** into the solution mix, *the entering variable*. One way of doing this is by identifying the column with the **largest positive value in the Cj - Zj row of the simplex table.** The **column with the largest positive entry** in the Cj - Zj row is called the **key or pivot column.** The non-basic variable at the top of the key column is the **entering variable** that will replace a basic variable.
* Next, we determine the **departing variable** to be replaced in the basis solution. This is **accomplished by dividing each number in the quantity column by the corresponding number in the key column** selected in identifying the entering variable. We compute the ratio b1/a1j, b2/a2j... bm/amn. This is called replacement ratio.

*Replacement Ratio (RR) = Solution Quantity (Q)*

*Corresponding values in pivot column*

* The row **with the minimum ratio is the key row or pivot row**. The corresponding variable in the key row (the departing variable) will leave the basis.
* We identify the *key or pivot element*. This is the number that lies at the **intersection of the key column and key row of a given simplex tableau.**

Step 7. Evaluate the new solution by constructing a second simplex tableau. After identifying the entering and departing variable, all that remains is to find the new basic feasible solution by constructing a new simplex tableau from the current one.

Now we evaluate or update the new solution in the following way:

* **New values for the key row** are computed by simply **dividing every element of the key row by the key element** to obtain a unit vector (1) in the key element.
* The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element (1) in the key column are zero, i.e. unit vector.

Step 8. If any of the numbers in Cj - Zj row are **positive, repeat the steps (6-7) again** until an **optimum solution has been obtained.**

☞***Not****e: Rules for Ties*. In choosing a key column and a key row, whenever there is a tie between two numbers, the following rules may be followed:

* The column farthest to the left may be selected if there is a tie between two numbers in the index row.
* The nearest ratio to the top may be selected whenever there is a tie between two replacement ratios in the ratio column.

**Example (Maximization Case)**

Finding the initial Feasible Solution

The initial feasible solution is found by setting the decision variables to zero.

Max Z = 60X1+50X2+0S1+0S2+0S3

Subject to 4X1+10X2+S1 = 100

2X1+ X2+S2 = 22

3X1+ 3X2+S3 = 39

X1, X2 > = 0

Initial Simplex tableau

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 60 | 50 | 0 | 0 | 0 | Quantity |
| X1 | X2 | S1 | S2 | S3 |
| S1 | 0 | 4 | 10 | 1 | 0 | 0 | 100 |
| S2 | 0 | 2 | 1 | 0 | 1 | 0 | 22 |
| S3 | 0 | 3 | 3 | 0 | 0 | 1 | 39 |
| Zj | | 0 | 0 | 0 | 0 | 0 | 0 |
| Cj-Zj | | 60 | 50 | 0 | 0 | 0 |

**Interpretation of the Initial Feasible Solution**

To be noted first is that the values of each basic variable (variables that are in solution) is composed of a single 1 and the rest 0’s. This is called a *unit vector*. Basic variables will have a unit vector. Moreover, ‘1’ will appear in the same row that the variable appears in. The unit vector concept will help us in developing subsequent tableaus when we want to change the list of variables that are in solution.

The Zj row in the **quantity column indicates** that the value of the objective function **is 0.**

A simplex solution in a maximization problem is optimal if the Cj-Zj row consists of entirely **zeros and negative numbers**, i.e. there are no positive values in the row.

So for our case in the initial tableau, we have **two positive values** under the non-basic variables, **which indicate that further improvement** of the solution is possible. As a result, we go for the **optimal solution** by developing the **second simplex tableau.**

**Developing the Second Simplex Tableau**

In the initial feasible solution, the slack variables form entirely the **basic variables and the decision variables entirely form the non-basic variables.** Since further improvement is possible, one of the decision variables will be brought into the solution and one of the current basic variables will be leaving the solution. ‘Which non-basic variable should be brought into the solution and which basic variable should leave the solution?’ is a major concern here.

In answering the **first question**, the non-basic variable that should **enter the solution** should be the one with the **highest positive value in the Cj-Zj row** since bringing that non-basic variable into the solution would make the **highest contribution to the solution** (objective function) and result in the **largest profit potential.** As a result, the variable with the highest value in the Cj-Zj row of the initial simplex tableau is **the X1 column** **with a value 60.** Therefore, **X1 should enter** the basis or the solution mix. The **X1 column** is now called the **pivot column.**

Since X1 has the highest profit potential, we need to make as much X1 units as possible. By **dividing the values in the pivot column by their respective values in the quantity column, we can identify the variable that is most limiting.(which has small value-11)/leaving var**

The values obtained by dividing will help us in determining the **variable that should leave the solution.**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 60 | 50 | 0 | 0 | 0 | Quantity | |
| X1 | X2 | S1 | S2 | S3 |
| S1 | 0 | 4 | 10 | 1 | 0 | 0 | 100 | 100 |
| S2 | 0 | 2 | 1 | 0 | 1 | 0 | 22 | 22/2 = 11 |
| S3 | 0 | 3 | 3 | 0 | 0 | 1 | 39 | 39 |
| Zj | | 0 | 0 | 0 | 0 | 0 | 0 | |
| Cj-Zj | | 60 | 50 | 0 | 0 | 0 |

Pivot Column

Pivot Row Pivot Element

In this particular case, there is only enough of the second constraint to make **11 units of X1**. In making the 11 units of X1, the second resource (S2) will be down to **zero indicating that S2 will leave out the solution mix.** The row of the leaving variable is called the **pivot row**. The **intersection** of the pivot row and the pivot column is called the **pivot element.** As a rule,

The leaving variable is the one with the **smallest non-negative ratio.**

Since we have determined the leaving and the entering variables and since the initial feasible solution can be improved further, we need to develop the **second tableau in order to find the optimal solution.** In developing the second tableau, we should compute for revised values of the constraint equations, the Zj row and the Cj-Zj row and remember that the variables in solution will have a unit vector, with a value of 1 in the intersection of the column and the row of the basis, i.e. the pivot element.

To obtain a unit vector in the basis column, we perform elementary row operations resulting in new row values by either multiplying/dividing all the elements in a row by a constant or adding/subtracting the multiple of a row to or from another row.

Having these **new row values**, we develop the second simplex tableau as shown below.

**Second Simplex Tableau**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 60 | 50 | 0 | 0 | 0 | Quantity |
| X1 | X2 | S1 | S2 | S3 |
| S1 | 0 | 0 | 8 | 1 | -2 | 0 | 56 |
| X1 | 60 | 1 | ½ | 0 | ½ | 0 | 11 |
| S3 | 0 | 0 | 3/2 | 0 | -3/2 | 1 | 6 |
| Zj | | 60 | 30 | 0 | 30 | 0 | 660 |
| Cj-Zj | | 0 | 20 | 0 | -30 | 0 |

*Interpretation of the Second Simplex Tableau*

The profit obtained at this point of solution is $660. In the Cj-Zj row, we search for the highest positive value. If there is, it means that we can further improve this solution.

Therefore, we have **a positive** value in the Cj-Zj row which indicates that this is **not the optimal solution**. As a result, we go for the **next tableau**.

Developing the Third Tableau

Here, we select the entering and the leaving variables. The entering variable is the one with the **highest Cj-Zj row value which is the X2 column**. This means that bringing one unit of X2 into the solution increases profit by $20. Therefore, the X2 will be the entering variable designated as the **pivot column.**

To **determine the leaving the variable**, we **divide** the values in the pivot column by their corresponding row values in the **quantity column**. The result obtained, as shown in the table below indicates that **S3** is the leaving variable with the **smallest non-negative ratio.** This means that S3 is the most **limiting resource** for how much units of X2 can be made.

Second tablue

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. Bj | | 60 | 50 | 0 | 0 | 0 | Quantity |  |
| X1 | X2 | S1 | S2 | S3 |
| S1 | 0 | 0 | 8 | 1 | -2 | 0 | 56 |
| X1 | 60 | 1 | ½ | 0 | ½ | 0 | 11 |
| S2 | 0 | 0 | 3/2 | 0 | -3/2 | 1 | 6 |
| Zj | | 60 | 30 | 0 | 30 | 0 | 660 |
| Cj-Zj | | 0 | 20 | 0 | -30 | 0 |

Pivot Row

Pivot Column Pivot Element

Third Simplex Tableau

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 60 | 50 | 0 | 0 | 0 | Quantity |
| X1 | X2 | S1 | S2 | S3 |
| S1 | 0 | 0 | 0 | 1 | 6 | -16/3 | 24 |
| X1 | 60 | 1 | 0 | 0 | 1 | -1/3 | 9 |
| X2 | 50 | 0 | 1 | 0 | -1 | 2/3 | 4 |
| Zj | | 60 | 50 | 0 | 10 | 40/3 | 740 |
| Cj-Zj | | 0 | 0 | 0 | -10 | -40/3 |

**Interpreting the Third Tableau**

All the values in the Cj-Zj row are **zero and negative** indicating that there is **no additional improvement.** This makes the third tableau to **contain the optimal solution** with the following basic variables:

S1=24 X1 = 9, and X2 = 4 producing a maximum profit of $740.

This means that in order to achieve the maximum profit, the company should produce 9 units of X1 and 4 units of X2 leaving 24 hours of unused resource of the second constraint.

# 2.3.2.2. MINIMIZATION LINEAR PROGRAMMING PROBLEMS

**Big M-method /Charnes Penalty Method/**

The Big-M Method is a technique**,** which is used in **removing artificial variables** from the basis. In this method; we assign **coefficients to artificial** variables, undesirable from the objective function point of view. If objective function Z is to be **minimized,** then a very **large positive** **price** (called ***penalty***) is assigned to each artificial variable. Similarly, if Z is to be **maximized**, then a very **large negative cost** (also called ***penalty***) is assigned to each of these variables. Following are the characteristics of Big-M Method:

1. High penalty cost (or profit) is assumed as M
2. M as a coefficient is assigned to **artificial variable A** in the objective function Z.
3. Big-M method can be applied to minimization as well as maximization problems with the following distinctions:
   * 1. ***Minimization problems***

-Assign **+M** as coefficient of artificial variable A in the objective function **Z** of the minimization problem.

* + 1. ***Maximization problems:***

-Here **–M** is assigned as coefficient of artificial variable **A** in the objective function **Z** of the maximization problem.

1. Coefficient of ***S*** (slack/surplus) takes zero values in the objective function **Z**
2. For minimization problem, the incoming variable corresponds to the **highest negative value** of ***Cj-Zj.***
3. The **solution is optimal** when there is **no negative value of *Cj-Zj.(****For minimization LPP* case)

The various steps involved in using simplex method for **minimization** problems are:

Step 1. Formulate the linear programming model, and express the mathematical model of L.P. problem in the standard form by **introducing surplus and artificial variables** in the left hand side of the constraints. Assign a 0 (zero) and +M as coefficient for surplus and artificial variables respectively in the objective function. M is considered a very large number so as to finally drive out the artificial variables out of basic solution.

Step 2. Next, an **initial solution** is set up. Just to initiate the solution procedure, the initial basic feasible solution is obtained by assigning zero value to decision variables. This solution is now summarized in the initial simplex table. Complete the initial simplex table by adding two final rows Z, and Cj - Zj. These two rows help us to know whether the current solution is **optimum or not.**

Step 3. Now; **we test for optimality** of the solution. If all the entries of **Cj - Zj, row are positive,** then the solution is optimum. But if at least one of the Cj - Zj values is less than zero, the current solution can be further improved by removing one basic variable from the basis and replacing it by some non-basic one.

Step 4. (i) Determine the variable to enter the basic solution. To do this, we identify the column with the **largest negative value in the Cj - Zj row** of the table.

(ii) Next we determine the **departing variable** from the basic solution. If an artificial variable goes out of solution, then we discard it totally and even this variable may not form part of further iterations.

Step 5. We update the new solution now. We evaluate the entries for next simplex table in exactly the same manner as was discussed earlier in the maximization case.

Step 6. Step (3—5) are repeated until an optimum solution is obtained.

So the following are the essential things to observe in solving for minimization problems:

* The **entering variable** is the one with the **largest negative value** in the Cj-Zj row while the **leaving variable** is the one with the **smallest non-negative ratio/+ve value.**
* The optimal solution is obtained when the Cj-Zj row contains entirely **zeros and positive values.**

**Example:** Assume the following minimization problem.

Min Z = 7X1+9X2

Subject to 3X1+6X2 >= 36

8X1+4X2 > = 64

X1, X2 > = 0

We introduce both surplus and artificial variables into both constraints as follows.

Min Z = 7X1+9X2+0S1+0S2+MA1+MA2

Subject to 3X1+6X2 –S1+A1 = 36

8X1+4X2-S2+A2 = 64

X1, X2 > = 0

So the subsequent tableaus for this problem are shown below.

**Initial Simplex Tableau**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 7 | 9 | 0 | 0 | M | M | Quantity |
| X1 | X2 | S1 | S2 | A1 | A2 |
| A1 | M | 3 | 6 | -1 | 0 | 1 | 0 | 36/3=12 |
| A2 | M | 8 | 4 | 0 | -1 | 0 | 1 | 64/8=8 |
| Zj | | 11M | 10M | -M | -M | M | M | 20M |
| Cj-Zj | | 7-11M | 9-10M | M | M | 0 | 0 |

From initial table the **large –ve value is 7-11M**, then it is key column and entering variable-X1

The **row** with **smallest** quantity ratio value is leaving variable-**A2**

**Second Simplex Tableau**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 7 | 9 | 0 | 0 | M | Quantity |
| X1 | X2 | S1 | S2 | A1 |
| A1 | M | 0 | 9/2 | -1 | 3/8 | 1 | 12 |
| X1 | 7 | 1 | ½ | 0 | -1/8 | 0 | 8 |
| Zj | | 7 | 7/2+9/2M | -M | 3/8M-7/8 | M | 56+12M |
| Cj-Zj | | 0 | 11/2-9/2M | M | 7/8-3/8M | 0 |

From this table the **large –ve value is x2** which becomes **entering variable**. And the **left** **artificial** variable is **A1- leaving** and which replaced by **X2.**

Third Simplex Tableau

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 7 | 9 | 0 | 0 | Quantity |
| X1 | X2 | S1 | S2 |
| X2 | 9 | 0 | 1 | -2/9 | 1/12 | 8/3 |
| X1 | 7 | 1 | 0 | 1/9 | -1/6 | 20/3 |
| Zj | | 7 | 9 | -11/9 | -5/12 | 212/3 |
| Cj-Zj | | 0 | 0 | 11/9 | 5/12 |

The third tableau represents a final tableau since it is the **optimal solution with entirely zeros and non-negative values in the Cj-Zj row.**

Therefore, the optimal solution is: X1 = 20/3 and X2 = 8/3 and value of objective function is 212/3.

|  |  |  |  |
| --- | --- | --- | --- |
| ***Types of***  ***constraint*** | ***Extra variables to be added*** | ***Coefficient of extra variables***  ***in the objective function***  ***MaxZ MinZ*** | ***Presence of variables in the initial solution mix*** |
| ***<*** | ***Add only slack variable*** | ***0 0*** | ***Yes*** |
| ***>*** | ***Subtract surplus variable and*** | ***0 0*** | ***No*** |
|  | ***Add artificial variable*** | ***-M +M*** | ***Yes*** |
| ***=*** | ***Add artificial variable*** | ***-M +M*** | ***Yes*** |

Summary

**Summary**

* The simplex method is an algebraic procedure that starts with a feasible solution that is not optimal and systematically moves from one feasible solution to another until an optimal solution is found
* The variations described in general simplex approach include; maximization and minimization problems, mixed constraint problems, problems with multiple optimal solution, problems with no feasible solution, undounded problems, tied pivot columns and rows.
* Multiple optimal solution are identified by Cj-Zj(or Zj- Cj) =0 for a non basic variable. To determine the alternate solution(s), enter the non basic variable(s) with a Cj-Zj value equal to zero.
* An infeasible problem is identified in the simplex procedure when an optimal solution is achieved and one or more of the basic **variables are artificial.**

# SIMPLEX MAXIMIZATION WITH MIXED CONSTRAINTS

(= AND >= CONSTRAINTS)

The Big M Method—introducing slack, surplus and artificial variables

* If any problem constraint has **negative constants** on the right side, **multiply** both sides **by -1** to obtain a constraint with a non-negative constant. (If the constraint is an inequality, this will reverse the direction of the inequality.)
* Introduce a **slack variable** in each <= constraints.
* Introduce a **surplus variable and an artificial** variable in each >= constraint.
* Introduce an **artificial variable** in each '='constraint.
* For each artificial variable Ai, **add -MAi** to the objective function in case of **maximization** and **+MAi in case of minimization**. Use the same **constant M** for all artificial variables.
* Form the simplex tableau for the modified problem.
* Solve the modified problem using the simplex method.
* Relate the solution of the modified problem to the original problem:
  + If the modified problem has no solution, then the original problem has no solution.
  + If any artificial variables are non-zero in the solution to the modified problem, then the original problem has no solution.

☞***Note****.* The **artificial** variables are introduced for the **limited purpose** of obtaining an initial solution and are required for the constraints of >= type or the constraints with '=' sign.

In addition to this, a **slack is not allowed in >= constraints** since they can only happen an **excess amount** that is more than the minimum amount of a constraint. This excess amount is represented by a **surplus variable and is subtracted** from the constraint.

**Example – Maximization Problems with Mixed Constraints**

Assume the following maximization problem with mixed constraints.

Max. Z = 6X1+8X2

Subject to X2 <= 4

X1+X2 = 9

6X1+2X2 >= 24

In solving for this problem, introducing a **slack variable** is not acceptable since they represent unused capacity and there is no unused capacity in = and >= constraints. Therefore, for >= constraints, we introduce a surplus variable and for both >= and = constraints we introduce artificial variables resulting as follows.

Max Z = 6X1+8X2+0S1+0S3-MA2-MA3

Subject to X2+ S1 = 4

X1+X2+A2 = 9

6X1+2X2-S3+A3 = 24

X1, X2 > = 0

Wouldn’t it be possible to set A2 as A1 and A3 as A2 in the above standardized LPM form?

**Developing** the Initial Simplex Tableau

In the initial tableau, we list the variable in the order of decision variable, **slack**/surplus variables and finally **artificial variables**. And for constraints, use artificial variables, and/or slack variables and/or surplus variables for the initial solution. Therefore the initial tableau for the problem is represented as follows.

**Initial Simplex Tableau**

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 6 | 8 | 0 | 0 | -M | -M | Quantity |
| X1 | X2 | S1 | S3 | A2 | A3 |
| S1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 4 |
| A2 | -M | 1 | 1 | 0 | 0 | 1 | 0 | 9 |
| A3 | -M | 6 | 2 | 0 | -1 | 0 | 1 | 24 |
| Zj | | -7M | -3M | 0 | +M | -M | -M | -33M |
| Cj-Zj | | 6+7M | 8+3M | 0 | -M | 0 | 0 |

Since this is a **maximization problem**, the **entering** variable is the one with the maximum Cj-Zj value. Therefore, the variable with the **maximum Cj-Zj is X1**=6+7M, **entering variable**. And the **leaving variable** is with the **minimum non-negative quantity ratio** which is A3, the artificial variable. Since this artificial variable **is not needed we remove** it from the next tableau.

**Developing the Second Simplex Tableau**

After identifying the entering and the leaving variable, the usual elementary row operations are performed to obtain a unit vector in the entering variable column. The end result after performing the row operations is as follows which shows the second tableau.

**Second Simplex Tableau**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 6 | 8 | 0 | 0 | -M | Quantity |
| X1 | X2 | S1 | S3 | A2 |
| S1 | 0 | 0 | 1 | 1 | 0 | 0 | 4 |
| A2 | -M | 0 | 2/3 | 0 | 1/6 | 1 | 5 |
| X1 | 6 | 1 | 1/3 | 0 | -1/6 | 0 | 4 |
| Zj | | 6 | 2-2/3M | 0 | -1-M/6 | -M | 24-5M |
| Cj-Zj | | 0 | 6+2/3M | 0 | 1+M/6 | 0 |

Again here, we identify the leaving the entering variables, i.e. the pivot row and the pivot column respectively. The **entering** variable with the highest Cj-Zj row value is **X2=6+2/3M**and the **leaving** variable with the **smallest ratio is S1**.

**Developing the Third Simplex Tableau**

Selecting the entering and leaving variables, we go for obtaining a unit vector in the pivot column by using the elementary row operations. We get the following third tableau.

**Third Simplex Tableau**

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 6 | 8 | 0 | 0 | -M | Quantity |
| X1 | X2 | S1 | S3 | A2 |
| X2 | 8 | 0 | 1 | 1 | 0 | 0 | 4 |
| A2 | -M | 0 | 0 | -2/3 | 1/6 | 1 | 7/3 |
| X1 | 6 | 1 | 0 | -1/3 | -1/6 | 0 | 8/3 |
| Zj | | 6 | 8 | 6+2/3M | -1-M/6 | -M | 48-7/3 |
| Cj-Zj | | 0 | 0 | -6-2/3M | 1+M/6 | 0 |

Similarly, we identify the entering and the leaving variables which are S3 =1+M/6 and A2 = 7/3 respectively representing the maximum Cj-Zj value and the minimum ratio.

**Developing the Fourth Simplex Tableau**

Perform elementary row operations usual to have a unit vector in the entering variable column and you will get the following fourth tableau.

**Fourth Simplex Tableau**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Cj  Basic V. | | 6 | 8 | 0 | 0 | Quantity |
| X1 | X2 | S1 | S3 |
| X2 | 8 | 0 | 1 | 1 | 0 | 4 |
| S3 | 0 | 0 | 0 | -4 | 1 | 14 |
| X1 | 6 | 1 | 0 | -1 | 0 | 5 |
| Zj | | 6 | 8 | 2 | 0 | 62 |
| Cj-Zj | | 0 | 0 | -2 | 0 |

This tableau represents the final tableau since we have only zeros and negative values in the Cj-Zj row which indicates that it is the optimal solution. So we have the following results for each of the variables and the profit obtained.

X1 = 5, X2 = 4, S3 = 14, and Profit = 62